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Extensions of abelian by hyper-(cyclic or finite) groups (II)

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Extensions of Abelian by Hyper-(Cyclic or Finite) Groups (II).

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ABSTRACT - If G is a hypercyclic (or hyperfinite and locally soluble) group and A a noetherian $\mathbb{Z}G$-module with no nonzero cyclic (or finite) $\mathbb{Z}G$-factors then Zaĭcev proved that any extension E of A by G splits conjugately over A. For G being a hyper-(cyclic or finite) locally soluble group, if A is a periodic artinian $\mathbb{Z}G$-module with no nonzero finite $\mathbb{Z}G$-factors, then we have shown that any extension E of A by G splits conjugately over A, too. Here we consider the noetherian case and prove the splitting theorem which generalizes that of Zaĭcev for G being a hyperfinite and locally soluble group.

In [1], we have proved: if G is a hyper-(cyclic or finite) locally soluble group and if A is a periodic artinian $\mathbb{Z}G$-module with no nonzero finite $\mathbb{Z}G$-factors, then any extension E of A by G splits conjugately over A. Now we continue the work and are going to prove the same result for A being noetherian.

The following lemma generalizes the corresponding one in Zaĭcev's paper [6] and is very important in our later proof.

LEMMA 1. Let H be a normal hyper-(cyclically or finitely) embedded subgroup of a group G, and let A be a nonzero noetherian $\mathbb{Z}G$-module. If $C_A(H) = 0$, then there is a subgroup K of H and a nonzero $\mathbb{Z}G$-submodule B of A such that K is normal in G, $C_B(K) = 0$, and K induces in B a cyclic or finite group of automorphisms.

PROOF. Suppose the lemma is false. Using the noetherian condition we may assume that the lemma is true in all proper $\mathbb{Z}G$-module A. We may also assume that G acts faithfully on A.

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There is a cyclic or finite subgroup $F \leq H$ with F being normal in G. If $C_A(F) = 0$ then the lemma is true taking F, A for K, B.

Consider the second possibility $C_A(F) \neq 0$. We let A_1 be the $\mathbb{Z}G$-submodule $C_A(F)$ and let $H_1 = C_H(F)$. Then H_1 is normal in G and $|H/H_1| < \infty$.

(1) Suppose that the centralizer $A_2/A_1 = C_{A/A_1}(H)$ is nonzero, i.e., $A_2 \neq A_1$. Consider the $\mathbb{Z}H_1$-isomorphism $A_2/C_{A_2}(f) \cong_{\mathbb{Z}H_1} A_2(f-1)$, where $f \in F$. Since $A_1 \leq C_{A_2}(f)$ and A_2/A_1 is H_1-trivial, we have that $A_2(f-1)$ is H_1-trivial for any $f \in F$. It follows that

$$[A_2, F] = \sum_{f \in F} A_2(f-1)$$

is H_1-trivial and so H induces a finite group of automorphisms on $[A_2, F]$. Since $A_2 \neq A_1$ the $\mathbb{Z}G$-submodule $[A_2, F] \neq 0$ and $C_{[A_2, F]}(H) = 0$ since $C_A(H) = 0$. Therefore the lemma is true with $K = H$, $B = [A_2, F]$.

(2) Suppose now that $A_2 = A_1$, i.e., $C_{A/A_1}(H_1) = 0$. Then the $\mathbb{Z}G$-module A/A_1 and the normal subgroup H_1 satisfy the hypotheses of the lemma and so there is a subgroup K_1 of H_1 and nonzero $\mathbb{Z}G$-submodule B_1/A_1 of A/A_1 such that K_1 is normal in G, $C_{B_1/A_1}(K_1) = 0$, and K_1 induces in B_1/A_1 a cyclic or finite group of automorphisms.

Put $G_1 = C_G(F)$; clearly $H_1 = H \cap G_1$, $|G/G_1| < \infty$.

(*a*) We consider firstly the case that $K_1/C_{K_1}(B_1/A_1)$ is cyclic.

Let $B_2 = [B_1, F]$ and let $K_0 = C_{K_1}(B_1/A_1)$. Since $A_1 = C_A(F)$, so

$$[K_0, B_1, F] = [[K_0, B_1], F] \leq [A_1, F] = 0;$$

also by $K_0 \leq K_1 \leq H_1 = C_H(F)$, we have

$$[F, K_0, B_1] = [[F, K_0], B_1] = [1, B_1] = 0.$$

Thus by the three subgroup lemma,

$$[B_2, K_0] = [[B_1, F], K_0] = [B_1, F, K_0] = 0.$$

Therefore $B_2 \leq C_A(K_0)$ and we then can view the noetherian $\mathbb{Z}G$-module B_2 as a noetherian $\mathbb{Z}(G/K_0)$-module. Applying Lemma 3 in [5] to the cyclic normal subgroup K_1/K_0 of G/K_0, there is an integer m such that

$$B_2(k-1)^m \cap C_{B_2}(k) = 0,$$

where k is an element such that $K_1 = K_0\langle k\rangle$.

If $B_2(k-1)^m = 0$, then

$$0 = B_2(k-1)^m = \Big(\sum_{f\in F} B_1(f-1)\Big)(k-1)^m = \sum_{f\in F} B_1((f-1)(k-1)^m) =$$

$$= \sum_{f\in F} B_1((k-1)^m(f-1)) = \sum_{f\in F} (B_1(k-1)^m)(f-1).$$

That is, $B_1(k-1)^m \leqslant C_A(F) = A_1$. But this is contrary to

$$C_{B_1/A_1}(k) = C_{B_1/A_1}(K) = 0.$$

So we have $B_2(k-1)^m$ and then the lemma is true by taking $B = B_2(k-1)^m$ and $K = K_1$.

(*b*) Secondly, we consider the case that $K_1/C_{K_1}(B_1/A_1)$ is finite. Choose in F a least set of elements $\{x_1, \ldots, x_n\}$ satisfying

$$A_1 = C_{B_1}(F) = C_{B_1}(x_1) \cap \ldots \cap C_{B_1}(x_n)$$

and put $B_2 = C_{B_1}(x_1) \cap \ldots \cap C_{B_1}(x_{n-1})$ if $n > 1$ and $B_2 = B_1$ if $n = 1$. Then

$$(1) \qquad B_2 \neq A_1$$

and $C_{B_2}(x_n) = C_{B_1}(x_1) \cap \ldots \cap C_{B_1}(x_n) = A_1$. Consider the $\mathbb{Z}G_1$-isomorphism

$$(2) \qquad B_2/A_1 = B_2/C_{B_2}(x_n) \cong_{\mathbb{Z}G_1} B_2(x_n - 1).$$

Since $K_1 \leqslant G_1$, $B_2 \leqslant B_1$, and K_1 indices a finite group of automorphisms on B_1/A_1, so K_1 induces a finite group of automorphism on B_2/A_1 and hence on $B_2(x_n-1)$. Since $C_{B_1/A_1}(K_1) = 0$ we also have $C_{B_2(x_n-1)}(K_1) = 0$.

Let $D = B_2(x_n - 1)$. Then D is a $\mathbb{Z}G_1$-submodule of B_1, $C_D(K_1) = 0$, and $|K_1/C_{K_1}(D)| < \infty$. Let $\overline{D}$ be the $\mathbb{Z}G$-module generated by D, then $\overline{D} = \sum_{g\in T} Dg$ is a finite sum of $\mathbb{Z}G_1$-submodules Dg, where T is a transversal to G_1 in G.

Note that since K_1 is normal in G, $C_{Dg}(K_1) = C_D(K_1)g = 0$, and $C_{K_1}(Dg) = g^{-1}C_{K_1}(D)g$. It follows that $|K_1/\bigcap_{g\in T} C_{K_1}(Dg)| < \infty$ and so K_1 induces a finite group of automorphisms in $\overline{D}$.

Now consider two cases.

(*A*) *D contains an element of finite order.*

Then D contains a maximal elementary abelian $\mathfrak{p}$-subgroup $D_1 (\neq 0)$

and we let $\overline{D}_1 = \sum_{g\in T} D_1 g$. Let S be the K_1-socle of the $\mathbb{Z}G_1$-submodule D_1, i.e., the sum of all irreducible $\mathbb{Z}G_1$-submodules (these irreducible $\mathbb{Z}G_1$-submodules are all finite since K_1 induces a finite group of automorphisms in D). Since D_1 is a $\mathbb{Z}G_1$-submodule and K_1 is normal in G so S is a $\mathbb{Z}G_1$-submodule and $\overline{S} = \sum_{g\in T} Sg$ is a $\mathbb{Z}G$-submodule. Now Sg is a sum of irreducible $\mathbb{Z}K_1$-submodules and so $\overline{S}$ is a sum of irreducible $\mathbb{Z}K_1$-submodules each being contained in some Sg. Since $C_{Dg}(K_1) = 0$ it follows that $C_{\overline{S}}(K_1) = 0$. Thus we can take K_1 and $\overline{S}$ satisfying the conclusion of the lemma.

(*B*) *The group D is torsion-free.*

Let $T(\overline{D})$ be the torsion part of $\overline{D}$. Since $\overline{D}$ is a noetherian $\mathbb{Z}G$-module, $T(\overline{D})$ has a finite exponent. Therefore $n\overline{D} \cap T(\overline{D}) = 0$ for some n and $n\overline{D}$ is torsion-free.

We put $m = |K_1/C_{K_1}(\overline{D})|$, $C = C_{\overline{D}}(K_1)$ and show that

$$[mn\overline{D}, K_1] \cap C = 0. \tag{3}$$

In fact, if $a \in [mn\overline{D}, K_1] \cap C$, then $a \in [mn\tilde{D}, K_1] \cap C$, for some finitely generated K_1-admissible subgroup $\tilde{D}$ of $\overline{D}$. Since $n\tilde{D} \cap C = C_{n\tilde{D}}(K_1)$, $\tilde{D} \leqslant \overline{D}$, and $n\overline{D}$ is torsion-free, so $n\tilde{D}/(n\tilde{D} \cap C)$ is torsion-free and then $n\tilde{D} = (n\tilde{D} \cap C) \oplus V$, where V is a free abelian subgroup. Applying Theorem 4.1 in [2], there is in $n\tilde{D}$ a K_1-admissible subgroup W such that $(n\tilde{D} \cap C) \cap W = 0$ and the factor group $n\tilde{D}/[(n\tilde{D} \cap C) \oplus W]$ has a finite exponent, dividing m. Thus $mn\tilde{D} \leqslant (n\tilde{D} \cap C) \oplus W$. It follows that $[mn\tilde{D}, K_1] \leqslant W$ and so $[mn\tilde{D}, K_1] \cap C = 0$. Hence $a = 0$ and (3) is proved.

Note now that $[mn\tilde{D}, K_1] \neq 0$. In fact, if $[mn\tilde{D}, K_1] = 0$, then $mn\overline{D} \leqslant C_{\overline{D}}(K_1) = C$. Therefore $mnD \leqslant C$ and since D is torsion-free, $D \leqslant C$. This shows that D is a K_1-trivial $\mathbb{Z}G_1$-module and since $D = B_2(x_n - 1)$ and is G_1-isomorphic to B_2/A_1 by (2) we have B_1/A_1 is also K_1-trivial. But $C_{B_1/A_1}(K_1) = 0$ and so $B_2 = A_1$ contrary to (1). Thus $[mn\overline{D}, K_1] \neq 0$. Since $[mn\overline{D}, K_1]$ is a $\mathbb{Z}G$-submodule and K_1 induces in it (as in $\overline{D}$) a finite group of automorphisms then it follows from (3) that the conditions of the lemma are satisfied by K_1 and $[mn\overline{D}, K_1]$. The lemma is proved.

As in the hyperfinite case, we need:

LEMMA 2. Let G be a hyper-(cyclic or finite) group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that B is of finite index in A and B has no nonzero finite $\mathbb{Z}G$-factors, then

B has a complement in A, i.e., $A = B \oplus C$ for some finite $\mathbb{Z}G$-sobmodule C of A.

PROOF. Suppose that B does not have a complement in A. By considering an appropriate factor-module of A we may assume that for every $\mathbb{Z}G$-submodule D of B with $D \neq 0$, B/D has a complement in A/D.

Put $H = C_G(A/B)$, then, since G/H is finite and the irreducible $\mathbb{Z}G$-factors of B are all infinite, we have $C_B H) = 0$ so we can apply Lemma 1 to the subgroup H and the $\mathbb{Z}G$-module B. So there is a subgroup K of H and a nonzero $\mathbb{Z}G$-submodule D of B such that K is normal in G, $C_D(K) = 0$ and K induces on D a cyclic or finite group of automorphisms, i.e., $K/C_K(D)$ is cyclic or finite.

We write A as a sum $A = B + A_1$ with $B \cap A_1 = D$ and we will consider the $\mathbb{Z}G$-submodule A_1 as a faithful $\mathbb{Z}G_0$-module, where $G_0 = G/C_G(A_1)$. It is clear that D is a $\mathbb{Z}G_0$-submodule of A_1 such that D is of finite index in A_1 and D has no nonzero finite $\mathbb{Z}G_0$-factors. Also D has no complements in A_1 for otherwise if $A_1 = D \oplus C_1$ for some $\mathbb{Z}G_0$-submodule C_1 of A_1 then C_1 can be viewed as a $\mathbb{Z}G$-submodule of A by $G_0 = G/C_G(A_1)$ and then $A = B + A_1 = B \oplus C_1$, a contradiction.

Since $C_D(K) = 0$ and $D \leqslant A_1$, so K is not contained in $C_G(A_1)$. Let $K_0 = (KC_G(A_1))/C_G(A_1)$, then $K_0 \neq 1$. Also, it is clear that $C_D(K_0) = 0$ and K_0 induces on the $\mathbb{Z}G_0$-submodule D of A_1 a cyclic or finite group of automorphisms. We prove that $C_{K_0}(D) = 1$. For suppose $C_{K_0}(D) \neq 1$ and let F_0 be a nontrivial cyclic or finite normal subgroup of G_0 contained in $C_{K_0}(D)$. If $x \in F_0$, then $D \leqslant C_{A_1}(x)$. Since $|A_1/D| = |A/B| < \infty$ and, as groups, $A_1/C_{A_1}(x) \cong A_1(x-1)$, we see that $A_1(x-1)$ is finite. Thus the $\mathbb{Z}G_0$-submodule $[A_1, F_0]$ is finite. Also

$$F_0 \leqslant C_{K_0}(D) \leqslant K_0 = (KC_G(A_1))/C_G(A_1) \leqslant (HC_G(A_1))/C_G(A_1) = \\ = (C_G(A/B)\,C_G(A_1))/C_G(A_1),$$

thus $[A_1, F_0] \leqslant B$, and then $[A_1, F_0] \leqslant D$. By D having no nonzero finite $\mathbb{Z}G_0$-factors, we have $[A_1, F_0] = 0$ contrary to G_0 acting faithfully on A_1. So $C_{K_0}(D) = 1$ and hence K_0 is cyclic or finite.

Now put

$$G_1 = C_{G_0}(K_0), \qquad K_0 = \langle x_1 = 1, x_2, \ldots, x_m \rangle, \qquad C_n = C_{A_1}(\langle x_1, \ldots, x_n \rangle), \\ n = 1, 2, \ldots, m.$$

We prove that $A_1 = D + C_n$, $n = 1, 2, \ldots, m$.

It is clear that $A_1 = D + C_1$. Suppose $A_1 = D + C_n$ we prove $A_1 =$

$= D + C_{n+1}$. Consider the isomorphism of $\mathbb{Z}G_1$-modules

$$C_n/C_{n+1} = C_n/C_{C_n}(x_{n+1}) \cong_{\mathbb{Z}G_1} C_n(x_{n+1} - 1),$$

where $C_n(x_{n+1} - 1)$ may not be contained in C_n if K_0 is nonabelian. Since

$$x_{n+1} \in K_0 = (KC_G(A_1))/C_G(A_1) \leqslant (HC_G(A_1))/C_G(A_1) \leqslant$$
$$\leqslant (C_G(A/B)\, C_G(A_1))/C_G(A_1),$$

the $\mathbb{Z}G_1$-module $C_n(x_{n+1} - 1)$ of A_1 is contained in B and then in D. Since $|G_0/G_1| < \infty$ it follows from Proposition 2 in [4] that the irreducible $\mathbb{Z}G_1$-factors of D are all infinite, hence so are the factors of C_n/C_{n+1}. But

$$C_n/(C_{n+1} + (D \cap C_n)) \cong_{\mathbb{Z}G_1} (C_n + D)/(C_{n+1} + D),$$

a factor module of the finite module A_1/D. Hence $C_n + D = C_{n+1} + D$ and so $A_1 = C_{n+1} + D$. Thus $A_1 = C_n + D$ for all $n = 1, 2, \ldots, m$. In particular, put $n = m$, $C_m = C_{A_1}(K_0)$ and $A_1 = D + C_{A_1}(K_0)$. Since $C_{A_1}(K_0)$ is clearly a $\mathbb{Z}G_0$-submodule of A_1 and since $D \cap C_{A_1}(K) = C_D(K) = 0$ we have $A_1 = D \oplus C_{A_1}(K)$, contrary to D having no complements in A_1. The proof is completed.

From the proof of Lemma 2, we have:

LEMMA 3. Let G be a hyper-(cyclic or finite) group, A a noetherian $\mathbb{Z}G$-module, and B a $\mathbb{Z}G$-submodule of A such that, as group, A/B is a finite $\mathfrak{p}$-group for some prime $\mathfrak{p}$ and the $\mathbb{Z}G$-submodule B contains no nonzero $\mathbb{Z}G$-factors being finite $\mathfrak{p}$-groups. Then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

Dual to Lemma 2, we have:

LEMMA 4. Let G be a hyper-(cyclic or finite) group, A a $\mathbb{Z}G$-module, and B a finite $\mathbb{Z}G$-submodule of A such that all irreducible $\mathbb{Z}G$-factors of A/B are infinite. Then B has a complement on A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

PROOF. By Zorn's Lemma, A has a $\mathbb{Z}G$-submodule D maximal with respect to $B \cap D = 0$. We show that $A = B \oplus D$. Suppose not, then by replacing A by A/D we may assume that for any nonzero $\mathbb{Z}G$-submodule C of A, $B \cap C \neq 0$. We also assume that G acts faithfully on A.

Put $H = C_G(B)$, $|G/H| < \infty$ so there is a normal subgroup K of G contained in H such that K is either cyclic or finite. Put $H_1 = C_H(K)$.

Since H_1 is normal in G and $|G/H_1| < \infty$ it follows from Proposition 2 in [4] that the irreducible $\mathbb{Z}H_1$-factors of A/B are infinite. If $x \in K$, then $B \leqslant C_A(x)$ and so the irreducible $\mathbb{Z}H_1$-factors of $A/C_A(x)$ and hence $A(x-1)$ are infinite.

We prove that $[A, K] \cap B = 0$. If not, then there is a minimal set of elements $x_1, \dots, x_n$ such that $B_1 = B \cap \sum_{i=1}^{n} A(x_i - 1) \neq 0$. Then

$$B_1 \cong_{\mathbb{Z}H_1} \left(b_1 \oplus \sum_{i=1}^{n-1} A(x_i - 1) \right) \Big/ \left(\sum_{i=1}^{n-1} A(x_i - 1) \right) =$$

$$= \left(\sum_{i=1}^{n} A(x_i - 1) \right) \Big/ \left(\sum_{i=1}^{n-1} A(x_i - 1) \right) \cong_{\mathbb{Z}H_1}$$

$$\cong_{\mathbb{Z}H_1} A(x_i - 1) \Big/ \left(A(x_n - 1) \cap \sum_{i=1}^{n-1} A(x_i - 1) \right).$$

This shows that $A(x_n - 1)$ has a nonzero finite $\mathbb{Z}H_1$-factor contrary to the fact that the irreducible $\mathbb{Z}H_1$-factors of $A(x-1)$ are all infinite. Thus $[A, K] \cap B = 0$ and hence $[A, K] = 0$, contrary to G acting faithfully on A. So the result is true.

An immediate consequence of Lemma 4 is:

COROLLARY 5. Let G be a hyper-(cyclic or finite) group, and A a noetherian $\mathbb{Z}G$-module. Then A has a nonzero finite $\mathbb{Z}G$-factor if and only if A has a nonzero finite $\mathbb{Z}G$-image.

PROOF. We only need to suppose that A has a finite $\mathbb{Z}G$-factor B/C, then using the noetherian condition we may assume that every irreducible $\mathbb{Z}G$-factor of A/B is infinite. Then applying Lemma 4 to A/C with the finite $\mathbb{Z}G$-submodule B/C we obtain a finite $\mathbb{Z}G$-image of A.

As before, we have:

LEMMA 6. Let G be a hyper-(cyclic or finite) group, A a $\mathbb{Z}G$-module and B a $\mathbb{Z}G$-submodule of A. If as a group B is a finite $\mathfrak{p}$-group for some prime $\mathfrak{p}$, and if the factor module A/B contains no nonzero finite $\mathbb{Z}G$-factors being $\mathfrak{p}$-groups, then B has a complement in A, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule C of A.

COROLLARY 7. Let G be a hyper-(cyclic or finite) group, and A a noetherian $\mathbb{Z}G$-module. Then A has a nonzero $\mathbb{Z}G$-image being a finite $\mathfrak{p}$-group for some prime $\mathfrak{p}$ if and only if A has such a nonzero $\mathbb{Z}G$- factor.

Before we prove the main splitting theorem, we need to prove the following three results.

LEMMA 8. Let G be a hyper-(cyclic or finite) group, B a $\mathbb{Z}G$-module, and A a noetherian $\mathbb{Z}G$-submodule of B such that all irreducible $\mathbb{Z}G$-factors of A are infinite. If B/A is torsion-free and G-trivial, then $B = A \oplus B_1$ for some $\mathbb{Z}G$-submodule B_1 of B.

PROOF. Suppose that A has no complements in B. Since A is noetherian, we may assume that for each nonzero $\mathbb{Z}G$-submodule C of A, A/C has a complement in B/C.

In B, we choose a $\mathbb{Z}G$-submodule M maximal with respect to $A \cap M = 0$. We show that if S is any $\mathbb{Z}G$-submodule such that $B = A + S$ then $M \leqslant S$.

Since $B/A \geqslant (A \oplus M)/A \cong_{\mathbb{Z}G} M$, we have M is a G-trivial $\mathbb{Z}G$-module and hence all of its irreducible $\mathbb{Z}G$-factors are finite. Also

$$A/(A \cap S) \cong_{\mathbb{Z}G} (A + S)/S = B/S \geqslant (M + S)/S \cong_{\mathbb{Z}G} M/(M \cap S).$$

Since A is noetherian and having no nonzero finite $\mathbb{Z}G$-factors, we must have $M = M \cap S$, i.e., $M \leqslant S$.

Consider the factor-module B/M. Every nonzero $\mathbb{Z}G$-submodule of B/M has nonzero intersection with $(A \oplus M)/M$. In particular, $(A \oplus M)/M$ has no complements in B/M. If V/M is a nonzero $\mathbb{Z}G$-submodule of $(A \oplus M)/M$ then $V = C \oplus M$, where $C = A \cap V$ is nonzero and so $B/C = A/C \oplus S_1/C$ for some $\mathbb{Z}G$-submodule S_1 of B. As above, $M \leqslant S_1$ and so $(A \oplus M) \cap S_1 = (A \cap S_1) \oplus M = C \oplus M = V$. Thus S_1/V is a complement to $(A \oplus M)/V$ in B/V.

By passing to the factor-module B/M we may assume that $M = 1$ so that: (a) A has no complements in B but for any nonzero $\mathbb{Z}G$-submodule C of A, A/C has a complement in B/C; (b) if N is a nonzero $\mathbb{Z}G$-submodule of B then $A \cap N \neq 0$.

We may assume that A is torsion-free. For otherwise, we may let $A[\mathfrak{p}]$ be the nonzero $\mathbb{Z}G$-submodule generated by all the elements of order $\mathfrak{p}$, where $\mathfrak{p}$ is a prime. By (a), $B/A[\mathfrak{p}] = A/A[\mathfrak{p}] \oplus B_1/A[\mathfrak{p}]$. Since $B_1/A[\mathfrak{p}] (\cong_{\mathbb{Z}G} B/A)$ is torsion-free, $\mathfrak{p}B_1 \neq 0$, then, by (b), $0 \neq A \cap \cap \mathfrak{p}B_1 \leqslant A[\mathfrak{p}] \cap B_1$. That is, B_1 has elements of order $\mathfrak{p}^2$, contrary to $B_1/A[\mathfrak{p}]$ being torsion-free. So A is torsion-free and then B is torsion-free. Since A has no nonzero finite $\mathbb{Z}G$-factors, we have $C_A(G) = 0$. By Lemma 1, G has a normal subgroup K and A has a nonzero $\mathbb{Z}G$-submodule A_1 such that $C_{A_1}(K) = 0$ and $K/C_K(A_1)$ is cyclic or finite. By (a),

$B/A_1 = A/A_1 \oplus B_1/A_1$. Consider the $\mathbb{Z}G$-module B_1 and we prove that $B_1 = A_1 \oplus B_2$ for some $\mathbb{Z}G$-submodule B_2 (and hence we get $B = A \oplus B_2$ as required).

Suppose $B_1 \neq A_1 \oplus B_2$ for any $\mathbb{Z}G$-submodule B_2 and suppose that G acts faithfully on B_1, i.e., $C_G(B_1) = 1$. It is clear that we still have that K is normal in G, $C_{A_1}(K) = 0$, and $K/C_K(A_1)$ is cyclic or finite. If $C_K(A_1) \neq \neq 1$, then, since $C_K(A_1) = K \cap C_G(A_1)$ is a normal subgroup of G, $C_K(A_1)$ contains a nontrivial cyclic or finite subgroup F being normal in G. Let $F = \langle f_i, \ldots, f_n \rangle$ and let $G_1 = C_G(F)$, then $|G/G_1| < \infty$. By Proposition 2 in [4], the irreducible $\mathbb{Z}G_1$-factors of A_1 are infinite. Since B_1/A_1 is G-trivial, it is also G_1-trivial. By $B_1/C_{B_1}(f_i) \cong_{\mathbb{Z}G_1} B_1(f_i - 1) \leq A_1$ and $A_1 \leq \leq C_{B_1}(f_i)$, we must have $B_1(f_i - 1) = 0$, for all i. That is, $1 \neq F \leq C_G(B_1)$, contrary to G acting faithfully on B_1. So $C_K(A_1) = 1$ and so K is a nontrivial cyclic or finite normal subgroup of G. Let $K = \langle k_1, \ldots, k_t \rangle$. Being similar with the above, we have $B_1/C_{B_1}(k_i) \cong_{\mathbb{Z}G_2} B_1(k_i - 1) \leq A_1$ for all i, where $G_2 = C_G(K)$. Thus $B_1/(A_1 + C_{B_1}(k_i))$ must be zero for all i. That is, $B_1 = A_1 + C_{B_1}(k_i)$ for any i. Let $C_m = C_{B_1}(\langle k_1, \ldots, k_m \rangle)$, $m = 1, \ldots, t$. Then we have $B_1 = A_1 + C_1$. Suppose that $B_1 = A_1 + C_m$; we prove that $B_1 = A_1 + C_{m+1}$.

Consider the $\mathbb{Z}G_2$-modules

$$C_m/C_{m+1} = C_m/C_{C_m}(k_{m+1}) \cong_{\mathbb{Z}G_2} C_m(k_{m+1} - 1).$$

Since B_1/A_1 is G-trivial, $C_m(k_{m+1} - 1) \leq A_1$ and so $C_m(k_{m+1} - 1)$ has no nonzero finite $\mathbb{Z}G_2$-factors; hence the irreducible $\mathbb{Z}G_2$-factors of C_m/C_{m+1} are all infinite. But

$$C_m/(C_{m+1} + (A_1 \cap C_m)) \cong_{\mathbb{Z}G_2} (C_m + A_1)/(C_{m+1} + A_1),$$

a factor module of the G_2-trivial $\mathbb{Z}G_2$-module B_1/A_1. Hence $A_1 + C_m = = A_1 + C_{m+1}$. That is, $B_1 = A_1 + C_{m+1}$. Therefore $B_1 = A_1 + C_m$ for all m. Put $m = n$, then $C_n = C_{B_1}(K)$ and $B_1 = A_1 + C_{B_1}(K)$, which implies that $C_{B_1}(K) \neq 0$. Hence, by (*b*) and $B/A_1 = A/A_1 \oplus B_1/A_1$, we have $C_{A_1}(K) = = A_1 \cap C_{B_1}(K) = A \cap C_{B_1}(K) = 0$, a contradiction. So $B_1 = A_1 \oplus B_2$ for some $\mathbb{Z}G$-submodule B_2 and hence the lemma is proved.

COROLLARY 9. *Let G be a hyper-(cyclic or finite) group, B a $\mathbb{Z}G$-module, and A a noetherian $\mathbb{Z}G$-submodule of B such that all irreducible $\mathbb{Z}G$-factors of A are infinite. If B/A is an infinite cyclic group, the $B = A \oplus B_1$ for some $\mathbb{Z}G$-submodule B_1 of B.*

PROOF. Let $G_1 = C_G(B/A)$, then $|G/G_1| \leq 2$ and B/A is torsion-free and G_1-trivial. By Lemma 8, $B = A \oplus B_1$ for some G_1-trivial $\mathbb{Z}G_1$-

submodule B_1 of B. For $g \in G$, if $B_1 g \neq B_1$, then $B_1 g$ is G_1-trivial and

$$0 \neq B_1 g/(B_1 \cap B_1 g) \cong_{\mathbb{Z}G_1} (B_1 + B_1 g)/B_1 \leqslant B/B_1 \cong_{\mathbb{Z}G_1} A.$$

That is, A has a nonzero G_1-trivial $\mathbb{Z}G_1$-factor and then a nonzero finite irreducible $\mathbb{Z}G_1$-factor, which will imply that A has a nonzero finite irreducible $\mathbb{Z}G$-factor, a contradiction. So $b_1 g = B_1$ for all $g \in G$. That is, B_1 is a $\mathbb{Z}G$-submodule of B. The result is proved.

LEMMA 10. Let E be an extension of the abelian group A by a hyper-(cyclic or finite) group G such that A is a noetherian $\mathbb{Z}G$-module and all irreducible $\mathbb{Z}G$-factors of A are infinite. Then if C/A is a normal subgroup of E/A and $C \leqslant C_E(A)$, then $C = A \times N$, where N is a normal subgroup of E and is contained in every supplement to A in E.

PROOF. Let N be a normal subgroup of E contained in C and maximal subject to $N \cap A = 1$. By considering the factor group E/N we may suppose that $N = 1$. Then E satisfies the following condition: if S is normal in E, $S \leqslant C$, and $S \neq 1$, then $S \cap A \neq 1$. We show that this implies that $A = C$.

Suppose that $A \neq C$. Since E/A is hyper-(cyclic or finite), there is a nontrivial finite subgroup $K/A \leqslant C/A$ such that K is normal in E or an infinite cyclic subgroup $L/A \leqslant C/A$ such that L is normal in E.

For K, by the hypothesis of the lemma, $K \leqslant C_E(A)$ and so K is a finite extension of its central subgroup A. Hence K' is finite (Theorem 10.1.4 in [3]). It follows that $A \cap K'$ is finite and so $A \cap K' = 1$ by A having no nonzero finite $\mathbb{Z}G$-factors. By the condition above, we have $K' = 1$ and so K is abelian. Apply Lemma 2 to the $\mathbb{Z}(E/K)$-module K and its submodule A, then $A = A \times K_1$ for some normal subgroup K_1 of E, contrary to the condition above.

For L, by the hypothesis of the lemma, $L \leqslant C_E(A)$ and so L is a cyclic extension of its central subgroup A. Thus L is abelian. By Corollary 9, $L = A \times L_1$ for some normal subgroup L_1 of E, contrary to the condition above.

Thus we have proved that $C = A \times N$, where N is normal in E.

Now let E_1 be a supplement to A in E so that $E = AE_1$, $C = A(C \cap \cap E_1)$ and $C \cap E_1$ is normal in AE_1. We have

$$N(C \cap E_1)/(C \cap E_1) \leqslant C/(C \cap E_1) = A(C \cap E_1)/(C \cap E_1).$$

Since N is hyper-(cyclically of finitely) embedded in E and the irreducible $\mathbb{Z}G$-factors of A are all finite, we must have $N(C \cap E_1)/(C \cap E_1) = 1$, i.e., $N(C \cap E_1) = C = E_1$. Hence $N \leqslant E_1$ as required.

Now we prove the main result of this paper.

THEOREM. Let G be a hyper-(cyclic or finite) locally soluble group and A a noetherian $\mathbb{Z}G$-module. If A has no nonzero finite $\mathbb{Z}G$-images, then the extension E of A by G splits conjugately over A and A has no nonzero finite $\mathbb{Z}G$-factors.

PROOF. By Corollary 5, A has no nonzero finite $\mathbb{Z}G$-factors.

Suppose the theorem is false, then using the fact that A is a noetherian $\mathbb{Z}G$-module we may assume that: A has conjugate complements in E modulo any nontrivial E-invariant subgroup of A.

Since A has no nonzero finite $\mathbb{Z}G$-factors, $C_A(E) = 1$. By Lemma 1, E/A has a normal subgroup K/A and A has a nontrivial E-invariant subgroup A_0 such that $C_{A_0}(K) = 1$ and $K/C_K(A_0)$ is cyclic or finite.

(1) If $K/C_K(A_0)$ is finite, then we may choose K and A_0 such that $K/C_K(A_0)$ is minimal and so $K/C_K(A_0)$ is a chief factor of E. (For if L is normal in E and $C_K(A_0) < L < K$ then if $C_{A_0}(L) = 1$ we have L, A_0 contrary to minimality of $|K/C_K(A_0)|$ and if $C_{A_0}(L) \neq 1$ then K, $C_{A_0}(L)$ is contrary to minimality of $|K/C_K(A_0)|$.) Hence $K/C_K(A_0)$ has order $\mathfrak{p}^k$ for some prime $\mathfrak{p}$ and integer $k \geq 1$. From $C_{A_0}(K) = 1$ it follows that $A_0[\mathfrak{p}] = 1$ and so $A_0^{\mathfrak{p}^k} \neq 1$.

By the assumption on A, we have E splits conjugately over A modulo $A_0^{\mathfrak{p}^k} \neq 1$.

Let E_1 be a complement to A in E modulo $A_0^{\mathfrak{p}^k} \neq 1$: $E = AE_1$, $A \cap E_1 = A_0^{\mathfrak{p}^k} \neq 1$; put $E_0 = A_0E_1$, $K_0 = K \cap E_0$, and $C_0 = C_{K_0}(A_0)$. By Lemma 10, $C_0 = A_0 \times N$, where N is normal in E_0 and is contained in E_1. Consider the factor group $\overline{E}_0 = E_0/N$ and the subgroups $\overline{K}_0, \overline{A}_0$. Since

$$\overline{K}_0/\overline{A}_0 = \overline{K}_0/\overline{C}_0 \cong K_0/C_0 \cong K/C_K(A_0),$$

we have $|\overline{K}_0/\overline{A}_0| = \mathfrak{p}^k$. Corresponding to $C_{A_0}(K) = 1$ we have $C_{\overline{A}_0}(\overline{K}_0) = 1$ and also $\overline{A}_0 \cap \overline{E}_1 = \overline{A}_0^{\mathfrak{p}^k}$. It follows, by applying Lemma 6 in [6] to $\overline{E}_0$ and its subgroups $\overline{K}_0, \overline{A}_0$, that $\overline{E}_0$ splits over $\overline{A}_0$: $\overline{E}_0 = \overline{A}_0 \overline{E}_2$, $\overline{A}_0 \cap \overline{E}_0 = 1$. The complete preimage E_2 of $\overline{E}_2$ in E_0 gives $E_0 = A_0E_2$ and $A_0 \cap E_2 = 1$. So that E_2 is a complement to A in E. Let S_1, S_2 be any two complements to A in E. Then, since E splits conjugately over A modulo $A_0^{\mathfrak{p}^k}$, we have S_1 and S_2 are conjugate modulo $A_0^{\mathfrak{p}^k}$ and we may assume that $A_0^{\mathfrak{p}^k} S_1 = A_0^{\mathfrak{p}^k} S_2$. Put $E_0 = A_0S_1 = A_0S_2$, $K_0 = K \cap E_0$, and $C_0 = C_{K_0}(A_0)$. By Lemma 10, $C_0 = A_0 \times N$, where N is normal in E_0 and is contained in every supplement to A_0 in E_0; in particular, $N \leq S_1 \cap S_2$. Consider the factor group $\overline{E}_0 = E_0/N$ and its subgroups $\overline{K}_0, \overline{A}_0$. Since $\overline{K}_0/\overline{A}_0 \cong K/C_K(A_0)$, so $\overline{K}_0/\overline{A}_0$ is a group of order $\mathfrak{p}^k$, and also $C_{\overline{A}_0}(\overline{K}_0) =$

$= \bar{1}$ by $C_{A_0}(K) = 1$. From $A_0^{p^k} S_1 = A_0^{p^k} S_2$ it follows that $\bar{S}_1$ and $\bar{S}_2$ are complements to $\bar{A}_0$ in $\bar{E}_0$ which coincide modulo $\bar{A}_0^{p^k}$. Applying Lemma 6 in [6] to the group $\bar{E}_0$ and its subgroups $\bar{K}_0$, $\bar{A}_0$, we have the conjugacy of the complements: $\bar{S}_1^{\bar{a}} = \bar{S}_2$, $a \in A_0$. Since $\bar{S}_1 = S_1/N$, $\bar{S}_2 = S_2/N$, and N is normal in E_0 it follows that $S_1^a = S_2$, i.e., E splits conjugately over A, a contradiction.

(2) Now we may suppose that $K/C_K(A_0)$ is cyclic.

In this case, we let $A_1 = [A_0, K] \leqslant A_0$, then, by $C_{A_0}(K) = 1$, we have $A_1 \neq 1$. Thus E splits conjugately over A modulo A_1, i.e., $E = AE_1$, $A \cap \cap E_1 = A_1$. Let $K_1 = K \cap E_1$ and $C_1 = C_{K_1}(A_0)$. It is clear that $A_1 \leqslant C_1 \leqslant \leqslant C_{K_1}(A_1) \leqslant C_{E_1}(A_1)$. By Lemma 10, $C_1 = A_1 \times N$ for some normal subgroup N of E_1. Since $K_1/C_1 \cong K/C_K(A_0)$, we have $K_1 = C_1\langle x\rangle$ for some $x \in K_1$. Let $M = N\langle x\rangle$, then $K_1 = C_1\langle x\rangle = A_1 M$. Since

$$[A_1 \cap M, K] = [A_1 \cap M, C_K(A_0)\langle x\rangle] = [A_1 \cap M, x] =$$

$$= [A_1 \cap M, \langle x\rangle] \leqslant [A_1, x] \cap [M, x] \leqslant A_1 \cap N = 1,$$

we have $A_1 \cap M \leqslant C_{A_0}(K) = 1$. Thus $K_1 = A_1 M$, i.e., M is a complement to A_1 in K_1.

Suppose that M_0 is also a complement to A_1 in K_1 with $N \leqslant M_0$; we show that M and M_0 are conjugate by an element of A_0. We can write $x = a_1 x_0$ with $a_1 \in A_1$ and $x_0 \in M_0$. Since

$$A_1 = [A_0, K] = [A_0, C_K(A_0)\langle x\rangle] = [A_0, \langle x\rangle] = [A_0, x^{-1}],$$

so $a_1 = [a_0^{-1}, x^{-1}]$ for some $a_0 \in A_0$, and therefore

$$x = a_1 x_0 = [a_0^{-1}, x^{-1}]x_0 = a_0 (a_0^{-1})^{x^{-1}} x_0 = (a_0^{-1})^{x^{-1}} a_0 x_0 = x(x^{-1})^{a_0} x_0,$$

i.e., $x_0 = x^{a_0}$. Since $N \leqslant M_0$ and $N \leqslant C_1 = C_{K_1}(A_0)$, we have

$$M^{a_0}(N\langle x\rangle)^{a_0} = N\langle x^{a_0}\rangle = N\langle x_0\rangle \leqslant M_0 .$$

As $C_K(A_0) = AC_{K_1}(A_0)$ and $K = K_1 C_K(A_0)$, so

$$AM_0 = A(A_1 M_0) = AK_1 = AC_{K_1}(A_0)K_1 = C_K(A_0)K_1 =$$

$$= K = K^{a_0} = (AM)^{a_0} = AM^{a_0},$$

also $A \cap M_0 = A_1 \cap M_0 = 1$ and $A \cap M = 1$ implies that $A \cap M^{a_0} = 1$. Thus $M_0 = M^{a_0}$.

We now prove that A has conjugate complements in E and that the complements are of the form $L = N_{E_0}(M)$, where $E_0 = A_0 E_1$ and M is, as above, a complement to A_1 in K_1 containing N.

If $g \in E_1$, then since N and K_1 are both normal in E_1 and the subgroup M^g is a complement to A_1 in K_1 containing N, thus $M_g = M^{a_0}$ for some $a_0 \in A_0$ and so $ga_0^{-1} \in N_{E_0}(M) = L$, hence $E = AE_1 = AL$. We show that L is a complement to A in E. That is, we need to prove that $A \cap L = 1$.

Since $L \leqslant E_0 = A_0 E_1$ and $A \cap E_1 = A_1$, so

$$A \cap L = A \cap (E_0 \cap L) = (A \cap E_0) \cap L = (A \cap A_0 E_1) \cap L =$$
$$= A_0(A \cap E_1) \cap L = A_0 A_1 \cap L = A_0 \cap L;$$

also A_0 is normal in E and $L = N_{E_0}(M)$, hence $[A_0 \cap L, M] \leqslant A_0 \cap M$.
Since

$$A_0 \cap M = A_0 \cap (E_1 \cap M) = (A_0 \cap E_1) \cap M = A_1 \cap M = 1,$$

so $A \cap L \leqslant C_{A_0}(M)$. Therefore, by $K = AM$ and $C_{A_0}(K) = 1$, we have $A \cap L \leqslant C_{A_0}(M) = C_{A_0}(K) = 1$. That is, $A \cap L = 1$ and so L is a complement to A in E.

Now let S be any complement to A in E. Thus S and L are conjugate modulo A_1 and we may assume that $A_1 L = A_1 S$. Therefore, we have

$$E_0 = E_0 \cap E = E_0 \cap AL = (E_0 \cap A)L = (A_0 E_1 \cap A)L =$$
$$= A_0 L = A_0 A_1 L = A_0 A_1 S = A_0 S.$$

Since $K_1 = A_1 M \leqslant A_1 L = A_1 S$, so $K_1 = A_1 M_1$, $A_1 \cap M_1 = 1$, where $M_1 = K_1 \cap S$; thus M and M_1 are complements to A_1 in K_1. We show that $N \leqslant M_1$. By $K_1 \leqslant A_1 S$ and $C_1 = C_{K_1}(A_0) \leqslant K_1$ we have $C_1 = C_1 \cap A_1 S = A_1(C_1 \cap S)$, thus $C_1 = A_1 \times N_1$, where $N_1 = C_1 \cap S \leqslant M_1$ and N_1 is normal in $A_0 S = E_0$ since $C_1 = C_{K_1}(A_0)$ is normal in $A_0 E_1 = E_0$. In particular, N_1 is normal in $E_1 \leqslant E_0$ and, since E_1/A_1 is hyper-(cyclic or finite), N_1 is hyper-(cyclically or finitely) embedded in E_1. Consider the product NN_1. If $NN_1 \neq N_1$ then, by $C_1 = A_1 \times N = A_1 \times N_1$, $NN_1 \cap A_1 \neq 1$ and so A_1 contains a nontrivial cyclic or finite subgroup normal in E_1. By $A_1 \leqslant A$ and $E_1/A_1 \cong E/A \cong G$, we have A has a nonzero cyclic or finite $\mathbb{Z}G$-module and hence contains a nonzero finite $\mathbb{Z}G$-factor, a contradition. Thus $NN_1 = N_1$, $N \leqslant N_1$ and so $N \leqslant M_1$.

This shows that M and M_1 are conjugate by anelement $a_0 \in A_0$, i.e., $M^{a_0} = M_1$, and hence $L^{a_0} = N_{E_0}(M)^{a_0} = N_{E_0}(M^{a_0}) = N_{E_0}(M_1)$. From $K_1 = A_1 M$ and M is normal in L it follows that K_1 is normal in $A_1 L$. Therefore, by $A_1 L = A_1 S$, we have K_1 is normal in $A_1 S$, and so $M_1 = K_1 \cap S$ is normal in S and $S \leqslant N_{E_0}(M_1)$. By $L^{a_0} = N_{E_0}(M_1)$, we have $S \leqslant L^{a_0}$ and so

$$L^{a_0} = AS \cap L^{a_0} = (A \cap L^{a_0})S = S.$$

That is, S and L are conjugate in E, i.e., E splits conjugately over A, a contradiction again.

Thus, we have finished the proof of the theorem.

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