RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

LAURA DE CARLI

L^p estimates for the Cauchy transforms of distributions with respect to convex cones

Rendiconti del Seminario Matematico della Università di Padova, tome 88 (1992), p. 35-53

<http://www.numdam.org/item?id=RSMUP_1992__88__35_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 88 (1992)

L^p Estimates for the Cauchy Transforms of Distributions with Respect to Convex Cones.

LAURA DE CARLI (*)

ABSTRACT - The main purpose of this work is to give L^p and L^p_{loc} estimates for the Cauchy transforms of a distribution with respect to a general convex cone. We discuss first the case of proper cones, and we show that global L^∞ estimates can be given for polygonal cones or for circular cones when $p < 2(n-1)/(n-2)$; in this case we show that the estimates that can be given are sharp.

Introduction.

The Cauchy transform gives the representation as boundary values of holomorphic functions of distributions that satisfy the dispersion relations with respect to a given casuality cone.

In wiew of applications to the study of the singularities of distributions of this kind that moreover satisfy some non-linear partial differential equation, we gather here some L^p results for this integral transform, that are seemingly new in the multidimensional case.

1. Basic properties of the Cauchy transform with respect to a proper cone.

In this paragraph the definition and some basic properties of the Cauchy transform will be recalled.

Let $\Gamma \subset \boldsymbol{R}^n$ be an open convex cone with vertex at 0; we define the

(*) Indirizzo dell'A.: University of California, Los Angeles, Department of Mathematics, Los Angeles, CA 90024-1555, USA.

dual cone of Γ as the cone:

$$\Gamma^0 = \{y \in \boldsymbol{R}^n : \langle y, \zeta \rangle > 0 \ \forall \zeta \in \Gamma\} \tag{1.1}$$

where $\langle , \rangle$ is the usual scalar product in $\boldsymbol{R}^n$.

We say that Γ is *proper* if Γ^0 is nonempty.

This is equivalent to say that Γ does not contain any line (see [H], pag. 257).

We say that Γ is *polygonal* (resp. *circular*) if Γ is the projection of a convex polyedron (resp. a sphere) with respect to the origin.

If Γ is a proper and convex cone in $\boldsymbol{R}^n$, we define the *Cauchy kernel with respect to* Γ as the function:

$$\begin{aligned} &K_\Gamma : \boldsymbol{R}^n + i\Gamma \to \boldsymbol{C}, \\ &K_\Gamma(x+iy) = \int_{\Gamma^0} e^{i\langle x+iy, \zeta \rangle} d\zeta . \end{aligned} \tag{1.2}$$

According to this definition, the function K_Γ is holomorphic in the tube domain $\boldsymbol{R}^n + i\Gamma$; if $n > 1$ we can write (see [V] pag. 149):

$$K_\Gamma(x+iy) = C(n)\, i^n \int_{\Gamma^0 \cap S^{n-1}} \frac{1}{\langle x+iy, \zeta \rangle} dS(\zeta) \tag{1.3}$$

where $C(n)$ is a constant depending only on n, and dS is the $(n-1)$-dimensional measure on S^{n-1} while when $n = 1$ and when $\Gamma_+ = (0, +\infty)$, (resp. $\Gamma_- = (-\infty, 0)$), an esplicit computation shows that:

$$K_{\Gamma_+}(x+iy) = \frac{i}{x+iy}, \qquad x \in \boldsymbol{R}, \ \ y > 0, \tag{1.4}$$

$$\left(\text{resp. } K_{\Gamma_-}(x+iy) = \frac{-i}{x+iy}, \ x \in \boldsymbol{R}, \ \ y < 0\right). \tag{1.4'}$$

Set $K_{\Gamma, y}(x) = K_\Gamma(x+iy)$; when $n = 1$ the following result holds:

PROPOSITION 1.1. i) $K_{\Gamma_+, y} \in L^p(\boldsymbol{R})$, $\forall p > 1$, $y > 0$, *and*

$$\|K_{\Gamma_+, y}\|_{L^p(\boldsymbol{R})} = C(p)\, y^{-1+1/p} \tag{1.5}$$

where $C(p) = \left(\int_{-\infty}^{+\infty} (t^2+1)^{-p/2} dt \right)^{1/p}$

ii) $K_{\Gamma_+, y} \in L^1_{\mathrm{loc}}(\boldsymbol{R})$, $\forall y > 0$, *and*

$$\|K_{\Gamma_+, y}\|_{L^1(-R, R)} = 2 \log (Ry^{-1} + \sqrt{1 + R^2 y^{-2}}). \tag{1.6}$$

(*Same for* $K_{\Gamma_-, y}$ *with* $-y$ *in place of* y.)

Given a function $f \in \mathcal{S}(\boldsymbol{R}^n)$, we define the *Cauchy transform of* f *with respect to an open, proper and convex cone* Γ as the function:

$$\begin{aligned} &T_\Gamma f \colon \boldsymbol{R}^n + i\Gamma \to \boldsymbol{C}, \\ &T_\Gamma f(x + iy) = K_{\Gamma, y} * f(x) \end{aligned} \tag{1.7}$$

which is equivalent to:

$$T_\Gamma f(x + iy) = (2\pi)^{-n} \int_{\Gamma^0} \hat{f}(\zeta) e^{i\langle \zeta, x + iy \rangle} d\zeta. \tag{1.7'}$$

We can see from def. (1.7′) that $T_\Gamma f$ is holomorphic in the tube domain $\boldsymbol{R}^n + i\Gamma$; moreover, we can see from def. (1.7) that when $\mathrm{supp}(\hat{f}) \subset \Gamma^0$, then $\lim_{\substack{y \to 0 \\ y \in \Gamma_1 \subset \Gamma}} T_\Gamma f(x + iy) = f(x)$, where Γ_1 is a closed cone in Γ and the limit is taken in distribution sense.

The above properties characterize the Cauchy transform with respect to a cone Γ of the functions whose Fourier transform are supported in the cone Γ^0, as specified by the following:

THEOREM 1.2. *Let* $f \in \mathcal{S}(\boldsymbol{R}^n)$ *be a function whose Fourier transform is supported in an open, proper and convex cone* Γ^0;

suppose that there exist a point $x_0 \in \boldsymbol{R}^n$, *a neighborhood* U *of* x_0 *in* $\boldsymbol{R}^n$, *a complex neighborhood* $\hat{U}$ *of* U *in* $\boldsymbol{C}^n$, *and a function* F, *holomorphic in an open and connected set* $\Omega \subset U + i\Gamma \cap \hat{U}$, *so that*

$$\lim_{\substack{y \to 0 \\ y \in \Gamma' \subset \Gamma}} F(x + iy) = f(x)$$

where Γ_1 *is a closed cone in* Γ *and the limit is taken in distribution sense; then,* $F = T_\Gamma f$.

PROOF. Since $\lim_{\substack{y \to 0 \\ y \in \Gamma' \subset \Gamma}} \big(F(x + iy) - T_\Gamma f(x + iy)\big) = 0$ in $\mathcal{D}'(U)$, the thesis comes from the classical Edge-of-the-wedge theorems (cfr. [R] or [Ko]).

Set $T_{\Gamma, y} f(x) = T_\Gamma f(x + iy)$; in the following we will consider Banach spaces of distributions $(X, \| \|_X)$ and $(Y, \| \|_Y)$ such that $\mathcal{S}(\boldsymbol{R}^n)$ is a dense subspace of X, and such that

$$\|T_{\Gamma, y} f\|_Y \leqslant C\|f\|_X \qquad \forall f \in \mathcal{S}(\boldsymbol{R}^n), \ y \in \Gamma$$

where C is a constant that can depend or not from y.

When the above inequality is satisfied, then $T_{\Gamma, y}$ can be extended to a bounded operator from X to Y, for every $y \in \Gamma$.

When $n = 1$ and $f \in \mathcal{S}(\boldsymbol{R})$, if we let $T_+ f$ (resp. $T_- f$) be the Cauchy transform of f with respect to the cone $\Gamma_+ = (0, +\infty)$ (resp. $\Gamma_+ = (-\infty, 0)$) from Proposition 1.1 ii) and definition (1.7) follows:

PROPOSITION 1.3. $T_{+,y} f \in L^p_{\text{loc}}(\boldsymbol{R})$, $\forall y > 0$, $p \in [1, +\infty]$ *and* $\forall r > 0$ *the following estimate holds*:

$$\|T_{+,y} f\|_{L^p(-r,r)} \leqslant 2\|f\|_{L^p(\boldsymbol{R})} \, |\log(ry^{-1})| \, .$$

(*Same for* $T_{-,y} f$.)

When $p \in (1, +\infty)$, a stronger result holds:

PROPOSITION 1.4 (M. Riesz). $T_{+,y} f \in L^p(\boldsymbol{R})$, $\forall y > 0$, $p \in (1, +\infty)$ *and the following estimate holds*:

$$\|T_{+,y} f\|_{L_p(\boldsymbol{R})} \leqslant A(p)\|f\|_{L^p(\boldsymbol{R})}$$

where $A(p)$ *is a constant depending only on* p.

(*Same for* $T_{-,y} f$.)

PROOF. See [Nr] pag. 68.

Proposition 1.4 says that $T_+ f$ (resp. $T_- f$) belong to the *Hardy space* $H^p(\boldsymbol{R} + i(0, +\infty))$ (resp. $H^p(\boldsymbol{R} + i(-\infty, 0))$) when $f \in L^p(\boldsymbol{R})$ and $p \in (1, +\infty)$; for the definition and the properties of the Hardy spaces see e.g. [SV].

The following proposition gives an estimate for the growth of the Cauchy transform near $\boldsymbol{R}$.

PROPOSITION 1.5. $T_{+,y} f \in L^\infty(\boldsymbol{R})$ $\forall y > 0$, *and the following estimate holds for every* $p \in [1, +\infty)$:

$$\|T_{+,y} f\|_{L^\infty(\boldsymbol{R})} \leqslant C(q)|y|^{-1/p}\|f\|_{L^p(\boldsymbol{R})} \, .$$

Here $1/q + 1/p = 1$ *and* $C(q)$ *is defined as in Proposition* 1.1.

(*Same for* $T_{-,y} f$.)

PROOF. Comes from Proposition 1.1 i) and definition (1.7).

Observe that the classical theory of H^p spaces on tubes yields Proposition 1.5, since the following result holds:

PROPOSITION 1.6. *Every $F \in H^p(\mathbf{R}^n + i\Gamma)$ satisfy the following estimate:*

$$|F(x+iy)| \leq C(F)\, \mathrm{dist}\,(y, \partial\Omega)^{-n/p} \qquad \forall x \in \mathbf{R}^n$$

where $C(F)$ is the norm of F in H^p.

PROOF. See e.g. [Kr].

The main purpose of this paper is to extend Propositions 1.1-1.5 to the case $n > 1$.

It is well known that the analogous of Proposition 1.4 does not hold in the multidimensional case when $p \neq 2$ and Γ is any cone, since the results of Fefferman (see [F]) and «loose theorem» (see [SV]) show that the characteristic function of the circular cone cannot be a Fourier multiplier.

In the next paragraph we will prove that Proposition 1.5 can be generalized to the case $n > 1$ when Γ is a circular cone in $\mathbf{R}^n$, and $p < (2(n-1))/(n-2)$, and we will give also a generalization of Propositions 1.1, 1.3 and 1.4.

In the thirth paragraph we will extend the definition and the basic properties of the Cauchy transform to the case of non-proper cones.

2. L^p estimates for the Cauchy transform with respect to a proper cone.

Our purpose is to find L^p estimates for the Cauchy transforms of a function with respect to a convex and proper cone in $\mathbf{R}^n$ that generalize Theorems 1.3, 1.4 and 1.5 to the case $n > 1$.

In what follows we will refer to Γ as a proper and convex cone in $\mathbf{R}^n$ and to K as a bounded measurable subset of $\mathbf{R}^n$.

By $C(u, v, ...)$ we will denote a generic constant depending on the parameters $(u, v, ...)$.

Assume first that Γ is a polygonal cone in $\mathbf{R}^n$; we can decompose Γ^0 into the union of a finite number of n-sided polygonal cones, $\Gamma_1^0, \ldots, \Gamma_N^0$, whose intersections have measure zero.

Using that decomposition, we can write:

$$K_\Gamma(x+iy) = \sum_{i=1}^{N} K_{\Gamma_i}(x+iy),$$

$$T_\Gamma f(x+iy) = \sum_{i=1}^{N} T_{\Gamma_i, K} f(x+iy).$$

Each Γ_i can be mapped onto the *positive quadrant*

$$Q_n = (0, +\infty)^n$$

by means of a linear transformation (see also [R] pag. 2); by using (1.4) and induction on n we can compute explictely the Cauchy kernel K_{Q_n}, and we have:

$$K_{Q_n}(x+iy) = \frac{(i)^n}{(x_1+iy_1)\dots(x_n+iy_n)}. \tag{2.1}$$

We reacall the following results:

(2.2) When a linear transformation ϕ: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ maps a cone Γ_1 onto a cone Γ_2, then the adjoint ϕ^* of ϕ maps Γ_2^0 onto Γ_1^0.

(2.3) When ϕ is as in (2.2), Ω is a measurable subset of $\boldsymbol{R}^n$, and $p \in [1, +\infty]$ we have:

$$\|K_{\Gamma_1, y}\|_{L^p(\Omega)} = (\det \phi)^{1-1/p} \|K_{\Gamma_2, \phi(y)}\|_{L^p(\phi(\Omega))}.$$

(2.4) For every $f \in L^p_{\text{loc}}(\boldsymbol{R}^n)$, and every $y \in \Gamma_1$, we have

$$\|T_{\Gamma_1, y} f\|_{L^p(\Omega)} = (\det \phi)^{1-1/p} \|T_{\Gamma_2, \phi(y)} f \circ \phi^{-1}\|_{L^p(\phi(\Omega))}.$$

This shows that every statement that holds for Q_n holds also for a general polygonal cone.

We can prove now the following results:

THEOREM 2.1. *When Γ is a polygonal cone, $T_{\Gamma, y} f \in L^p(\boldsymbol{R}^n)$, $\forall y \in \Gamma$ and the following inequality holds*:

$$\|T_{\Gamma, y} f\|_{L^p(\boldsymbol{R}^n)} \leq C(n, p, \Gamma)\|f\|_{L^p(\boldsymbol{R}^n)}.$$

PROPOSITION 2.2. i) *When Γ is a polygonal cone, $K_{\Gamma, y} \in L^p(\boldsymbol{R}^n)$, $\forall p > 1$, $y \in \Gamma$, and the following inequalities holds*:

$$\|K_{\Gamma, y}\|_{L^p(\boldsymbol{R}^n)} \leq C(n, p, \Gamma) K_\Gamma(iy)^{1-1/p}.$$

ii) $K_{\Gamma, y} \in L^1_{\text{loc}}(\boldsymbol{R}^n)$ *for every* $y \in \Gamma$, *and* $AK \subset\subset \boldsymbol{R}^n$ *we have*:

$$\|K_{\Gamma, y}\|_{L^1(K)} \leq C(\Gamma, n)\left[\log\left(\frac{\text{diam}(K)}{\text{dist}(y, \partial\Gamma)}\right)\right]^n,$$

PROOF. Since we can assume $\Gamma = Q_n$, the thesis of Theorem 2.1 (resp. Proposition 2.2) comes from Theorem 1.4 (resp. Proposition 1.1) by induction on n.

From Proposition 2.2 ii) and definition (1.7), we deduce the following:

COROLLARY 2.3. *When* Γ *is a polygonal cone* $T_{\Gamma, y} \in L^p_{\text{loc}}(\boldsymbol{R}^n)$ $\forall p \in [1, +\infty]$, $y \in \Gamma$, *and* $\forall K \subset\subset \boldsymbol{R}^n$ *there exists a constant*

$$C = C(\Gamma, n)\left[\log\left(\frac{\text{diam}(K)}{\text{dist}(y, \partial\Gamma)}\right)\right]^n,$$

such that the following inequality holds:

$$\|T_{\Gamma, y} f\|_{L^p(K)} \leq C\|f\|_{L^p(\boldsymbol{R}^n)}.$$

In what follows we will use a well-known result whose proof can be found e.g. in [V].

LEMMA 2.4. *For every* $u \in \boldsymbol{R}^n$ *and for every proper convex cone* $\Gamma \subset \boldsymbol{R}^n$, *we have*:

$$\text{dist}(y, \partial\Gamma) = \min_{\zeta \in \Gamma^0:\ |\zeta| = 1} \langle y, \zeta\rangle.$$

PROPOSITION 2.5. i) *When* Γ *is a polygonal cone,* $T_{\Gamma, y} \in L^\infty(\boldsymbol{R}^n)$ $\forall y \in \Gamma$ *and* $\forall p < \infty$ *there exists a constant* $C = C(n, p, \Gamma)$ *such that the following inequality holds*:

$$\|T_{\Gamma, y} f\|_{L^\infty(\boldsymbol{R}^n)} \leq C\|f\|_{L^p(\boldsymbol{R}^n)} \text{dist}(y, \partial\Gamma)^{-n/p}.$$

ii) *When* $p \in [1, 2]$, *the above statement holds for every cone* Γ, *with* $C(n, p, \Gamma) = C(n, p)$ *meas* $(\Gamma^0 \cap S^{n-1})^{1/p}$.

PROOF. By definition (1.2) we have:

$$K_\Gamma(iy) = i^n C(n) \int_{\Gamma^0 \cap S^{n-1}} \frac{dS(\zeta)}{\langle y, \zeta\rangle^n}$$

and by Lemma 1.4, we have:

$$K_\Gamma(iy) \leq C_n \operatorname{meas}(\Gamma^0 \cap S^{n-1}) \operatorname{dist}(y, \partial\Gamma)^{-n} . \tag{2.5}$$

From Proposition 2.2 i) and definition (1.7) comes the first part of the theorem.

To prove the second part, observe that $K_{\Gamma, y} \in L^\infty(\boldsymbol{R}^n)$ $\forall y \in \Gamma$, and since

$$|K_{\Gamma, y}(x)| \leq \int_{\Gamma^0} e^{-\langle y, \zeta\rangle} d\zeta = K_\Gamma(iy) \qquad \forall x \in \boldsymbol{R}^n$$

we have:

$$\|K_{\Gamma, y}\|_{L^\infty(\boldsymbol{R}^n)} = K_\Gamma(iy) . \tag{2.6}$$

Moreover, $K_{\Gamma, y}$ is the Fourier transform of the function

$$\phi_y(\zeta) = e^{-\langle y, \zeta\rangle} \chi_{\Gamma^0}(\zeta) \tag{2.7}$$

where χ_{Γ^0} is the characteristic function of the cone Γ^0.

Since $\phi_y \in L^2(\boldsymbol{R}^n)$, and:

$$\|\phi_y\|_{L^2(\boldsymbol{R}^n)} = \left(\int_{\Gamma^0} e^{-\langle 2y, \zeta\rangle} d\zeta \right)^{1/2} = 2^{-n/2} K_\Gamma(iy)^{1/2}$$

by Plancherel theorem we have that $K_{\Gamma, y} \in L^2(\boldsymbol{R}^n)$ and $\|K_{\Gamma, y}\|_{L^2(\boldsymbol{R}^n)} = 2^{-n/2} K_\Gamma(iy)$; by interpolation we have that $K_{\Gamma, y} \in L^p(\boldsymbol{R}^n)$ $\forall p \in [2, +\infty]$, and

$$\|K_{\Gamma, y}\|_{L^p(\boldsymbol{R}^n)} \leq 2^{-n/p} K_\Gamma(iy)^{1 - 1/p} .$$

From Lemma 2.4 and def. (1.7) we have the thesis.

Since the statements discussed above hold for every proper cone in $\boldsymbol{R}^2$, in what follows we will assume $n \geq 3$.

Consider a circular cone Γ in $\boldsymbol{R}^n$; after a rotation σ we can move Γ onto the cone:

$$\Gamma_c = \{(\zeta', \zeta_n) \in \boldsymbol{R}^n : \ \zeta_n \geq c|\zeta|'\} ; \tag{2.8}$$

the Cauchy kernel with respect to Γ_c can be computed esplicitely (see [V] pag. 64) and it is:

$$K_{\Gamma_c}(x+iy) = \frac{C(n)\, c^{n-1}}{\left(c^2 \sum_{i=1}^{n-1} (x_i + y_i)^2 - (x_n + iy_n)^2\right)^{n/2}}, \tag{2.9}$$

let $y \in \Gamma$, $\bar{y} = (\bar{y}', \bar{y}_n) = \sigma(y)$.

The linear transformation

$$h\colon \boldsymbol{R}^n \to \boldsymbol{R}^n, \qquad h(x', x_n) = (cx', x_n) \tag{2.10}$$

maps the cone Γ_c onto Γ_1 and $\det h = c^{n-1}$.

With a rotation ρ around the axis of Γ_1, we can move the point $h(\bar{y}) = (c\bar{y}', \bar{y}_n)$ onto the point $\rho(h(\bar{y})) = (0, \ldots 0, c|\bar{y}'|, \bar{y}_n)$, and with the linear transformation $L\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that L^{-1} is represented by the matrix:

$$A = \begin{pmatrix} \sqrt{\bar{y}_n^2 - c^2|\bar{y}'|^2} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \sqrt{\bar{y}_n^2 - c^2|\bar{y}'|^2} & 0 & 0 \\ 0 & \cdots & 0 & \bar{y}_n & c|\bar{y}'| \\ 0 & \cdots & 0 & c|\bar{y}'| & \bar{y}_n \end{pmatrix}$$

we can move the point $\rho(h(\bar{y}))$ onto the n-th vector of the canonical basis of $\boldsymbol{R}^n$, e_n; the transformation L leaves Γ_1 fixed, and $\det(L) = (\bar{y}_n^2 - c^2|\bar{y}|^2)^{-n/2}$.

The composite transformation, $\Lambda_y = L \circ \rho \circ h \circ \sigma$, maps Γ onto Γ_1, y onto e_n and:

$$\det \Lambda_y = \frac{c^{n-1}}{(\bar{y}_n^2 - c^2|\bar{y}'|^2)^{n/2}} = C(n)^{-1} K_\Gamma(iy). \tag{2.11}$$

Moreover, by computing the matrix representing Λ_y with respect to the canonical basis of $\boldsymbol{R}^n$, we can check that:

$$\|\Lambda_y\| = \sup_{x \in \boldsymbol{R}^n\colon |x| = 1} \Lambda_y(x) = \tag{2.12}$$

$$= \frac{c}{\sqrt{\bar{y}_n^2 - c^2|\bar{y}'|^2}} = (cC(n)^{-1})^{1/n} (K_\Gamma(iy))^{1/n}.$$

By (2.3) we have:

$$\|K_{\Gamma, y}\|_{L^p(\Omega)} = C(n)^{-1+1/p} K_\Gamma(iy)^{1-1/p} \|K_{\Gamma_1, e_n}\|_{L^p(\Lambda_y(\Omega))} . \tag{2.13}$$

We can prove now the following result:

THEOREM 2.8. i) *When Γ is a circular cone $K_{\Gamma, y} \in L^p(\boldsymbol{R}^n)$ for $p > 2(1 - 1/n)$, $y \in \Gamma$ and*

$$\|K_{\Gamma, y}\|_{L^p(\boldsymbol{R}^n)} = C(n, p)\, K_\Gamma(iy)^{1-1/p} .$$

ii) *$K_{\Gamma, y} \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$ for every $p \leqslant 2(1 - 1/n)$, $y \in \Gamma$ and $\forall K \subset\subset \boldsymbol{R}^n$ we have:*

$$\|K_{\Gamma, y}\|_{L^p(K)} \leqslant C(n, p, K, c)\, K_\Gamma(iy)^{1-1/p} \log (cR^n K_\Gamma(iy)) \qquad \text{when } p = 2(1 - 1/n),$$

$$\|K_{\Gamma, y}\|_{L^p(K)} \leqslant C(n, p, K, c)\, K_\Gamma(iy)^{1-1/p} [RK_\Gamma(iy)^{1/n}]^{(n-1)/p - n/2} \qquad \text{when } p < 2(1 - 1/n).$$

Here $C(n, p, K, c)$ is a bounded constant and R is the diameter of K.

iii) *When Γ is the form (2.8), $y = te_n$, and $K = \boldsymbol{R}^{n-1} \times [-R, R]$, with $R > 0$, the equalities in* ii) *hold.*

PROOF. By (2.13), we have that i) holds if and only if $K_{\Gamma_1, e_n} \in L^p(\boldsymbol{R}^n)$; since i) holds when $p = \infty$ (see (2.7)) we can assume $p < \infty$.

We have to estimate:

$$\|K_{\Gamma_1}(\cdot + ie_n)\|_{L^p(\boldsymbol{R}^n)} = \tag{2.14}$$

$$= \left(\int_{\boldsymbol{R}^n} \frac{1}{\left(\left| \left(\sum_{i=1}^{n-1} x_i^2 - x_n^2 + 1 \right) \right| + 2|x_n| \right)^{np/2}} \, dx \right)^{1/2} =$$

(if we set $x_n = \eta$ and polar coordinates on $\boldsymbol{R}^{n-1}$)

$$= \mathrm{meas}\,(S^{n-2})^{1/p} \left(\int_{-\infty}^{+\infty} d\eta \int_0^{+\infty} \frac{r^{n-2}}{(|r^2 - \eta^2 + 1| + 2|\eta|)^{np/2}} \, dr \right)^{1/p} =$$

$$= C(n, p)\, I^{1/p} ;$$

where

(2.15) $$I = 2\int_0^{+\infty} d\eta \int_0^{+\infty} \frac{r^{n-2}\,dr}{(|r^2-\eta^2+1|+2\eta)^{np/2}}\,.$$

Assume first $n = 2k+3$; if we set:

$$R_1 = \{(r,\eta)\colon\ \eta \in [0,1],\ r \geqslant 0\},$$
$$R_2 = \{(r,\eta)\colon\ \eta \geqslant 1,\ \ r \geqslant \sqrt{\eta^2-1}\},$$
$$R_3 = \{(r,\eta)\colon\ \eta \geqslant 1,\ \ r \leqslant \sqrt{n^2-1}\},$$

we can write I as the sum of three integrals that we will estimate separately.

We recall that:

(2.16) $$2\int_{\boldsymbol{R}} \frac{r^{2k+1}\,dr}{(a \pm r^2)^{np/2}} = \sum_{m=0}^{k} (\pm 1)^{m+1} A_m (\mp a)^{k-m}(a \pm r^2)^{m+1-np/2}$$

where $a \in \boldsymbol{R}$, $A_m = \binom{k}{m}(m+1-np/2)^{-1}$ then:

$$I_1 = 2\int_{R_1} \frac{r^{n-2}}{(r^2-\eta^2+1+2\eta)^{np/2}}\,dr\,d\eta =$$

$$= 2\int_0^1 d\eta \int_0^{+\infty} \frac{r^{n-2}}{(r^2-\eta^2+1+2\eta)^{np/2}}\,dr\,d\eta =$$

(by (2.16))

$$= -\sum_{m=0}^{k}(-1)^{k-m}A_m \int_0^1 (2\eta+1-\eta^2)^{k+1-np/2}\,d\eta = C_1(n,p) < \infty$$

because the parentesis never vanishes in the interval $[0,1]$.

$$I_2 = 2\int_{R_2} \frac{r^{n-2}}{(r^2-\eta^2+1+2\eta)^{np/2}}\,dr\,d\eta =$$

$$= 2\int_1^{+\infty} d\eta \int_{\sqrt{\eta^2-1}}^{+\infty} \frac{r^{n-2}}{(r^2-\eta^2+1+2\eta)^{np/2}}\,dr =$$

(by (2.16))

$$= - \sum_{m=0}^{k} (-1)^{k-m} A_m \int_1^{+\infty} (2\eta + 1 - \eta^2)^{k-m} (2\eta)^{m+1-np/2} d\eta =$$

$$= \int_1^{+\infty} C_2(n, p)\, \eta^{2k+1-np/2} + (\ldots \text{lower degree terms})\, d\eta .$$

When $p > 2(1 - 1/n)$ we have that $I_2 = C_2(n, p) < \infty$.

The same computation can be repeated for I_3 and gives the same results; this proves the first part of the theorem when $n = 2k + 3$.

When $n = 2k$ we set in (2.14) $x_n = \eta$, $x_{n-1} = \eta_1$, and polar coordinates on $\boldsymbol{R}^{n-2}$ so to have:

$$\|K_{\Gamma_1}(\cdot + ie_n)\|_{L^p(\boldsymbol{R}^n)} = C(n, p) I^{1/p}$$

where:

$$I = 4 \int_0^{+\infty} d\eta \int_0^{+\infty} d\eta_1 \int_0^{+\infty} \frac{r^{n-3}}{(|r^2 + \eta_1^2 - \eta^2 + 1| + 2\eta)^{np/2}}\, dr .$$

If we set:

$$R_1 = \{(r, \eta, \eta_1):\ r \geqslant 0,\ \eta \geqslant 0,\ \eta_1 \geqslant 0,\ \eta_1^2 - \eta^2 + 1 \geqslant 0\},$$

$$R_2 = \left\{(r, \eta, \eta_1):\ \eta_1 \geqslant 0,\ \eta_1^2 - \eta^2 + 1 \leqslant 0,\ r \geqslant \sqrt{\eta^2 - \eta_1^2 - 1}\right\},$$

$$R_3 = \left\{(r, \eta, \eta_1):\ \eta_1 \geqslant 0,\ \eta_1^2 - \eta^2 + 1 \leqslant 0,\ 0 \leqslant r \leqslant \sqrt{\eta^2 - \eta_1^2 - 1}\right\},$$

we can write I as the sum of three integrals that can be estimate with the same technique we used before; part i) of the theorem is thus proven.

To prove iii), observe that when h is as in (2.8), the linear transformation $\Lambda_t(x) = t^{-1} h(x)$ maps the cone Γ onto the cone Γ_1 and the point te_n onto the point e_n.

Since $\Lambda_t(\boldsymbol{R}^{n-1} \times [-R, R]) = \boldsymbol{R}^{n-1} \times [-Rt^{-1}, Rt^{-1}]$ and $t^{-1} = K_\Gamma(ite_n)^{1/n} C(n)^{-1/n} c^{-1/n}$, we only have to repeat the computations we have developed in the first part, starting from (2.11) with $\boldsymbol{R}^{n-1} \times [-RK_\Gamma(ite_n)^{1/n} c^{-1+1/n} C(n)^{-1/n},\ RK_\Gamma(ite_n)^{1/n} c^{-1+1/n} C(n)^{-1/n}]$ in place of $\boldsymbol{R}^n$.

For ii) take $K \subset\subset R^n$ and take $A > 0$ such that $\Lambda_y(K) \subset \boldsymbol{R}^{n-1} \times [-A, A]$; by (2.12), we can see that we can choose $A =$

$= R(cC(n)^{-1})^{1/n}(K_\Gamma(iy))^{1/n}$, where R is the diameter of K; this concludes the proof of the theorem.

When Γ is a circular cone, from Theorem 2.8, and Lemma 2.4, follows:

COROLLARY 2.9. i) $T_{\Gamma, y} f \in L^p_{loc}(\boldsymbol{R}^n)$, $\forall y \in \Gamma$, $p \in [1, +\infty]$, *and the following inequality holds for every* $K \subset\subset \boldsymbol{R}^n$:

$$\|T_{\Gamma, y} f\|_{L^p(K)} \leq C(n, p, K, \Gamma) R^{n/2-1} K_\Gamma(iy)^{1/2-1/n} \|f\|_{L^p(\boldsymbol{R}^n)} \,. \tag{2.17}$$

Here $C(n, p, K, \Gamma)$ *is the same constant as in Theorem* 2.8 ii), *and* R *is the diameter of* K

ii) $T_\Gamma f \in L^\infty(\boldsymbol{R}^n) \forall y \in \Gamma$, $p < \dfrac{2(n-1)}{n-2}$ *and the following inequality holds*:

$$|T_\Gamma f(x+iy)| \leq C(n, p, \Gamma) \operatorname{dist}(y, \partial\Gamma)^{-n/p} \|f\|_{L^p(\boldsymbol{R}^n)} \tag{2.17'}$$

where $C(n, p, \Gamma)$ *is the same constant as in Theorem* 2.8 i).

The following theorem shows that (2.17) does not hold globally for every $f \in L^p(\boldsymbol{R}^n)$ when p is not in the interval

$$(2(1-1/n), (2(n-1))/(n-2))$$

and that (2.17′) holds for every $f \in L^p(\boldsymbol{R}^n)$ only when $p < (2(n-1))/(n-2)$.

THEOREM 2.10. i) *Let* Γ_c *be a circular cone in the form* (2.8) *and let* $2((n-1)/(n-2)) \leq p < \infty$.

Set $T(0, R) = \boldsymbol{R}^{n-1} \times [-R, R]$, *with* $R > 0$; *then there exists* $y \in \Gamma$ *such that for every* $R > 0$ *there exist a bounded constant* $C = C(n, p, c, R)$ *and a function* $\phi_R \in L^p(R(0, R))$ *such that* $\|\phi_R\|_{L^p(T(0, R))} = 1$ *and such that the following inequalities hold*:

$$\|T_{\Gamma_c, y}\phi_R\|_{L^\infty(T(0,R))} \geq C(R|y|^{-1})^{(n-1)(1-1/p)-n/2} K_{\Gamma_c}(iy)^{1/p} \qquad \textit{when } p \geq 2\,\frac{n-1}{n-2},$$

$$\|T_{\Gamma_c, y}\phi_R\|_{L^\infty(T(0,R))} \geq C \log(R|y|^{-1}) K_{\Gamma_c}(iy)^{1/p} \qquad \textit{when } p = 2\,\frac{n-1}{n-2};$$

ii) *when* $p \notin \left(2\,\dfrac{n-1}{n},\ 2\,\dfrac{n-1}{n-2}\right)$, *the ratio*:

(2.18) $$\frac{\|T_{\Gamma_c y} f\|_{L^p(T(0,R))}}{\|f\|_{L^p(\mathbf{R}^n)}} \quad \textit{where } f \in L^p(\mathbf{R}^n)\text{: } f \neq 0$$

cannot be bounded by a constant independent on R and on f for every $y \in \Gamma$.

PROOF. In what follows we will refer to q as the conjugate exponent of p, and to d as the distance of y from $\partial\Gamma$.

Since:

$$\|K_{\Gamma_c y}\|_{L^q(T(0,R))} = \sup_{\psi:\ \|\psi\|_{L^p(T(0,R))}=1} \left| \int_{\mathbf{R}^n} K_{\Gamma_c}(x+iy)\,\psi(x)\,dx \right|$$

and since $L^p(T(0,R))$ is reflexive, there exists a function ψ_R, with $\|\psi_R\|_{L^p(T(0,R))} = 1$, such that

$$\|K_{\Gamma_c, y}\|_{L^q(T(0,R))} = \left| \int_{\mathbf{R}^n} K_{\Gamma_c}(x+iy)\,\psi_R(x)\,dx \right| .$$

If we set $\phi_R(x) = \psi_R(-x)$, by (1.7) we have that:

(2.19) $$\|K_{\Gamma_c, y}\|_{L^q(T(0,R))} = |T_{\Gamma_c}\phi_R(iy)| .$$

For $p > 2((n-1)/(n-2))$ and $y = te_n$, with $t > 0$, by Theorem 2.8 iii) we have:

$$\|T_{\Gamma_c, y}\phi_R\|_{L^\infty(T(0,R))} \geqslant |T_{\Gamma_c}\phi_R(iy)| =$$

$$= CK_{\Gamma_c}(iy)^{1/p}[K_{\Gamma_c}(iy)^{1/n}R]^{(n-1)(1-1/p)-n/2},$$

where $C = C(n, p, R, c)$ is the same constant as in Theorem 2.8 ii); recalling that $K_{\Gamma_c}(ite_n) = t^{-n}C(n)c^{n-1}$, we have:

$$\|T_{\Gamma_c, y}\phi_R\|_{L^\infty(T(0,R))} > C_1 K_{\Gamma_c}(iy)^{1/p}(Rt^{-1})^{(n-1)(1-1/p)-n/2}$$

where $C_1 = C_1(n, p, R, c)$ is a bounded constant.

This proves part i) of the theorem when $p > 2((n-1)/(n-2))$; the same argumentations yields the case $p = 2((n-1)/(n-2))$; part i) of the theorem is thus proven.

In order to prove part ii), suppose that for some $p \notin \left(2\frac{n-1}{n},\ 2\frac{n-1}{n-2}\right)$, the ratio (2.18) is bounded by a constant independent on R and on f for every $y \in \Gamma$; set:

$$C(y) = \sup_{f \in L^p(\boldsymbol{R}^n),\ f \neq 0} \frac{\|T_{\Gamma_c, y} f\|_{L^p(T(0,R))}}{\|f\|_{L^p(\boldsymbol{R}^n)}} . \tag{2.20}$$

$C(y)$ is continuous and positive in the cone Γ_c, hence it is bounded on the compact subsets of Γ_c.

For $y = te_n$, with $t > 0$ and for $p > 2((n-1)/(n-2))$, we consider again (2.19); since the Cauchy transform of a distribution with respect to the cone Γ_c is holomorphic in the tube domain $\boldsymbol{R}^n + i\Gamma_c$ the mean property holds and for every $\varepsilon < 1$ we have:

$$|T_{\Gamma_c} \phi_R(iy)| = \frac{1}{\text{meas}\,(B_{2n}(iy, \varepsilon d))} \left| \int_{B_{2n}(iy,\varepsilon d)} T_{\Gamma_c} \phi_R(\zeta + i\eta)\, d\zeta\, d\eta \right| \leqslant$$

$$\leqslant \frac{(\varepsilon d)^{-2n}}{\text{meas}\,(B_{\boldsymbol{R}^{2n}}(0,1))} \int_{T(0,R) \times B_n(y,\varepsilon d)} |T_{\Gamma_c} \phi_R(\zeta + i\eta)|\, d\zeta\, d\eta \leqslant$$

$$\leqslant C(n,p)(\varepsilon d)^{n/q - 2n} \int_{B_n(y,\varepsilon d)} \|T_{\Gamma_c} \phi_R(\cdot + i\eta)\|_{L^p(T(0,R))}\, d\eta \leqslant$$

$$\leqslant C(n,p)(\varepsilon d)^{n/q - n} \sup_{\eta \in B_n(y,\varepsilon d)} \|T_{\Gamma_c} \phi_R(\cdot + i\eta)\|_{L^p(T(0,R))} .$$

Take $y_R \in B_{\boldsymbol{R}^n}(y, \varepsilon d)$ such that

$$\sup_{\eta \in B_n(y,\varepsilon d)} \|T_{\Gamma_c} \phi_R(\cdot + i\eta)\|_{L^p(T(0,R))} = \|T_{\Gamma_c, y_R} \phi_R\|_{L^p(T(0,R))}$$

we have:

$$\|T_{\Gamma_c, y_R} \phi_R\|_{L^p(T(0,R))} \geqslant C(n,p)(\varepsilon d)^{n/p} \|K_{\Gamma_c, y_R}\|_{L^q(T(0,R))}$$

and by Theorem 2.8 iii) we have:

$$\|T_{\Gamma_c, y_R} \phi_R\|_{L^p(T(0,R))} \geqslant C(\varepsilon d)^{n/p} K_{\Gamma_c}(iy)^{1/p} [K_{\Gamma_c}(iy)^{1/n} R]^{(n-1)(1-1/p) - n/2}$$

where $C = C(n, p, R, c)$ is the same constant as in Theorem 2.8 ii).

Since $d = (1 + c^2)^{-1/2} t$, we have:

$$\|T_{\Gamma_c, y_R} \phi_R\|_{L^p(T(0,R))} \geqslant C_1 \varepsilon^{n/p} (Rt^{-1})^{(n-1)(1-1/p) - n/2}$$

where $C_1 = C_1(n, p, R, c)$ is a bounded constant.

This shows that for every $R > 0$, $\varepsilon > 0$, we can find $y_R \in B_n(y, \varepsilon d)$, so that the following inequality holds:

$$C(y_R) \geqslant C_1 \varepsilon^{n/p} (Rt^{-1})^{(n-1)(1-1/p)-n/2}$$

hence $C(y)$ is not bounded on the ball $B_{R^n}(y, \varepsilon d)$.

The same argument yields also the case $p = 2((n-1)/(n-2))$; for $1 < p < 2(1-1/n)$, $y = te_n$, with $t > 0$ and $\phi \in L^q(T(0, R))$ we have:

$$\|T_{\Gamma_c}\phi(\cdot + iy)\|_{L^q(T(0,R))} = \sup_{\psi:\, \|\psi\|_{L^p(T(0,R))}=1} \left| \int_{R^n} T_{\Gamma_c}\phi(x+iy)\,\psi(x)\,dx \right|.$$

Since $L^q(T(0, R))$ is reflexive, we have:

$$\|T_{\Gamma_c}\phi(\cdot + iy)\|_{L^q(T(0,R))} = \left| \int_{R^n} T_{\Gamma_c}\phi(x+iy)\,\psi_R(x)\,dx \right|$$

for some $\psi_R \in L^p(T(0, R))$ such that $\|\psi_R\|_{L^p(T(0,R))} = 1$, and if we set $h_R(x) = \psi_R(-x)$, by definition (2.7) we have:

$$\int_{R^n} T_{\Gamma_c}\phi(x+iy)\,\psi_R(x)\,dx = \int_{R^n} T_{\Gamma_c}h_R(\zeta+iy)\,\phi(\zeta)\,d\zeta.$$

By Holder inequality we have:

$$\|T_{\Gamma_c}\phi(\cdot + iy)\|_{L^q(T(0,R))} \leqslant \|T_{\Gamma_c}h_R(\cdot iy)\|_{L^p(T(0,R))}\,\|\phi\|_{L^q(T(0,R))}\,.$$

If we set $\phi = \phi_R$ and $y = y_R$, we have:

$$\|T_{\Gamma_c}h_R(\cdot + iy_R)\|_{L^p(T(0,R))} \geqslant C(n, p, R)\,\varepsilon^{n(1-1/p)}(Rt^{-1})^{(n-1)/p-n/2}$$

from the above inequality we deduce that the function defined in (2.20) cannot be bounded on the compact sets of Γ_c.

The same argumentation yields also the case $p = 2((n-1)/n)$; the theorem is thus proven.

3. The case of non proper cones.

Let Γ be a non proper cone; this is equivalent to say that after rotation Γ can be written in the form:

$$\Gamma = G \times \boldsymbol{R}^{n-k} \tag{3.1}$$

where G is a proper and convex cone in $\boldsymbol{R}^k$.

In the following when we will write:

$$x = (x', x'')$$

we will assume $x \in V^n$, $x' \in V^k$, $x'' \in V^{n-k}$, where V is any vector space.

First of all, we want to generalize the definition of Cauchy kernel given in the first paragraph to the case of non proper cones.

Since $\Gamma^0 = G^0 \times \{0\}$, we can define the Cauchy kernel with respect to Γ as the function:

$$K_\Gamma: \boldsymbol{R}^n + i\Gamma \to \boldsymbol{C},$$

$$K_\Gamma(x' + iy', x'' + iy'') = \int_{G^0} e^{i\langle \zeta', x' + iy' \rangle} \, d\zeta'. \tag{3.2}$$

By our definition, K_Γ is holomorphic in the tube domain $\boldsymbol{R}^n + i\Gamma$, and in particular it is independent on the variables $x'' + iy''$.

For this reason we cannot define the Cauchy transform of any function $f \in \mathcal{S}(\boldsymbol{R}^n)$ with respect to Γ as in (1.7), since we want that the following properties that characterize the Cauchy transform with respect to proper cones hold also for the Cauchy transform with respect to non proper cones (see Proposition 1.2):

i) $T_\Gamma f$ is holomorphic in the domain $\boldsymbol{R}^n + i\Gamma$,

ii) $\lim\limits_{\substack{y \to 0 \\ y \in \Gamma}} T_{\Gamma, y} f(x) = f$ where the limit is taken in distribution sense,

whenever $\operatorname{supp}(\hat{f}) \subset \Gamma^0$.

When $f \in \mathcal{S}(\boldsymbol{R}^n)$ and $\operatorname{supp}(\hat{f}) \subset \Gamma^0$, by i) and ii) we have that f must be holomorphic with respect to the variables x''.

We restrict the definition of the Cauchy transform with respect to the cone Γ in the form (3.1) to the class of all functions f having the following properties:

(3.3) $f \in \boldsymbol{C}^\infty(\boldsymbol{R}^n)$ and $f(x', \cdot)$ can be extended to a holomorphic function in a neighborhood U of $\boldsymbol{R}^{n-k}$ for every $x' \in \boldsymbol{R}^k$, and $f(\cdot, x'' + iy'')$ is in the Schwartz space $\mathcal{S}(\boldsymbol{R}^k)$ for every $x'' + iy'' \in U$.

When $f \in \boldsymbol{C}^\infty(\boldsymbol{R}^n)$ is as in (3.1), we define the *Cauchy transform of f with respect to the cone Γ* as the function:

$$T_\Gamma f: \boldsymbol{R}^n + i\Gamma \to \boldsymbol{C},$$

$$T_\Gamma f(x' + iy', x'' + iy'') = \int_{G^0} \tilde{f}(\zeta', x'' + iy'') \, e^{i\langle \zeta', x' + iy' \rangle} \, d\zeta', \tag{3.4}$$

where $\tilde{f}$ is the Fourier transform of f with respect to the variables in $\boldsymbol{R}^k$.

According to this definition, ii) holds; i) holds only in a neighborhood of $\boldsymbol{R}^k$, but this is fine enough since we are interested in the growth of the Cauchy transform of f near $\boldsymbol{R}^n$.

Moreover, when f is as in (3.3), we can see that:

$$T_\Gamma f(x' + iy', x'' + iy'') = (K_{\Gamma, y'} * L(\cdot, x'' + iy''))(x'). \tag{3.5}$$

With the above notation we mean that the convolution is computed with respect to the variables in $\boldsymbol{R}^k$.

The estimates we have proven in the previous paragraph can be immediately generalized to the Cauchy transform with respect to non proper cones; the following theorem gives a generalization of the L^∞ and L^p estimates.

PROPOSITION 3.1. *When Γ is a non proper cone in the form* (3.1), *and when f is as in* (3.3), *the following estimates hold*:

(i) $\|T_{\Gamma, y'} f(\cdot, x'' + iy'')\|_{L^p(\boldsymbol{R}^k)} \leq C(\cdot, x'' + iy'')\|f(\cdot, x'' + iy'')\|_{L^p(\boldsymbol{R}^k)}$

where G is polygonal, $p \in (1, +\infty)$, $y = y(y', y'') \in \Gamma$.

(ii) $\|T_{\Gamma, y'} f(\cdot, x'' + iy'')\|_{L^\infty(\boldsymbol{R}^k)} \leq$

$$\leq C(\cdot, x'' + iy'')\, \|f(\cdot, x'' + iy'')\|_{L^p(\boldsymbol{R}^k)} \operatorname{dist}(y', \partial G)^{-k/p}$$

when $y = (y', y'') \in \Gamma$, G is polygonal and $p \in [1, +\infty)$, G is circular and $p < 2((n-1)/(n-2))$ or when G is any cone and $p \in [1, 2]$.

Here $C(\cdot, x'' + iy'') = C(k, p, G, x'' + iy'')$ is continuous with respect to $x'' + iy''$ and the norms are with respect to the variables in $\boldsymbol{R}^k$.

PROOF. Comes from the theorems proven in the second paragraph and (3.5).

REFERENCES

[CM] R. CARMICHAEL - D. MITROVIC, *Distributions and Analytic Functions*, Pitman Research Notes in Mathematics Series, Longman Scientific and Technical, Harlow (1989).

[D] L. DE CARLI, *Γ-quasi analytic solutions for non linear systems of partial differential equations*, Preprint of the Math. Dept. of the University of Pisa, 1.44 (568) (1991).

[F] C. FEFFERMAN, *The multiplier problem for the ball*, Ann. Math., (2), **94** (1971), pp. 330-336.

[H] L. HORMANDER, *The Analysis of Linear Partial Differential Operators*, Vol. 1, Springer-Verlag, Berlin, Heidelberg, New York (1983).

[Kr] S. KRANZ, *Notes on Stein's Course*, Princeton (1971).

[Mo] M. MORIMOTO, *Edge of the wedge theorem and hyperfunctions*, in: H. KOMATSU (ed.), *Hyperfunctions and Pseudo-Differential Equations. Proceedings, Katata 1971* (Lecture Notes Mathematics, Vol. 287), Springer-Verlag, Berlin, Heidelberg, New York (1973).

[Nl] Y. NELIPA, *Physique des particules elementaires*, MIR, Mosca.

[Nr] U. NERI, *Singular Integrals*, Lecture Notes Mathematics, Vol. 200, Springer-Verlag, Berlin, Heidelberg, New York (1971).

[O] G. OKIKIOLU, *Aspects of the Theory of Bounded Integral Operators in L^p Spaces*, Academic Press, London (1971).

[R] W. RUDIN, *Lectures on the Edge-of-the-Wedge Theorem*, Regional Conference Series in Mathematics, American Mathematical Society, Providence (1971).

[SV] E. M. STEIN - G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press (1971).

[V] V. S. VLADIMIROV, *Le distribuzioni nella Fisica Matematica*, MIR, Mosca (1981).

Manoscritto pervenuto in redazione il 19 marzo 1991.