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On the exterior Dirichlet problem for
$\Delta u - u + f(x, u) = 0$

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On the Exterior Dirichlet Problem for $\Delta u - u + f(x, u) = 0$.

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1. Introduction.

In this paper we study the existence of a nonnegative nontrivial solution of the Dirichlet problem

$$(1.1)\qquad \begin{cases} \Delta u - u + f(x, u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is an exterior domain (i.e. $\Omega = \mathbb{R}^n \setminus \overline{O}$, and 0 is a regular bounded open set in $\mathbb{R}^n$) and f is a continuous function such that there exists

$$(1.2)\qquad \lim_{|x| \to +\infty} f(x, u) = f_\infty(u) \in \mathbb{R} \qquad \forall u \in \mathbb{R}.$$

Since the domain Ω is unbounded, the Sobolev embedding $L^p(\Omega) \subset \subset H_0^1(\Omega)$, $2 \leq p < 2n/(n-2)$, is not compact, and standard variational techniques do not apply. However, it is possible to use comparison methods between equation (1.1) and the «equation at infinity»

$$(1.3)\qquad \Delta u - u + f_\infty(u) = 0 \quad \text{in } \mathbb{R}^n,$$

for which existence results are well known, because of its radial structure (see for example [1], and [2]).

These methods were first introduced by Ding and Ni [3], and P. L. Lions [4], [5], who proved an existence result for (1.1), when

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$\Omega = \mathbb{R}^n$, and

$$f_\infty(u) = \inf_{x \in \mathbb{R}^n} f(x, u),$$

using a simple variant of the concentration compactness method.

Then Benci and Cerami [6] and P. L. Lions [7] proved a representation theorem for all the Palais-Smale sequences of (1.1), which gives a precise estimate of the energy levels where the Palais-Smale condition for the functional related to equation (1.1) can fail.

Hence, using a minimax method Benci and Cerami [6] obtained an existence result for (1.1), when Ω is «almost equal» to $\mathbb{R}^n$,

$$f(x, u) = f_\infty(u) = |u|^{p-2}u, \qquad 2 < p < \frac{2n}{n-2},$$

and (1.3) has an unique positive solution. (This last condition is actually a consequence of the hypothesis $f_\infty(u) = |u|^{p-2}u$, because of a later result of [8]). Very recently Bahri and Lions [9] improved this result, and showed that (1.1) has a solution when Ω is an arbitrary exterior domain,

$$f(x, u) = b(x)|u|^{p-2}u, \qquad 2 < p < \frac{2n}{n-2}$$

and $b(x) \to b_\infty > 0$ as $|x| \to +\infty$ exponentially fast. The proof is obtained by studying the energy levels found in the representation theorem, with very powerful algebraic topology methods.

Let's finally recall that Coffman and Marcus [10] obtained some existence theorems, for (1.1), when Ω is an almost spherically symmetric exterior domain, by using a completely different approach.

In this work using the technique introduced by Bahri and Lions, we study problem (1.1) for general exterior domains and general functions f satisfying (1.2). The paper is organized as follows: in paragraph 2 we state our main Theorem (Theorem 2.1), and give some examples; in 3 we introduce some notations, and recall some well known results we will apply in the following section; in 4 we give the proof of Theorem 2.1; the proofs of some rather technical lemma are collected in 5.

Acknowledgment. I'm very grateful to Prof. M. Ferri, who kindly helped me in understanding some tricky algebraic topology arguments in [9].

2. Let's consider the following problem

$$(2.1) \qquad \begin{cases} \Delta u - a(x)\, u + f(x, u) = 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega), \quad u \geq 0, & u \not\equiv 0, \end{cases}$$

where Ω is an exterior domain. Assume that a is bounded, measurable, nonnegative and such that

$$(2.2) \qquad a(x) \to a_\infty > 0 \quad \text{as } |x| \to +\infty;$$

$$(2.3) \qquad \exists c > 0,\ \delta > 0,\ R > 0 \text{ such that } a(x) \leq$$

$$\leq a_\infty + c \exp(-\delta |x|)\, |x|^{-(n-1)/2} \qquad \forall x\colon |x| > R.$$

We will also assume that $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous, strictly positive for $t > 0$, and $t \to f(x, t)$ is of class C^1 and *odd* for every $x \in \Omega$. Moreover there exist p, q such that $2 < p < q < 2n/(n-2)$, and

$$(2.4) \qquad t \to \frac{f(x,t)}{t^{q-1}} \text{ is decreasing and } t \to \frac{f(x,t)}{t^{p-1}} \text{ is increasing on } [0, +\infty)$$

$$\text{for every fixed } x \in \Omega.$$

Finally, we will suppose that there exists a *convex* function f_∞ of class C^1 such that

$$(2.5) \qquad \frac{f(x,t)}{f_\infty(t)} \to 1 \quad \text{as } |x| \to +\infty \text{ uniformly in } t;$$

$$(2.6) \qquad f(x,t) \geq (1 - \exp(-\delta |x|)\, |x|^{-(n-1)/2})\, f_\infty(t) \quad \forall t \in \mathbb{R}^+,\ \forall x,\ |x| > R.$$

Under these hypotheses we prove the following theorem, which is our main result.

THEOREM 2.1. *If the «equation at infinity» associated to* (2.1)

$$(2.7) \qquad \Delta u - a_\infty u + f_\infty(u) = 0 \quad \text{in } \mathbb{R}^n,$$

has a unique ground state solution, then (2.1) *has at least a solution.*

We recall that a ground state solution to equation (2.7) is a classical strictly positive solution ω such that $\omega(x) \to 0$ as $x \to +\infty$. It is well known that every ground state of (2.7) is radially symmetric with respect to a point in $\mathbb{R}^n$. In Theorem 2.1 we mean uniqueness of the ground state radially symmetric with respect to the origin.

With the same methods we are going to use in Theorem 2.1, it is also possible to prove

THEOREM 2.1′. *Under the same hypotheses as in Theorem* 2.1 *the Dirichlet problem*

$$(2.1)' \qquad \begin{cases} \sum_{i=1}^{n} \partial_i (a_{ij} \partial_j u) - a(x) u + f(x, u) = 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega), \quad u \geq 0, \quad u \not\equiv 0 \end{cases}$$

has at least a solution if $a_{ij} = a_{ji} \in L^{\infty}(\Omega)$ *for every* i, j, *and*

$$a_{ij}(x) \to \delta_{ij} \quad as \ |x| \to +\infty,$$

$\exists c > 0$, $\delta > 0$, $R > 0$ *such that* $a_{ij}(x) \leq \delta_{ij} + c \exp(-\delta |x|) |x|^{-(n-1)/2}$

$$\forall x \colon |x| > R,$$

where δ_{ij} *is the Kronecker function.*

We will prove only Theorem 2.1; with easy adaptations, it is possible to obtain also the proof of 2.1′.

Using the uniqueness theorems for ground state solutions proved by Kwong and Zhang [11], we can give more explicit hypotheses on f_∞, which are sufficient to apply Theorem 2.1 (or Theorem 2.1′).

COROLLARY 2.2. *Assume that* f *and* f_∞ *satisfy all the previous hypotheses, and let* $\theta > 0$ *be such that* $-u + f_\infty(u) < 0$ *in* $(0, \theta)$, *and* $-u + f_\infty(u) > 0$ *in* $(\theta, +\infty)$. *If*

$$(2.8) \qquad \frac{-u + u f'_\infty(u)}{-u + f_\infty(u)} \text{ is decreasing in } (\theta, +\infty),$$

then (2.1) *or* (2.1)′ *has at least a solution.*

PROOF. By (2.4) the function $f_\infty(u)/u^{p-1}$ is increasing on $(0, \infty)$, hence, since f_∞ is C^1, $u f'_\infty(u) \geq (p-1) f_\infty(u) > f_\infty(u)$. Consequently

$$\lim_{u \to +\infty} \frac{-u + u f'_\infty(u)}{-u + f_\infty(u)} > 1.$$

Besides $((-u + u f'_\infty(u))/(-u + f_\infty(u)) \leq 1$ in the interval $(0, \theta)$. Hence, by Theorem 1 in [11], (2.7) has a unique positive solution.

Significative examples coming within Corollary 2.2 are the following ones:

$$f_\infty(u) = u^p \qquad \text{for } p \in (1, 2n/(n-2)),$$

$$f_\infty(u) = u^4 - u^3 + u^2.$$

extended as odd functions to $u < 0$.

For somewhat different examples we also refer the reader to the paper on the uniqueness of radial ground states of MacLeod and Serrin [12].

3. Preliminary remarks and sketch of the proof of Theorem 2.1.

Let us call

$$F(x, u) = \int_0^u f(x, t)\, dt, \qquad F_\infty(u) = \int_0^u f_\infty(t)\, dt,$$

and

$$I: H_0^1(\Omega) \to \mathbb{R} \qquad I(u) = \frac{1}{2}\int_\Omega |Du|^2 + \frac{1}{2}\int_\Omega au^2 - \int_\Omega F(x, u),$$

I is of class C^1, and its critical points are the weak solutions of (2.1). Analogously we define I_∞ the functional associated to the equation at infinity:

$$I_\infty : H_0^1(\Omega) \to \mathbb{R} \qquad I_\infty(u) = \frac{1}{2}\int_\Omega |Du|^2 + \frac{1}{2}\int_\Omega a_\infty u^2 - \int_\Omega F_\infty(u).$$

Finally let's call

$$M = \{u \neq 0: dI(u)(u) = 0\}, \qquad \text{and} \qquad M_\infty = \{u \neq 0: dI_\infty(u)(u) = 0\};$$

$$M^+ = \{u \in M: u(x) > 0 \text{ for all } x \in \Omega\}.$$

REMARK 3.1. u is a weak solution of (2.1) if and only if u is a critical point of $I|_M$ (cf. [5], page 266). Moreover $I|_M$ satisfies the following properties:

$$\inf_M I > 0, \qquad \inf_M \|u\| > 0.$$

$\forall u \in H_0^1(\Omega)$, $u \geqslant 0$, $\exists! k(u) > 0$ such that $k(u)\, u \in M$, and $\forall m > 0$,

$$\text{(3.1)}\qquad \min\left(\left(\frac{k(u)}{m}\right)^{p-2},\left(\frac{k(u)}{m}\right)^{q-2}\right)\leqslant$$

$$\leqslant \frac{m^2\int_\Omega(|Du|^2+au^2)\,dx}{\int_\Omega f(x,mu)\,mu\,dx}\leqslant \max\left(\left(\frac{k(u)}{m}\right)^{p-2},\left(\frac{k(u)}{m}\right)^{q-2}\right).$$

Let us recall the proof of (3.1). Since $k(u)\,u\in M$,

$$\text{(3.2)}\qquad k^2(u)\int_\Omega|Du|^2\,dx+k^2(u)\int_\Omega au^2\,dx=\int_\Omega f(x,k(u)\,u)\,k(u)\,u\,dx\,.$$

On the other side, if $k(u)/m\leqslant 1$, by (2.4)

$$\text{(3.3)}\qquad \left(\frac{k(u)}{m}\right)^q\int_\Omega f(x,mu)\,mu\,dx\leqslant$$

$$\leqslant\int_\Omega f(x,k(u)\,u)\,k(u)\,u\,dx\leqslant\left(\frac{k(u)}{m}\right)^p\int_\Omega f(x,mu)\,mu\,dx\,.$$

Analogously, if $(k(u)/m\geqslant 1)$, then

$$\text{(3.4)}\qquad \left(\frac{k(u)}{m}\right)^p\int_\Omega f(x,mu)\,mu\,dx\leqslant$$

$$\leqslant\int_\Omega f(x,k(u)\,u)\,k(u)\,u\,dx\leqslant\left(\frac{k(u)}{m}\right)^q\int_\Omega f(x,mu)\,mu\,dx$$

(3.1) immediately follows from (3.2), (3.3) and (3.4).

REMARK 3.2. *If problem* (2.1) *has no solutions, then*

$$\text{(3.5)}\qquad I(u)>\inf_M I\qquad\forall u\in M\,,$$

and

$$\text{(3.6)}\qquad \inf_M I=\inf_{M_\infty}I_\infty\,.$$

Indeed if (3.5) is not true, then there exists $u\in M$ such that $I(u)=\inf_M I$, hence, by the first part of Remark 3.1, u is a solution of (2.1). Moreover, if (3.6) does not hold, $\inf_M I<\inf_{M_\infty}I_\infty$, and (2.1) has a solution by the comparison theorems of [3] and [5].

In the sequel we will always assume that equation (2.7) has an unique positive solution ω (up to a translation in the independent variable).

This function is radially symmetric about some point in $\mathbb{R}^n$, radially decreasing, and if r is the radial coordinate, there exists $C > 0$ such that

$$\omega(r)\, r^{(n-1)/2} \exp(\sqrt{a_\infty}\, r) \to C \quad \text{as } r \to +\infty\,, \tag{3.7}$$

$$\omega'(r)\, r^{(n-1)/2} \exp(\sqrt{a_\infty}\, r) \to -\sqrt{a_\infty}\, C \quad \text{as } r \to +\infty\,, \tag{3.8}$$

$$I_\infty(\omega) = \inf_{M_\infty} I_\infty\,. \tag{3.9}$$

((3.9) is well known, for (3.7) and (3.8) cf. [13]).

We also recall the following representation theorem for the Palais Smale sequences related to the functional I:

Proposition 3.3. *Let (u_h) be a nonnegative sequence in $H_0^1(\Omega)$ such that $I(u_h) \to c$ and $dI|_M(u_h) \to 0$ as $h \to +\infty$. Then there exist a number $m \in \mathbb{N}$, and m sequences $(y_h^j)_{h\in\mathbb{N}}$ in $\mathbb{R}^n$, $j = 1, \ldots, m$, such that $|y_h^j| \to +\infty$, for every $i \neq j$ $|y_h^i - y_h^j| \to +\infty$ as $h \to +\infty$ and*

$$u_h(x) = u^0(x) + \sum_{j=1}^{m} \omega(x - y_h^j) + \psi_h(x)\,,$$

where u_0 satisfies $dI(u^0) = 0$, $\psi_h \to 0$ in $H_0^1(\Omega)$ as $h \to +\infty$ and

$$I(u_h) \to I(u^0) + mI_\infty(\omega) \quad \text{as } h \to +\infty\,.$$

We omit the proof of this proposition, which can be obtained using concentration compactness principle as in [6], [9] (see also [14]). Let us only note explicitly that the Palais Smale sequences of the functional $I|_M$ are also Palais Smale sequences of I.

Corollary 3.4. *If (2.1) has no solutions, and (u_h) satisfies the same assumptions as in Proposition 3.3, then*

$$u_h = \sum_{i=1}^{m} \omega(\cdot - y_h^i) + o(1) \quad \text{as } h \to +\infty\,,$$

$$I(u_h) \to mI_\infty(\omega) \quad \text{as } h \to +\infty\,.$$

These results point out the crucial role played by the level sets of I, which corresponds to an integer multiple of $I_\infty(\omega)$, and by the sets of finite sums of translations of ω. Hence the idea of [9] is to define

$$W_m = \{u \in M^+ : I(u) \leq (m+1) I_\infty(\omega)\} \tag{3.10}$$

and

(3.11) $V(m, \varepsilon) =$

$$= \left\{u \in M^+ : \left|u - \sum_{i=1}^{m} \omega(\cdot - y_i)\right| < \varepsilon,\ |y_i| > \frac{1}{\varepsilon},\ |y_i - y_j| > \frac{1}{\varepsilon}\ \forall i \neq j\right\},$$

and to study their topological properties with a deformation argument.

For every $u_0 \in W_m$ we consider the Cauchy problem

$$\begin{cases} u' = \dfrac{-I'|_M(u)}{(1 + |I'|_M(u)|^2)^{1/2}}, \\ u(0) = u_0 \in W_m . \end{cases}$$

This has a unique solution $t \to u(t, u_0)$ defined on all of $\mathbb{R}$, such that $u(t, u_0) \in M^+$ for every $t \in \mathbb{R}$. Let's denote

$$T_\delta(u_0) = \begin{cases} \inf\{s \geq 0 : I(u(s, u_0)) \leq m I_\infty(\omega) + \delta\}, \\ +\infty \text{ if the set is empty}. \end{cases}$$

$T_\delta(u_0)$ is continuous as a function of u_0, for every $\delta \geq 0$ and, if (2.1) has no solutions, $T_\delta(u_0) < +\infty$ for every $\delta > 0$. Hence we can define

$$T(u_0) = \min\{T_\delta(u_0) + \sqrt{\delta},\ T_0(u_0)\},$$

$$W^\delta_{m-1} = \{u(T(u_0), u_0) : u_0 \in W_m\},$$

and

$$r : [0, 1] \times W_m \to W_m, \qquad r(t, u_0) = u(tT(u_0), u_0).$$

Then the following lemma holds:

LEMMA 3.5. *If equation* (2.1) *has not a solution, then* (W_m, W_{m-1}) *retracts by deformation through* r *on* $(W^\delta_{m-1}, W_{m-1})$, *and for every* $\varepsilon > 0$ *there exist* $\delta > 0$, $\varepsilon_1 \in (0, \varepsilon)$ *such that*

$$W^\delta_{m-1} \setminus W_{m-1} \subseteq V(m, \varepsilon) \tag{3.12}$$

and

$$V(m, \varepsilon_1) \subseteq W^{\partial}_{m-1} \cap V(m, \varepsilon) . \tag{3.13}$$

PROOF. This lemma can be proved with the same techniques as (4.7) in [9]. For the sake of brevity we omit the proof.

Now, to complete the sketch of the proof of our Theorem 2.1, we introduce some algebraic topology arguments. To explain the main idea, we first describe it in a particular case which is however already contained in [9]. We assume that a is constant, $f(x, u) = b|u|^{p-2}u$, and Ω is an exterior domain. Let's denote

$$B_m = \left\{ k\left(\varphi \sum_{i=1}^{m} t_i \, \omega(\cdot - x_i)\right) \varphi \sum_{i=1}^{m} t_i \, \omega(\cdot - x_i) : \sum_{i=1}^{m} t_i = 1, \, t_i \geq 0, \, x_i \in S^{n-1} \right\}, \tag{3.14}$$

where k is defined in Remark 3.1 and φ is a C^∞ function such that $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \Omega'$, $\varphi \equiv 1$ in a neighborhood of $+\infty$, and $0 \leq \varphi \leq 1$. In order to study the homology groups of B_m Bahri and Lions also need to introduce suitable manifolds T_m such that $\forall m \in \mathbb{N}$

$$(T_m, \partial T_m) \text{ has the same homology groups as } (B_m, B_{m-1}) . \tag{3.15}$$

and call h_* the isomorphism between these groups.

With these notations they prove that there exists a natural number μ such that

$$(B_m, B_{m-1}) \subseteq (W_m, W_{m-1}) \qquad \forall m \leq \mu , \tag{3.16}$$

$$B_\mu \subseteq W_{\mu - 1} .$$

Then they assume by contradiction that (2.1) has no solutions, and they built a suitable commutative diagram involving the groups $H_{mn-1}(B_m, B_{m-1})$ (which are not trivial because $H_{mn-1}(B_m, B_{m-1}) = H_{mn-1}(T_m, \partial T_m) = \mathbb{Z}_2$ for every $m \leq \mu$). This is done in two steps.

The first is based on a simple property of B_m. As $B_{m-2} \subseteq B_{m-1} \subseteq B_m$, [9] can consider the following commutative diagram,

$$\begin{array}{ccc} H_{(m-1)n}(B_m, B_{m-1}) & \xrightarrow{\ \partial\ } & H_{(m-1)n-1}(B_{m-1}, B_{m-2}) = Z_2 \\ \Big\downarrow i_{(m-1)n} & & \Big\downarrow i_{(m-1)n-1} \\ H_{(m-1)n}(W_m, W_{m-1}) & \xrightarrow{\ \partial\ } & H_{(m-1)n-1}(W_{m-1}, W_{m-2}) \end{array}$$

where ∂ is the connecting homomorphism and i is the embedding

(3.16). By (3.15) a connecting homomorphism is also induced on $(T_m, \partial T_m)$, and the following diagram commutes

$$
(3.17)\qquad
\begin{array}{ccc}
H_{(m-1)n}(T_m, \partial T_m) & \xrightarrow{\ \partial\ } & H_{(m-1)n-1}(T_{m-1}, \partial T_{m-1}) = \mathbb{Z}_2 \\
\Big\downarrow h_{(m-1)n} & & \Big\downarrow h_{(m-1)n-1} \\
H_{(m-1)n}(W_m, W_{m-1}) & \xrightarrow{\ \partial\ } & H_{(m-1)n-1}(W_{m-1}, W_{m-2})
\end{array}
$$

In the second step Bahri and Lions use a deformation lemma (which is analogous to our Lemma 3.5), the cup product and some properties of T_m to define two homomorphisms d_α and $\overline{d}_\alpha$ such that the following diagram is commutative

$$
(3.18)\qquad
\begin{array}{ccc}
\mathbb{Z}_2 = H_{mn-1}(T_m, \partial T_m) & \xrightarrow{\ d_\alpha\ } & H_{(m-1)n}(T_m, \partial T_m) \\
\Big\downarrow h_{mn-1} & & \Big\downarrow h_{(m-1)n} \\
H_{mn-1}(W_m, W_{m-1}) & \xrightarrow{\ \overline{d}_\alpha\ } & H_{(m-1)n}(W_m, W_{m-1})
\end{array}
$$

and $\partial \cdot d_\alpha$ sends the generator of H_{mn-1} to the generator of $H_{(m-1)n-1}$.

In this way they build this chain of homomorphisms:

$$
\begin{array}{ccccc}
H_{\mu n-1}(B_{\mu n-1}, B_{\mu n-1}) & & H_{(\mu-1)n-1}(B_{(\mu-1)n-1}, B_{(\mu-1)n-1}) & & H_{n-1}(B_1) \\
\| & & \| & & \| \\
H_{\mu n-1}(T_{\mu n-1}, \partial T_{\mu n-1}) & \xrightarrow{\ \partial\cdot d_\alpha\ } & H_{(\mu-1)n-1}(T_{(\mu-1)n-1}, \partial T_{(\mu-1)n-1}) & \xrightarrow{\ \partial\cdot d_\alpha\ } \quad \longrightarrow & H_{n-1}(S^{n-1}) \\
\Big\downarrow h_{\mu n-1} & & \Big\downarrow h_{(\mu-1)n-1} & & \Big\downarrow h_{n-1} \\
H_{\mu n-1}(W_{\mu n-1}, W_{\mu n-1}) & \longrightarrow & H_{(\mu-1)n-1}(W_{(\mu-1)n-1}, W_{(\mu-1)n-1}) & \longrightarrow \quad \longrightarrow & H_{n-1}(W_1)
\end{array}
$$

Since $h_{\mu n-1}$ is zero (by (3.16)) and $\partial \cdot d_\alpha$ is surjective, then also $h_{n-1}: H_{n-1}(S^{n-1}) \to H_{n-1}(W_1)$ has to be zero. However a direct computation shows that $H_{n-1}(W_1) = \mathbb{Z}_2$ and h_{n-1} is an isomorphism. This contradiction proves that (2.1) has a solutión.

In the general case, when f is not a power, but it merely is a function verifying the behavior hypotheses (2.4) with f_∞ convex, we can not prove (3.16), and we can not work as before. However we show that there exist two neighborhoods of B_{m-1}, respectively called $\hat{B}_{m-1}$ and $\hat{\hat{B}}_{m-1}$ such that

$$(B_m \setminus \hat{B}_{m-1}, \hat{\hat{B}}_m \setminus \hat{B}_{m-1}) \text{ has the same homology groups as } (B_m, B_{m-1}),$$

$$(B_m \setminus \hat{B}_{m-1}, \hat{\hat{B}}_m \setminus \hat{B}_{m-1}) \subseteq (W_m, W_{m-1})$$

and

$$B_\mu \setminus \hat{B}_{\mu-1} \subsetneq W_\mu,$$

for a suitable $\mu \in \mathbb{N}$. Hence with more technical arguments, we built a commutative diagram analogous to the preceding one and we conclude the proof of Theorem 2.1.

4. Proof of Theorem 2.1.

We argue by contradiction, and we assume that (2.1) has no solution. Hence in particular (3.5) and (3.6) in Remark 3.2 hold, and $W_0 = \emptyset$.

STEP 0. *Definition of the manifold* $T_m = S_0^m \underset{\sigma_m}{\times} \Delta_{m-1}$ *and of an homomeorphism between a suitable subsets of it and* $B_m \setminus B_{m-1}$ *(this argument is analogous to* (3.15)).

S is a $(n-1)$-dimensional sphere embedded in $\bar{\Omega}$ such that $\lambda S \subseteq \bar{\Omega}$ for every $\lambda \geq 1$ (it is not restrictive to assume that S is a sphere of radius 1, because this may be achieved by a simple scaling). Let σ_m be the group of permutation of $\{1, ..., m\}$, $D_m = \{(x_i, \dots, x_m) \in S^m \colon \exists i \neq j \colon x_i = x_j\}$, $\hat{D}_m$ and $\hat{\hat{D}}_m$ be a σ_m-invariant tubular neighborhoods of D_m, such that $\hat{D}_m \subset\subset \hat{\hat{D}}_m$ and $S_0 = \overline{S^m \setminus \hat{D}_m}$. Next we denote

$$\Delta_{m-1} = \left\{(t_1, \dots t_m) \in \mathbb{R}^m \colon \sum_{i=1}^m t_i = 1,\ t_i \geq 0\ \forall i = 1 \dots m\right\}$$

and

$$\Delta_{m-1}^\eta = \left\{(t_1, \dots, t_m) \in \mathbb{R}^m \colon \sum_{i=1}^m t_i = 1,\ t_i \geq 0\ \forall i = 1 \dots m,\ \sup \left| t_i - \frac{1}{m} \right| \leq \eta\right\}.$$

The group σ_m acts on $S^m \times \Delta_{m-1}$ and $S_0^m \times \Delta_{m-1}$; we will denote respectively $S^m \underset{\sigma_m}{\times} \Delta_{m-1}$ and $S_0^m \underset{\sigma_m}{\times} \Delta_{m-1}$ the quotient under the action of σ_m.

Let us finally introduce the continuous map

$$h \colon S^m \underset{\sigma_m}{\times} \Delta_{m-1} \to M^+, \tag{4.1}$$

$$h(x_1, \dots, x_m, t_1, \dots, t_m) = k\left(\varphi \sum_{i=1}^m t_i\,\omega(\cdot - \lambda x_i)\right) \varphi \sum_{i=1}^m t_i\,\omega(\cdot - \lambda x_i),$$

where ω is the unique solution of (2.7), k is defined in Remark 3.1 and φ

is a fixed function of class C^∞, which is identically zero in $\mathbb{R}^n \setminus \Omega$, 1 in a neighborhood of $+\infty$, and $0 \leq \varphi \leq 1$ in $\mathbb{R}^n$.

REMARK 4.1. *If we set*

$$h(S^m \underset{\sigma_m}{\times} \Delta_{m-1}) = B_m,$$

then $B_{m-1} \subseteq B_m \subseteq M^+$ $\forall m \in \mathbb{N}$. *Moreover* h *defines an homomeorphism between*

$$S^m \underset{\sigma_m}{\times} \Delta_{m-1} \setminus (D_m \underset{\sigma_m}{\times} \Delta_{m-1} \cup S^m \underset{\sigma_m}{\times} \partial\Delta_{m-1}) \quad \text{and} \quad B_m \setminus B_{m-1}.$$

PROOF. We recall the proof of this remark, already contained in [9] (Lemma III.1), because we are using a different notation. We will only prove that, if $(x_1, \ldots, x_N)$ are distinct points in $\mathbb{R}^n$, $\gamma_1, \ldots, \gamma_N \in \mathbb{R}^n$ and

$$\varphi \sum_{j=1}^N \gamma_j \omega(\cdot - x_j) = 0,$$

then $\gamma_1 = \ldots = \gamma_N = 0$.

Integrating the previous expression on $\mathbb{R}^n$, we get

$$\int_{\mathbb{R}^n} \sum_{j=1}^N \gamma_j \varphi(y) \omega(y - x_j) \exp(-i\langle \xi, y \rangle) \, dy = 0.$$

Let's make the change of variable $y - x_j = z$

$$\int_{\mathbb{R}^n} \sum_{j=1}^N \gamma_j \varphi(z + x_j) \omega(z) \exp(-i\langle \xi, z + x_j \rangle) \, dz = 0.$$

Differentiating with respect to ξ, for every polynomial p we have

$$\int_{\mathbb{R}^n} p(z) \sum_{j=1}^N \gamma_j \varphi(z + x_j) \omega(z) \exp(-i\langle \xi, z + x_j \rangle) \, dz = 0.$$

Consequently, (taking into account that ω is always strictly positive)

$$\sum_{j=1}^N \gamma_j \varphi(z + x_j) \exp(-i\langle \xi, x_j \rangle) = 0,$$

and we conclude, since the points x_j are distinct and $\varphi(z + x_j) = 1$ if $|z|$ is big.

STEP 1. *Proposition 4.2, and the subsequent assertion* (4.4) *are analogous to* (3.16), *and allows us to define a diagram analogous to* (3.17).

PROPOSITION 4.2. *Let* ε, ε_1, δ *be the real value defined in the deformation Lemma* 3.5. *Then there exists* $\mu \in \mathbb{N}$, $\eta \in [0,1]$ *and* $\lambda > 0$ *such that*

$$h: S_0^{\mu} \underset{\sigma_\mu}{\times} \Delta_{\mu-1}^{\eta} \to W_{\mu-1}, \tag{4.2}$$

$$\forall m \leqslant \mu,\ m \geqslant 1\ h(S_0^m \underset{\sigma_m}{\times} \Delta_{m-1}^{\eta}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta_{m-1}^{\eta})) \subseteq \subseteq (V(m, \varepsilon_1), W_{m-1}) \subseteq (W_m, W_{m-1}), \tag{4.3}$$

where h is defined in (4.1), W_m *in* (3.10) *and* $V(m, \varepsilon)$ *in* (3.11). *h also sends any sufficiently small neighborhood of* $\partial(S_0^m \underset{\sigma_m}{\times} \Delta_{m-1}^{\eta})$ *to* W_{m-1}.

(We'll prove this proposition in Section 5.)

Let $\hat{\Delta}_{m-1}$ and $\hat{\hat{\Delta}}_{m-1}$ be neighborhood of $\partial\Delta_{m-1}$, which retract by deformation of $\partial\Delta_{m-1}$, and such that the closure of $\hat{\Delta}_{m-1}$ si embedded in the interior $\hat{\hat{\Delta}}_{m-1}$. Then also $\hat{B}_{m-1} = h(\hat{D}_m \underset{\sigma_m}{\times} \Delta_{m-1} \cup S^m \underset{\sigma_m}{\times} \hat{\Delta}_{m-1})$ and $\hat{\hat{B}}_{m-1} = h(\hat{\hat{D}}_m \underset{\sigma_m}{\times} \Delta_{m-1} \cup S^m \underset{\sigma_m}{\times} \hat{\hat{\Delta}}_{m-1})$ are neighborhoods of B_{m-1} which retract by deformation on B_{m-1}. If we call r^{η} the dilatation,

$$r^{\eta}: (\Delta_{m-1}, \partial\Delta_{m-1}) \to (\Delta_{m-1}^{\eta}, \partial\Delta_{m-1}^{\eta}),$$

from Remark 4.1 and Proposition 4.2, it follows that

$$(h \cdot r^{\eta} \cdot h^{-1})(B_m \setminus \hat{B}_{m-1}, \hat{\hat{B}}_{m-1} \setminus \hat{B}_{m-1}) \subseteq (W_m, W_{m-1}). \tag{4.4}$$

Hence we can consider the following diagram, which is commutative:

$$\begin{array}{ccccccccc}
B_m \setminus \hat{B}_{m-1}, \hat{\hat{B}}_{m-1} \setminus \hat{B}_{m-1}) & \xrightarrow{\partial_*} & H_*(\hat{\hat{B}}_{m-1} \setminus \hat{B}_{m-1}) & \xrightarrow{j_*} & H_*(\hat{\hat{B}}_{m-1} \setminus \hat{B}_{m-2}) & \xrightarrow{r_*^B} & H_*(B_{m-1} \setminus \hat{B}_{m-2}) & \xrightarrow{p_*} & H_*(B_{m-1} \setminus \hat{B}_{m-2}, \hat{\hat{B}}_{m-2} \setminus \hat{B}_{m-2}) \\
\Big\downarrow {\scriptstyle h^{-1})_{*+1}} & & & \searrow {\scriptstyle (h \cdot r^{\eta} \cdot h^{-1})_*} & & \swarrow & & & \Big\downarrow {\scriptstyle (h \cdot r^{\eta} \cdot h^{-1})_*} \\
H_{*+1}(W_m, W_{m-1}) & & \xrightarrow{\partial_*} & & H_*(W_{m-1}) & & \xrightarrow{q_*} & & H_*(W_{m-1}, W_{m-2})
\end{array}$$

where ∂_* are the connecting homomorphisms, $j_*: H_*(\hat{\hat{B}}_{m-1} \setminus \hat{B}_{m-1}) \to \to H_*(\hat{\hat{B}}_{m-1} \setminus \hat{B}_{m-2})$ is induced by the natural embedding, r_*^B is induced by the retraction of $\hat{\hat{B}}_{m-1}$ on B_{m-1}, p_* and q_* are the projection on the quotient. Let us recall that $q_* \cdot \partial_*$ is exactly the natural connection homomorphism of the triple (W_m, W_{m-1}, W_{m-2}). On the other side, since $h \cdot (r^{\eta})^{-1}$ is an homomorphism,

$$H_*(B_m \setminus \hat{B}_{m-1}, \hat{\hat{B}}_{m-1} \setminus \hat{B}_{m-1}) =$$

$$= H_*(S^m \underset{\sigma_m}{\times} \Delta^{\eta}_{m-1} \setminus (\hat{D}_m \times \Delta^{\eta}_{m-1} \underset{\sigma_m}{\cup} S^m \times \hat{\Delta}^{\eta}_{m-1}),$$

$$\hat{\hat{D}}_m \times \Delta^{\eta}_{m-1} \underset{\sigma_m}{\times} S^m \times \hat{\hat{\Delta}}^{\eta}_{m-1} \setminus (\hat{D}_m \times \Delta^{\eta}_{m-1} \underset{\sigma_m}{\cup} S^m \times \hat{\Delta}^{\eta}_{m-1})) =$$

$$= H_*(S^m_0 \underset{\sigma_m}{\times} \Delta^{\eta}_{m-1}, \partial(S^m_0 \underset{\sigma_m}{\times} \Delta^{\eta}_{m-1})).$$

Hence the function $p_* \cdot r^B_* \cdot j_* \cdot \partial_*$ naturally induces an homomorphism, which we call again ∂_*,

$$\partial_*: H_{*+1}(S^m_0 \underset{\sigma_m}{\times} \Delta^{\eta}_{m-1}, \partial(S^m_0 \underset{\sigma_m}{\times} \Delta^{\eta}_{m-1})) \to H_*(S^{m-1}_0 \underset{\sigma_{m-1}}{\times} \Delta^{\eta}_{m-2}, \partial(S^{m-1}_0 \underset{\sigma_{m-1}}{\times} \Delta^{\eta}_{m-2})),$$

such that the following diagram is commutative

$$\begin{array}{ccc} H_*(S^m_0 \underset{\sigma_m}{\times} \Delta^{\eta}_{m-1}, \partial(S^m_0 \underset{\sigma_m}{\times} \Delta^{\eta}_{m-1})) & \xrightarrow{\partial_*} & H_{*-1}(S^{m-1}_0 \underset{\sigma_{m-1}}{\times} \Delta^{\eta}_{m-2}, \partial(S^{m-1}_0 \underset{\sigma_{m-1}}{\times} \Delta^{\eta}_{m-2})) \\ \downarrow h_* & & \downarrow h_{*-1} \\ H_*(W_m, W_{m-1}) & \xrightarrow{\partial_*} & H_{*-1}(W_{m-1}, W_{m-2}) \end{array}$$

Since we have assumed that (2.1) has no solutions, from the deformation Lemma 3.5 it follows that

$$H_*(W_m, W_{m-1}) = H_*(W^{\delta}_{m-1}, W_{m-1}) = (\text{by excision Theorem}) =$$

$$= H_*(W^{\delta}_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)).$$

The natural embedding

$$i_*: H_*(W^{\delta}_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) \to H_*(W_m, W_{m-1})$$

is an isomorphism, and induces an homomorphism

$$\partial_*: H_*(W^{\delta}_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) \to$$

$$\to H_{*-1}(W^{\delta}_{m-2} \cap V(m, \varepsilon), W_{m-2} \cap V(m, \varepsilon)),$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
H_*(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1})) & \xrightarrow{\partial_*} & H_{*-1}(S_0^{m-1} \underset{\sigma_{m-1}}{\times} \Delta^\eta_{m-2}, \partial(S_0^{m-1} \underset{\sigma_{m-1}}{\times} \Delta^\eta_{m-2})) \\
\downarrow h_* & & \downarrow h_{*-1} \\
H_*(W_m, W_{m-1}) & \xrightarrow{\partial_*} & H_{*-1}(W_{m-1}, W_{m-2}) \\
\downarrow i_*^{-1} & & \downarrow i_{*-1}^{-1} \\
H_*(W^\delta_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) & \xrightarrow{\partial_*} & H_{*-1}(W^\delta_{m-2} \cap V(m, \varepsilon), W_{m-2} \cap V(m, \varepsilon))
\end{array}$$

On the other side, by (4.3) and (3.11), h_*sends $H_*(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}))$ to $H_*(W^\delta_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon))$ and the previous diagram simply reduces to

$$\begin{array}{ccc}
H_*(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1})) & \xrightarrow{\partial_*} & H_{*-1}(S_0^{m-1} \underset{\sigma_{m-1}}{\times} \Delta^\eta_{m-2}, \partial(S_0^{m-1} \underset{\sigma_{m-1}}{\times} \Delta^\eta_{m-2})) \\
\downarrow h_* & & \downarrow h_{*-1} \\
H_*(W^\delta_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) & \xrightarrow{\partial_*} & H_{*-1}(W^\delta_{m-2} \cap V(m, \varepsilon), W_{m-2} \cap V(m, \varepsilon))
\end{array}$$

STEP 2. *Definition of two homomorphisms d_α and $\bar{d}_\alpha$ analogous to* (3.18).

PROPOSITION 4.3. *There exists a continuous function*

$$s\colon V(m, \varepsilon) \to S^m/\sigma_m,$$

such that the following diagram is commutative

$$\begin{array}{ccc}
S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1} & \xrightarrow{t} & S^m/\sigma_m \\
\downarrow h & \nearrow s & \\
W^\delta_m \cap V(m, \varepsilon) & &
\end{array}$$

where t is the projection on the first coordinate (cf. [9], Proposition III.1).

Let's denote

$$s^{n-1}\colon H^{n-1}(S^m/\sigma_m) \to H^{n-1}(V(m, \varepsilon)),$$

$$t^{n-1}\colon H^{n-1}(S^m/\sigma_m) \to H^{n-1}(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}),$$

the homomorphism induced on the cohomology groups. Then for every

$\alpha \in H^{n-1}(S^m/\sigma_m)$ we can define

$$\bar{d}_\alpha : H_{mn-1}(W^\delta_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) \to$$
$$\to H_{mn-n}(W^\delta_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)),$$
$$\bar{d}_\alpha(\xi) = s^*(\alpha) \cap \xi$$

and

$$d_\alpha : H_{mn-n}(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1})) \to$$
$$\to H_{mn-n}(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1})),$$
$$\bar{d}_\alpha(\xi) = t^*(\alpha) \cap \xi,$$

where $\cap$ is the cap product, and the following diagram is commutative:

$$\begin{array}{ccc}
H_{mn-1}(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1})) & \xrightarrow{\quad d_\alpha \quad} & H_{mn-n}(S_0^{m-1} \underset{\sigma_{m-1}}{\times} \Delta^\eta_{m-2}, \partial(S_0^{m-1} \underset{\sigma_{m-1}}{\times} \Delta^\eta_{m-2})) \\
{\scriptstyle h_{mn-1}} \downarrow & & \downarrow {\scriptstyle h_{mn-n}} \\
H_{mn-1}(W^\delta_{m-1} \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) & \xrightarrow{\quad \bar{d}_\alpha \quad} & H_{mn-n}(W^\delta_{m-2} \cap V(m, \varepsilon), W_{m-2} \cap V(m, \varepsilon))
\end{array}$$

Moreover (cf. [9], (5.9)), we have

$$H_{mn-1}(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1})) = \mathbb{Z}_2,$$

and there exists $\alpha \in H^{n-1}(S^m/\sigma_m)$ such that

$$\partial \cdot d_\alpha : H_{mn-1}(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1})) \to$$
$$\to H_{(m-1)n-1}(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}, \partial(S_0^m \underset{\sigma_m}{\times} \Delta^\eta_{m-1}))$$

sends the generator in the generator.

In this way we have defined a commutative diagram

$$\begin{array}{ccc}
H_{\mu n-1}(S_0^m \underset{\sigma_\mu}{\times} \Delta^\eta_{\mu-1}, \partial(S_0^\mu \underset{\sigma_\mu}{\times} \Delta^\eta_{\mu-1})) & \xrightarrow{\quad \partial \cdot d_\alpha \ldots \partial \cdot d_\alpha \quad} & H_{n-1}(S_0 \underset{\sigma_1}{\times} \Delta^\eta_0, \partial(S_0 \underset{\sigma_1}{\times} \Delta^\eta_0)) \\
{\scriptstyle h_{\mu n-1}} \downarrow & & \downarrow {\scriptstyle h_{n-1}} \\
H_{\mu n-1}(W^\delta_{\mu-1} \cap V(m, \varepsilon), W_{\mu-1} \cap V(m, \varepsilon)) & \xrightarrow{\quad \partial \cdot \bar{d}_\alpha \ldots \partial \cdot \bar{d}_\alpha \quad} & H_{n-1}(W_0^\delta \cap V(1, \varepsilon), W_0 \cap V(1, \varepsilon))
\end{array}$$

where $(\partial \cdot d_\alpha) \cdot \ldots \cdot (\partial \cdot d_\alpha)$ is an isomorphism.

STEP 3. *Conclusion of the proof.*

Since $W_0 = \emptyset$, and $S_0 = S$, this diagram becomes:

$$\begin{array}{ccc}
H_{\mu n-1}(S_0^\mu \underset{\sigma_\mu}{\times} \Delta_{\mu-1}^\eta, \partial(S_0^\mu \underset{\sigma_\mu}{\times} \Delta_{\mu-1}^\eta)) & \xrightarrow{\partial\cdot d_\alpha \dots \partial\cdot d_\alpha} & H_{n-1}(S) \\
\Big\downarrow h_{\mu n-1} & & \Big\downarrow h_{n-1} \\
H_{\mu n-1}(W_{\mu-1}^\delta \cap V(m, \varepsilon), W_{\mu-1} \cap V(m, \varepsilon)) & \xrightarrow{\partial\cdot \bar{d}_\alpha \dots \partial\cdot \bar{d}_\alpha} & H_{n-1}(W_0^\delta \cap V(1, \varepsilon))
\end{array}$$

By (4.2) $h_{n\mu-1} = 0$. Hence, by the commutativity of this diagram, also

$$h_{n-1}: H_{n-1}(S) \to H_{n-1}(W_0^\delta \cap V(1, \varepsilon)),$$

is identically zero.

On the other side by Proposition 4.3, on $V(1, \varepsilon)$ is defined, a function

$$s: V(1, \varepsilon) \to S,$$

which is a left inverse of h, hence h_{n-1} can not be zero. This contradiction proves that (2.1) has a solution.

5. Proof of Proposition 4.2.

Let's begin with proving some remarks.

PROPOSITION 5.1. *From the hypotheses we have made on f, if follows that*

$$(5.1) \qquad (p-1)f(x, t) \leqslant tf'(x, t) \leqslant (q-1)f(x, t) \qquad \forall t > 0, \forall x \in \Omega,$$

$$(5.2) \qquad f_\infty\left(\sum_{i=1}^m a_i\right) \sum_{i=1}^m a_i \geqslant \sum_{i=1}^m f_\infty(a_i)\, a_i + (p-1) \sum_{\substack{i,j=1 \\ i \neq j}}^m f_\infty(a_i)\, a_j,$$

$$(5.3) \qquad F_\infty\left(\sum_{i=1}^m a_i\right) \geqslant \sum_{i=1}^m F_\infty(a_i) + \frac{2^p - 2}{p + 2^p - 2} \sum_{\substack{i,j=1 \\ i \neq j}}^m f_\infty(a_i)\, a_j,$$

for every m-uple $(a_1, \dots, a_m)$ such that $a_i \geqslant 0$ for every $i = 1, \dots, 0$. Let us note explicitly that $(2^p - 2)/(p + 2^p - 2) > 1/2$; this will be crucial in the proof of Proposition 4.2.

PROOF. (5.1) immediately follows from (2.4). Let's prove (5.2).

Since f_∞ is convex, for every fixed $j \in \{1, ..., m\}$, we have

$$f_\infty\left(a_j + \sum_{\substack{i=1\\ i\neq j}}^{m} a_i\right) \geqslant f_\infty(a_j) + f'_\infty(a_j) \sum_{\substack{i=1\\ i\neq j}}^{m} a_i .$$

Multiplying by a_j we get

$$f_\infty\left(\sum_{i=1}^{m} a_i\right) a_j = f_\infty\left(a_j + \sum_{\substack{i=1\\ i\neq j}}^{m} a_i\right) a_j \geqslant f_\infty(a_j)\, a_j + f'_\infty(a_j)\, a_j \sum_{\substack{i=1\\ i\neq j}}^{m} a_i \geqslant$$

$$\geqslant f_\infty(a_j)\, a_j + (p-1) f_\infty(a_j) \sum_{\substack{i=1\\ i\neq j}}^{m} a_i .$$

Hence, summing on j, we get (5.2). In particular

$$f_\infty\left(\sum_{i=1}^{m} a_i\right) \sum_{j=1}^{m} a_j \geqslant \sum_{i,j=1}^{m} f_\infty(a_j)\, a_i .$$

Then, dividing by $\sum_{j=1}^{m} a_j$, we have

$$f_\infty\left(\sum_{i=1}^{m} a_i\right) \geqslant \sum_{i=1}^{m} f_\infty(a_i) . \tag{5.4}$$

Let's now prove (5.3) when $m = 2$; in other words we will prove that for every $a, b > 0$,

$$F_\infty(a+b) \geqslant F_\infty(a) + F_\infty(b) + \frac{2^p - 2}{p + 2^p - 2}\,(f_\infty(a)\, b + f_\infty(b)\, a) \cdot$$

$$F_\infty(a+b) - F_\infty(a) - F_\infty(b) = \int_0^1 f_\infty(a + \theta b)\, b\, d\theta - \int_0^1 f_\infty(\theta b)\, b\, d\theta =$$

$$= \int_0^1 \int_0^1 f'_\infty(\tau a + \theta b)\, ab\, d\tau\, d\theta =$$

$$= \frac{2^p - 2}{p + 2^p - 2} \int_0^1 \int_0^1 f'_\infty(\tau a + \theta b)\, ab\, d\tau\, d\theta +$$

$$+ \frac{p}{p + 2^p - 2} \int_0^1 \int_0^1 f'_\infty(\tau a + \theta b)\, ab\, d\tau\, d\theta \geqslant$$

(since f' is increasing)

$$\geqslant \frac{2^p-2}{p+2^p-2}\int_0^1\int_0^1 f'_\infty(\tau a)\,ab\,d\tau\,d\theta +$$

$$+\frac{p}{p+2^p-2}\int_0^1\int_0^1 f'_\infty((\tau+\theta)\,b)\,ab\,d\tau\,d\theta \geqslant$$

(if we make the change of variable $s=\tau+\theta$ in the second integral)

$$\geqslant \frac{2^p-2}{p+2^p-2}f_\infty(a)\,b+\frac{p}{p+2^p-2}\int_0^1\int_\tau^{\tau+1} f'_\infty(sb)\,ab\,ds\,d\tau \geqslant$$

$$\geqslant \frac{2^p-2}{p+2^p-2}f_\infty(a)\,b+\frac{p}{p+2^p-2}\int_0^1 (f_\infty((1+\tau)\,b)-f_\infty(\tau b))\,a\,d\tau \geqslant$$

(by (2.4))

$$\geqslant \frac{2^p-2}{p+2^p-2}f_\infty(a)\,b+\frac{p}{p+2^p-2}\int_0^1 ((1+\tau)^{p-1}-\tau^{p-1})\,f_\infty(b)\,a\,d\tau \geqslant$$

$$\geqslant \frac{2^p-2}{p+2^p-2}(f_\infty(a)\,b+f_\infty(b)\,a)\,.$$

(5.3) can now be proved by induction on m, using this inequality and (5.4).

PROPOSITION 5.2. *Let $\varphi\in C(\mathbb{R}^n)$, and $\psi\in C(\mathbb{R}^n)$, be radially symmetric, and assume that $\exists\alpha,\ \beta>0\ \exists C\in\mathbb{R}$ such that*

$$\varphi(x)\exp(\alpha|x|)|x|^\beta\to C \quad \textit{as}\ \ |x|\to+\infty\,,$$

$$\int_{\mathbb{R}^n} |\psi(x)|\exp(\alpha|x|)(1+|x|^\beta)\,dx<+\infty\,,$$

then

$$\left(\int_{\mathbb{R}^n} \varphi(x+y)\psi(x)\,dx\right)\exp(\alpha|y|)|y|^{\beta}\,dx \to$$

$$\to C\int_{\mathbb{R}^n} \psi \exp(-\alpha\langle\nu, x\rangle)\,dx \qquad as \ |y| \to +\infty,$$

for an arbitrary $\nu \in \mathbb{R}^n$, $|\nu| = 1$ (*the left hand side is independent of* ν, *since* ψ *is a radial function*).

PROOF. If $y = \lambda\nu$, with $\nu \in \mathbb{R}^n$, $|\nu| = 1$, then

$$\varphi(x+y)\psi(x)\exp(\alpha|y|)|y|^{\beta} \to C\psi\exp(-\alpha\langle\nu, x\rangle) \qquad \text{as } \lambda \to +\infty.$$

Now the result follows by dominated convergence Theorem, as in [9] Lemma II.2.

Let us now prove Proposition 4.2.

By the definition of $(S_0^{n-1})^m$ for every $m \in \mathbb{N}$ there exists $\gamma_m > 0$ such that

$$\min_{i\neq j} |x_i - x_j| \geq \gamma_m \qquad \forall (x_1, \ldots, x_m, t_1, \ldots, t_m) \in (S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}^{\eta}. \tag{5.5}$$

We can also assume that

$$\min_{i\neq j} |x_i - x_j| \leq \frac{\delta}{2\sqrt{a_\infty}} \qquad \forall (x_1, \ldots, x_m, t_1, \ldots, t_m) \in \partial(S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}^{\eta}, \tag{5.6}$$

where δ is defined in (2.6) and a_∞ in (2.2). Because of the definition of h (see (4.1)), we have only to prove that

(5.7) $\exists \bar\lambda, \bar\eta > 0$ such that

$\forall \lambda \geq \bar\lambda$, $\forall \eta \leq \bar\eta$ if $\min |x_i - x_j| \geq \gamma_m$, and $(t_1, \ldots, t_m) \in \Delta_{m-1}^{\eta}$,

then $k\left(\varphi \sum_{i=1}^{m} t_i\,\omega(\cdot - \lambda x_i)\right) \varphi \sum_{i=1}^{m} t_i\,\omega(\cdot - \lambda x_i) \in V(m, \varepsilon_1)$, (5.8) $\forall m \geqslant 2\ \exists \bar{\lambda}, \bar{\eta} > 0$ such that

$$\forall \lambda \geqslant \bar{\lambda}, \forall \eta \leqslant \bar{\eta} \text{ if } \gamma_m \leqslant \min |x_i - x_j| \leqslant \frac{\delta}{2\sqrt{a_\infty}}, \text{ and}$$

$$(t_1, \ldots, t_m) \in \Delta^{\eta}_{m-1}, I\left(k\left(\varphi \sum_{i=1}^{m} t_i \omega(\cdot - \lambda x_i)\right)\right) \varphi \sum_{i=1}^{m} t_i \omega(\cdot - \lambda x_i) < mI_\infty(\omega),$$

(5.9) $\forall m \geqslant 2\ \exists \bar{\eta} > 0$ such that

$$\forall \eta \leqslant \bar{\eta}, \exists \bar{\lambda} > 0: \forall \lambda \geqslant \bar{\lambda} \text{ if } \gamma_m \leqslant \min |x_i - x_j|, \text{ and}$$

$$(t_1, \ldots, t_m) \in \partial \Delta^{\eta}_{m-1}, I\left(k\left(\varphi \sum_{i=1}^{m} t_i \omega(\cdot - \lambda x_i)\right)\right) \varphi \sum_{i=1}^{m} t_i \omega(\cdot - \lambda x_i) < mI_\infty(\omega),$$

(5.10) $\exists \mu \in \mathbb{N}$ such that $\forall (x_1, \ldots x_\mu) \in (S_0^{n-1})^\mu \min\limits_{i \neq j} |x_i - x_j| \leqslant \dfrac{\delta}{2\sqrt{a_\infty}}$.

Indeed (4.3) is very easy if $m = 1$. If $m \geqslant 2$ and $\chi = (x_1, \ldots, x_m, t_1, \ldots, t_m) \in (S_0^{n-1})^m \times \Delta^{\eta}_{m-1}$, from (5.5) and (5.7) it follows that $\chi \in V(m, \varepsilon_1)$; if $\chi \in \partial(S_0^{n-1})^m \times \Delta^{\eta}_{m-1}$, then, by (5.6) and (5.8) $\chi \in W_{m-1}$; if $\chi \in (S_0^{n-1})^m \times \partial \Delta^{\eta}_{m-1}$, then $\chi \in W_{m-1}$ by (5.9).

Finally (5.10), together with (5.8) proves (4.2).

We first prove (5.10). If by contradiction

$$\forall \mu \in \mathbb{N}\ \exists (x_1, \ldots, x_\mu) \in (S_0^{n-1})^\mu\, |x_i - x_j| > \frac{\delta}{2\sqrt{a_\infty}}, \qquad \forall i \neq j$$

then

$$S^{n-1} \subseteq \bigcup_{i=1}^{\mu} \left(S^{n-1} \cap B\left(x_i, \frac{\delta}{4\sqrt{a_\infty}}\right)\right).$$

On the other side the $(n-1)$-measure of $S^{n-1} \cap B(x_i, \delta/(4\sqrt{a_\infty}))$ is independent of x_i. Hence if we call it $|S^{n-1} \cap B(x_i, \delta/(4\sqrt{a_\infty}))|_{n-1} = C(\delta)$, then

$$|S^{n-1}|_{n-1} \geqslant \mu C(\delta) \qquad \forall \mu \in \mathbb{N}$$

and this is a contradiction.

Let's now prove (5.7).

We'll first show that $k\left(\varphi \sum\limits_{i=1}^{m} t_i \omega_i\right) \to m$ as $t_i \to 1/m$ and $\lambda \to +\infty$, where k is the function defined in Remark 3.1

$$\min\left(\left(\frac{k\left(\varphi\sum_{i=1}^m t_i\omega_i\right)}{m}\right)^{p-2}, \left(\frac{k\left(\varphi\sum_{i=1}^m t_i\omega_i\right)}{m}\right)^{q-2}\right) \leqslant$$

$$\leqslant \frac{\int_\Omega \left|D\left(\varphi\sum_{i=1}^m mt_i\omega_i\right)\right|^2 + \int_\Omega a\left(\varphi\sum_{i=1}^m mt_i\omega_i\right)^2}{\int_\Omega f\left(x, \varphi\sum_{i=1}^m mt_i\omega_i\right)\varphi\sum_{i=1}^m mt_i\omega_i} \leqslant$$

(by (5.5))

$$= \frac{\sum_{i=1}^m \int_\Omega |D(mt_i\omega_i)|^2 + \int_\Omega a(mt_i\omega_i)^2}{\sum_{i=1}^m \int_\Omega f(x, mt_i\omega_i)\, mt_i\omega_i} + o(1) \quad \text{as } \lambda \to +\infty$$

(since ω is a solution of (2.7))

$= 1 + o(1)$ as $\lambda \to +\infty$ and $mt_i \to 1$

uniformly with respect to $(x_1, \ldots, x_m) \in (S_0^{n-1})^m$. Analogously

$$\max\left(\left(\frac{k}{m}\right)^{p-2}, \left(\frac{k}{m}\right)^{q-2}\right) \geqslant$$

$$\geqslant \frac{\int_\Omega \left|D\left(\varphi\sum_{i=1}^m mt_i\omega_i\right)\right|^2 + \int_\Omega a\left(\varphi\sum_{i=1}^m mt_i\omega_i\right)^2}{\int_\Omega f\left(x, \varphi\sum_{i=1}^m mt_i\omega_i\right)\varphi\sum_{i=1}^m mt_i\omega_i} \to 1$$

as $\lambda \to +\infty$ and $mt_i \to 1$. Hence $kt_i \to 1$ as $\lambda \to +\infty$, and $t_i \to 1/m$, and we conclude that $k\varphi\sum_{i=1}^m t_i\omega_i \in V(m, \varepsilon)$ $\left(\text{here and in the sequel we write } k \text{ instead of } k\left(\varphi\sum_{i=1}^m t_i\omega_i\right)\right)$.

Let's prove (5.8). We will denote

$$\langle u, v\rangle = \int_\Omega Du \cdot Dv + \int_\Omega a_\infty uv,$$

$$\|u\| = \left(\int_\Omega |Du|^2 + \int_\Omega a_\infty u^2\right)^{1/2} \quad \text{and} \quad \|u\|_a = \left(\int_\Omega |Du|^2 + \int_\Omega au^2\right)^{1/2}.$$

First recall the following estimates (see [9], 6.20)

$$\int_\Omega \left| D\left(\varphi \sum_{i=1}^m kt_i \omega_i\right)\right|^2 + \int_\Omega a\left(\varphi \sum_{i=1}^m kt_i \omega_i\right)^2 \leqslant$$

$$\leqslant \left\|\sum_{i=1}^m kt_i \omega_i\right\|^2 + \int_\Omega |\Delta\varphi| \left(\sum_{i=1}^m kt_i \omega_i\right)^2 + \int_\Omega (a - a_\infty)\left(\varphi \sum_{i=1}^m k\, t_i\ \omega_i\right)^2 =$$

$$= \sum_{i=1}^m (kt_i)^2 \|\omega\|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^m t_i t_j k^2 \langle \omega_i, \omega_j\rangle + C\sum_{i=1}^m \int_\Omega (|\Delta\varphi| + a - a_\infty)\, \omega_i^2 .$$

Now, since ω is a solution of (2.7), we have

$$\langle \omega_i, \omega_j\rangle = \int_\Omega f_\infty(\omega_i)\, \omega_j, \qquad \text{and} \qquad \|\omega\|^2 = \int_\Omega f_\infty(\omega)\, \omega$$

and, if we assume that $\delta < \sqrt{a_\infty}$, from Proposition 5.2 we have

$$\sum_{i=1}^m \int_\Omega (|\Delta\varphi| + a - a_\infty)\, \omega_i^2 \leqslant C \exp(-\lambda\delta)\, \lambda^{-(n-1)/2}.$$

Hence we get

$$\int_\Omega \left| D\left(\varphi \sum_{i=1}^m kt_i \omega_i\right)\right|^2 + \int_\Omega a\left(\varphi \sum_{i=1}^m kt_i \omega_i\right)^2 \leqslant$$

$$\leqslant \sum_{i=1}^m (kt_i)^2 \int_\Omega f_\infty(\omega)\, \omega + \sum_{\substack{i,j=1\\ i\neq j}}^m t_i t_j k^2 \int_\Omega f_\infty(\omega_i)\, \omega_j + C\exp(-\delta\lambda)\, \lambda^{-(n-1)/2}.$$

Let us estimate the second therm in the functional I

$$\int_\Omega F\left(x, \varphi \sum_{i=1}^m kt_i \omega_i\right) \geqslant$$

$$\geqslant \int_\Omega F_\infty\left(\sum_{i=1}^m kt_i \omega_i\right) - \int_\Omega \left(F_\infty\left(\sum_{i=1}^m kt_i \omega_i\right) - F_\infty\left(\varphi \sum_{i=1}^m kt_i \omega_i\right)\right) +$$

$$- \int_\Omega \left(F_\infty\left(\varphi \sum_{i=1}^m kt_i \omega_i\right) - F\left(x, \varphi \sum_{i=1}^m kt_i \omega_i\right)\right) \geqslant$$

(by (2.4) and (2.6))

$$\geqslant \int_\Omega F_\infty\left(\sum_{i=1}^m kt_i \omega_i\right) + C\int_\Omega (\varphi^{q-1} - 1) F_\infty\left(\sum_{i=1}^m \omega_i\right) -$$

$$- \int_\Omega \exp(-\delta|x|)|x|^{-(n-1)/2} F_\infty\left(\sum_{i=1}^m \omega_i\right) \geqslant$$

(by Proposition 5.2)

$$\geqslant \int_\Omega F_\infty\left(\sum_{i=1}^m kt_i \omega_i\right) - C\exp(-\delta\lambda)\lambda^{-(n-1)/2} \geqslant$$

(by 5.3))

$$\geqslant \sum_{i=1}^m \int_\Omega F_\infty(kt_i \omega_i) + \frac{2^p - 2}{2^p + p - 2} \sum_{\substack{i,j=1 \\ i \neq j}}^m \int_\Omega f_\infty(kt_i \omega_i) kt_j \omega_j -$$

$$- C\exp(-\delta\lambda)\lambda^{-(n-1)/2} \geqslant$$

(by 2.4))

$$\sum_{i=1}^m \int_\Omega F_\infty(kt_i \omega_i) + \frac{2^p - 2}{2^p + p - 2} \sum_{\substack{i,j=1 \\ i \neq j}}^m \min((kt_i)^{p-1}, (kt_i)^{q-1}) kt_j \int_\Omega f_\infty(\omega_i) \omega_j -$$

$$- C\exp(-\delta\lambda)\lambda^{-(n-1)/2}.$$

Hence

$$I\left(\varphi \sum_{i=1}^{m} kt_i\,\omega_i\right) \leqslant \frac{1}{2}\sum_{i=1}^{m}(kt_i)^2 \int_\Omega f_\infty(\omega)\,\omega - \sum_{i=1}^{m}\int_\Omega F_\infty(kt_i\,\omega_i) +$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{m}\left(\frac{1}{2}t_i k - \frac{2^p-2}{2^p+p-2}\min\left((kt_i)^{p-1},(kt_i)^{q-1}\right)\right)kt_j\int_\Omega f_\infty(\omega_i)\,\omega_j +$$

$$+\,C\exp(-\delta\lambda)\,\lambda^{-(n-1)/2}.$$

By the proof of (5.7), for every $i = 1, \dots, m$, $kt_i \to 1$ as $\lambda \to +\infty$ and $\eta \to 0$, hence

$$\frac{1}{2}t_i k - \frac{2^p-2}{2^p+p-2}\min\left((kt_i)^{p-1},(kt_i)^{q-1}\right) \to \frac{p+2-2^p}{2(2^p-2+p)} < 0.$$

On the other side, $\min|x_i - x_j| \leqslant \delta/(2\sqrt{a_\infty})$; hence from Proposition 5.2, it follows that there exist $i, j \in \{1, \dots, m\}$ such that

$$\int_{\mathbb{R}^n} f_\infty(\omega_i)\,\omega_j \geqslant C\exp\left(-\lambda\sqrt{a_\infty}\,|x_i - x_j|\right)\left(\lambda|x_i - x_j|\right)^{-(n-1)/2} \geqslant$$

$$\geqslant C\exp\left(-\frac{\lambda\delta}{2}\right)\left(\frac{\lambda\delta}{2}\right)^{-(n-1)/2}.$$

Consequently, if λ is big enough we get:

$$I\left(\varphi \sum_{i=1}^{m} kt_i\,\omega_i\right) \leqslant \frac{1}{2}\sum_{i=1}^{m}(kt_i)^2 \int_\Omega f_\infty(\omega)\,\omega - \sum_{i=1}^{m}\int_\Omega F_\infty(kt_i\,\omega_i) \leqslant$$

(since the function $(t_1, \dots, t_m) \to (1/2)\sum_{i=1}^{m} t_i^2 \int_\Omega f_\infty(\omega)\,\omega - \sum_{i=1}^{m}\int_\Omega F_\infty(t_i\,\omega_i)$ has a maximum at the point $(1, \dots, 1)$)

$$\leqslant \frac{1}{2}\sum_{i=1}^{m}\int_\Omega f_\infty(\omega)\,\omega - \sum_{i=1}^{m}\int_\Omega F_\infty(\omega) =$$

(since ω is a solution of (2.7))

$$= m\left(\frac{1}{2}\|\omega\|^2 - \int_\Omega F_\infty(\omega)\right) = mI_\infty(\omega).$$

Let's now prove (5.9).

We'll begin with estimating

$$\frac{k}{m} = \frac{k\left(\varphi \sum_{i=1}^{m} t_i \omega_i\right)}{m}.$$

Arguing as before we get

$$\int_\Omega f\left(x, \varphi \sum_{i=1}^{m} mt_i \omega_i\right) \varphi \sum_{i=1}^{m} mt_i \omega_i \geqslant$$

$$\geqslant \int_\Omega f_\infty \left(\sum_{i=1}^{m} mt_i \omega_i\right) \sum_{i=1}^{m} mt_i \omega_i - C \exp(-\delta\lambda) \lambda^{-(n-1)/2} \geqslant$$

(b y (5.2))

$$\geqslant \sum_{i=1}^{m} \int_\Omega f_\infty (mt_i \omega_i) mt_i \omega_i + (p-1) \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \int_\Omega f_\infty (mt_i \omega_i) mt_j \omega_j -$$

$$- C \exp(-\delta\lambda) \lambda^{-(n-1)/2} \geqslant \sum_{i=1}^{m} \int_\Omega f_\infty (mt_i \omega_i) mt_i \omega_i - C \exp(-\lambda\delta) \lambda^{-(n-1)/2}.$$

Then, by Remark 3.1

$$\min\left(\left(\frac{k}{m}\right)^{p-2}, \left(\frac{k}{m}\right)^{q-2}\right) \leqslant \frac{\sum_{i=1}^{m} (mt_i)^2 \int_\Omega f_\infty(\omega)\omega + o(\lambda)}{\sum_{i=1}^{m} \int_\Omega f_\infty (mt_i \omega_i) mt_i \omega_i + o(\lambda)}.$$

It is not difficult to prove that the function

$$(t_1, \ldots, t_m) \to \sum_{i=1}^{m} (mt_i)^2 \int_\Omega f_\infty(\omega)\omega - \sum_{i=1}^{m} \int_\Omega f_\infty (mt_i \omega_i) mt_i \omega_i$$

constrained on the manifold $\sum_{i=1}^{m} t_i = 1$, has a strict maximum in the point $(1/m, \ldots, 1/m)$, then $\exists \bar\eta > 0$ such that $\forall \eta < \bar\eta\ \exists \theta(\eta)\ \exists \lambda(\eta)$: $\forall (t_1, \ldots, t_m) \in \in \partial \Delta_{m-1}^{\eta}$, $\forall \lambda > \lambda(\eta)$,

$$\min\left(\left(\frac{k}{m}\right)^{p-2}, \left(\frac{k}{m}\right)^{q-2}\right) \leqslant \frac{\sum_{i=1}^{m} (mt_i)^2 \int_\Omega f_\infty(\omega)\omega + o(\lambda)}{\sum_{i=1}^{m} \int_\Omega f_\infty (mt_i \omega_i) mt_i \omega_i - o(\lambda)} < 1 - \frac{\theta}{2}.$$

Consequently $\forall \eta \leqslant \bar{\eta}$ there exists $\lambda(\eta)$ and $\bar{\theta}(\eta)$ such that $k < m - \bar{\theta}(\eta)$. Now

$$I\left(\varphi \sum_{i=1}^{m} kt_i \omega_i\right) \leqslant \frac{1}{2} \sum_{i=1}^{m} (kt_i)^2 \int_{\Omega} f_\infty(\omega)\, \omega - \sum_{i=1}^{m} \int_{\Omega} F_\infty(kt_i \omega_i) + o(\lambda).$$

The function

$$(kt_1, \dots kt_m) \rightarrow \frac{1}{2} \sum_{i=1}^{m} (kt_i)^2 \int_{\Omega} f_\infty(\omega)\, \omega - \sum_{i=1}^{m} \int_{\Omega} F_\infty(kt_i \omega_i),$$

has a strict maximum at the point $(1 \dots 1)$. Since $\min |x_i - x_j| \geqslant \gamma_m$, we can assume that kt_i belongs to an arbitrary small neighborhood of 1 for every $i = 1, \dots, m$. On the other side, from the estimate we have just obtained, we get

$$\sum_{i=1}^{m} |kt_i - 1| \geqslant m - \sum_{i=1}^{m} kt_i = m - k > \bar{\theta}(\eta)\, m.$$

Hence $\exists \bar{\bar{\theta}} > 0$ such that

$$I\left(\varphi \sum_{i=1}^{m} kt_i \omega_i\right) \leqslant \frac{1}{2} \sum_{i=1}^{m} \int_{\Omega} f_\infty(\omega)\, \omega - \sum_{i=1}^{m} \int_{\Omega} F_\infty(\omega_i) - \bar{\bar{\theta}}(\eta) + o(\lambda) < m I_\infty(\omega),$$

if λ is big enough.

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