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E. OSSANNA

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Comparison between the Generalized Mean Curvature according to Allard and Federer's Mean Curvature Measure.

E. OSSANNA (*)

ABSTRACT - We compare two well known generalized notions of mean curvature for the boundary of a convex body: Federer's mean curvature measure, defined via Steiner's formula, and Allard's generalized mean curvature, which is a vector measure obtained via the first variation of the area. The comparison is got by a suitable approximation lemma for convex sets.

Our purpose is to compare the generalized mean curvature (according to Allard) of a convex body K to its mean curvature measure (according to Federer). We recall that a body K of $\boldsymbol{R}^n$ is a compact subset of $\boldsymbol{R}^n$ such that $\overset{\circ}{K} \neq \phi$. Our result, which is stated in Theorem 1, follows approximating K by a suitable sequence of regular convex bodies, whose existence is assured by Lemma 2.

After this work was completed we have been informed by Joseph Fu that he has recently obtained [FU], by different methods, a similar (unpublished) result in the context of sets with a generalized unit normal bundle.

First we recall some useful facts.

Let K be a convex body of $\boldsymbol{R}^n$, let $p(K, \cdot)\colon \boldsymbol{R}^n \to K$ be the nearest-point map for K and define

$$u(K, x) = \frac{p(K, x) - x}{\|p(K, x) - x\|}.$$

If one considers the set $A_\varepsilon(K, E) = \{x \in K_\varepsilon \mid p(K, x) \in E\}$, where E is

(*) Indirizzo dell'A.: Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento), Italy.

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any Borel subset of $\boldsymbol{R}^n$ and $K_\varepsilon = \{x \in \boldsymbol{R}^n \mid d(K, x) < \varepsilon\}$, then [SCH1] there are the so called Federer's *curvature measures* of K, denoted by $\phi_m(K, \cdot)$, $m = 0, \ldots, n$ and defined on the Borel sets of $\boldsymbol{R}^n$ in such a way that

$$H^n(A_\varepsilon(K, E)) = \sum_{m=0}^{n} \varepsilon^{n-m} \alpha(n-m) \phi_m(K, E),$$

where $\alpha(k) = H^k(\{x \in \boldsymbol{R}^k \mid \|x\| \leq 1\})$. In particular we call $\phi_{n-2}(K, \cdot)$ the mean curvature measure.

One can also consider the set $M_\varepsilon(K, A) = \{x \in K_\varepsilon \setminus K \mid (p(K, x), u(K, x)) \in A\}$, where A is any Borel subset of $\boldsymbol{R}^n \times S^{n-1}$. In this case one finds that [SCH2] there are n measures, $\Theta_m(K, \cdot)$, $m = 0, \ldots, n-1$, defined on the Borel sets of $\boldsymbol{R}^n \times S^{n-1}$ and called the *generalized curvature measures* of K, such that

$$H^n(M_\varepsilon(K, A)) = \sum_{m=0}^{n-1} \varepsilon^{n-m} \alpha(n-m) \Theta_m(K, A).$$

Moreover let $\overline{H}$ be the *generalized mean curvature* of K according to Allard, defined so that

$$\int_{\partial K} \operatorname{div}_{\partial K} X \, dH^{n-1} = -(n-1) \int_{\partial K} X \cdot d\overline{H}$$

for each vector field $X \in C_0^1(U)$, where U is an open subset of $\boldsymbol{R}^n$.

We shall give also the definition of *stratification* of a measure defined on the product of two sets.

LEMMA 1 [SIM]. *Let α be a Radon measure on $\boldsymbol{R}^n \times S^{n-1}$ and consider the Radon measure σ on $\boldsymbol{R}^n$ such that $\sigma(A) = \alpha(A \times S^{n-1})$ for each Borel set $A \subset \boldsymbol{R}^n$. Then for almost all $x \in \boldsymbol{R}^n$ there is a Radon measure λ_x on S^{n-1} such that for each Borel set $B \subset S^{n-1}$ the function $\lambda_x(B)$ is the density of the measure ρ_B with respect of σ, where $\rho_B(A) = \alpha(A \times B)$. From this decomposition of the measure α one gets*

$$\int_{\boldsymbol{R}^n \times S^{n-1}} g(x, y) \, d\alpha = \int_{\boldsymbol{R}^n} \left(\int_{S^{n-1}} g(x, y) \, d\lambda_x \right) d\sigma$$

for each function $g(x, y) \in C_0^0(\boldsymbol{R}^n \times S^{n-1})$.

We call (σ, λ_x) the stratification of the measure α.

THEOREM 1. *Let K be a convex body of $\boldsymbol{R}^n$, let $\overline{H}$ be the generalized mean curvature of K and $\phi_{n-2}(K, \cdot)$ the mean curvature measure of K.*

Then for each Borel set $E \subset \mathbf{R}^n$ we have

$$\frac{n-1}{2\pi}\overline{H}(E) = -\int_{\partial K \cap E}\left(\int_{S^{n-1}} y\, d\lambda_x\right) d\phi_{n-2}(K) = -\int_{\partial K \cap E} b_{\lambda_x}\, d\phi_{n-2}(K),$$

where $(\phi_{n-2}(K), \lambda_x)$ is the stratification of the generalized curvature measure $\Theta_{n-2}(K)$ and b_{λ_x} is the barycenter of the measure λ_x.

For the proof of the theorem we need the approximation lemma below. First we recall that [HUT] the oriented varifold associated to an oriented hypersurface M of $\mathbf{R}^n$ is the Radon measure μ on $\mathbf{R}^n \times S^{n-1}$ such that for each function $g \in C_0^0(\mathbf{R}^n \times S^{n-1})$ one has

$$\int_{\mathbf{R}^n \times S^{n-1}} g(x, y)\, d\mu = \int_M g(x, \nu)\, dH^{n-1},$$

where ν is the unit normal field to the surface M.

LEMMA 2. *Let K be a convex body of $\mathbf{R}^n$, then there is a sequence of convex bodies $\{K_j\}_{j \in N}$ such that $\partial K_j = M_j$ is a smooth surface and such that the following properties hold:*

i) $K_j \xrightarrow{d_H} K$, *where d_H stays for Hausdorff distance;*

ii) *the oriented varifolds μ_j associated to the surfaces M_j converge weakly to the varifold μ associated to the surface $\partial K = M$.*

PROOF. Let K be a convex body of $\mathbf{R}^n$ and M its boundary, suppose that $B_r(0) \subset \overset{\circ}{K}$ and $K \subset B_R(0)$, for suitable $r > 0$ and $R > 0$. Consider the function $u: \mathbf{R}^n \to \mathbf{R}$ defined in the following way:

$$u(0) = 0,$$
$$u(x) = \frac{\|x\|}{\|i_M(x)\|} \qquad \text{if } x \neq 0,$$

where $i_M(x) = \{tx \mid t \geq 0\} \cap M$. Note that the function i_M is well defined and $i_M(x) \neq 0$ for every $x \in \mathbf{R}^n \setminus \{0\}$.

In this case M consists of the points $x \in \mathbf{R}^n$ such that $u(x) = 1$.

Clearly the function u is convex and lipschitz continuous, moreover the gradient $\nabla u(x)$ exists for almost all x, and where ∇u is defined we have $1/R \leq \|\nabla u(x)\| \leq 1/r$.

Now let $\{\eta_j\}_{j \in N}$ be a sequence of mollifiers and define the functions $u_j = u * \eta_j$. This way we get a sequence of smooth convex functions such that:

(*a*) $\{u_j\}_{j \in N}$ converges uniformly to the function u;

(*b*) for every $x \in \boldsymbol{R}^n$ we have $1/R \leq \|\nabla u_j(x)\| \leq 1/r$.

Now consider the sets $K_j = \{x \in \boldsymbol{R}^n \mid u_j(x) \leq 1\}$ $j \in \boldsymbol{N}$. One can see easily that the sets K_j are convex and uniformly bounded, hence they are convex bodies.

The statement i) of the lemma follows from (*a*) after some calculations. Moreover observe that, being K_j convex bodies, convergence i) implies

$$K_j \xrightarrow{L^1} K. \tag{1}$$

From (*b*) we get that each set $M_j = \{x \in \boldsymbol{R}^n \mid u_j(x) = 1\}$, boundary of the convex body K_j, is a smooth hypersurface of $\boldsymbol{R}^n$.

In what follows we denote by ν the unit outward normal to M (ν is defined almost everywhere) and by ν_j the unit outward normal to M_j.

Now we shall prove statement ii) of the lemma, that is $\mu_j \rightharpoonup \mu$.

As a *first step* we consider the vector measures

$$\beta_j = \nu_j(x)(H^{n-1} \llcorner M_j) \quad j \in \boldsymbol{N}, \qquad \beta = \nu(x)(H^{n-1} \llcorner M)$$

and their total variation measures

$$|\beta_j| = H^{n-1} \llcorner M_j, \qquad |\beta| = H^{n-1} \llcorner M$$

and we shall prove that

$$\beta_j \rightharpoonup \beta, \tag{2}$$

$$|\beta_j| \rightharpoonup |\beta|. \tag{3}$$

To obtain $\beta_j \rightharpoonup \beta$ we use the divergence theorem and the convergence (1), getting for each vector field $X \in C_0^1(\boldsymbol{R}^n)$

$$\lim_{j \to +\infty} \int_{\boldsymbol{R}^n} X \cdot d\beta_j = \int_{\boldsymbol{R}^n} X \cdot d\beta, \tag{4}$$

and we extend (4) to vector fields belonging to C_0^0, using the density of C_0^1 in C_0^0 and the uniform boundness of the convex bodies K_j.

Next we show that $|\beta_j| \rightharpoonup |\beta|$. By convergence (2) we get

$$\liminf_{j \to +\infty} |\beta_j|(A) \geq |\beta|(A) \quad \forall \text{ open } A \subset \boldsymbol{R}^n, \tag{5}$$

and on the other hand by i) we have

(6) $$|\beta_j|(\boldsymbol{R}^n) \to |\beta|(\boldsymbol{R}^n),$$

in fact, i) implies that for every $\varepsilon > 0$ there are $r > 0$ and $J \in \boldsymbol{N}$ such that, if we set $\delta = 1 + \varepsilon/r$, it results $\delta^{-1} K \subset K_j \subset \delta K$ for all $j > J$, from which, again for $j > J$, $H^{n-1}(\partial K) \leqslant H^{n-1}(\partial K_j) \leqslant H^{n-1}(\partial K)$ follows. Thus, being $|\beta_j|$ and $|\beta|$ finite measures, (3) follows from (5) and (6) [HAL].

The *second step* consists in proving that, if there is a subsequence $\{\mu_{j_h}\}_{h \in \boldsymbol{N}}$ of the sequence of Radon measures $\{\mu_j\}_{j \in \boldsymbol{N}}$ and a Radon measure α such that

(7) $$\mu_{j_h} \rightharpoonup \alpha,$$

then $\alpha = \mu$. For this we consider the stratification (σ, λ_x) of the measure α and a function $g \in C_0^0(\boldsymbol{R}^n)$, then we have

$$\int_{\boldsymbol{R}^n \times S^{n-1}} g(x)\, d\alpha = \int_{\boldsymbol{R}^n} g(x) \left(\int_{S^{n-1}} d\lambda_x \right) d\sigma = \int_{\boldsymbol{R}^n} g(x)\, d\sigma,$$

and from (3) and (7) we get

$$\int_{\boldsymbol{R}^n \times S^{n-1}} g(x)\, d\alpha = \int_{\boldsymbol{R}^n} g(x)\, d|\beta|,$$

which implies

(8) $$\sigma = |\beta|.$$

Moreover since $\beta_j \rightharpoonup \beta$ and $\mu_{j_h} \rightharpoonup \alpha$, between the two measures β and α there is the relationship [RES] $\beta = b(x)\sigma$, where $b(x) = \int_{S^{n-1}} y\, d\lambda_x$. But $\beta = \nu(x)(H^{n-1} \llcorner M)$ and $\sigma = |\beta|$, hence $\nu(x)|\beta| = b(x)|\beta|$, which implies $\int_{S^{n-1}} y\, d\lambda_x = \nu(x)$ for $|\beta|$-almost all $x \in \boldsymbol{R}^n$, that is

(9) $$\lambda_x = \delta_{\nu(x)},$$

$|\beta|$-almost everywhere. Now, using (8) and (9), we can prove $\alpha = \mu$, in fact for each function $g \in C_0^0(\boldsymbol{R}^n \times S^{n-1})$ we have

$$\int_{\boldsymbol{R}^n \times S^{n-1}} g(x, y)\, d\alpha = \int_{\boldsymbol{R}^n} \left(\int_{S^{n-1}} g(x, y)\, d\delta_{\nu(x)} \right) d|\beta| =$$

$$= \int_{\boldsymbol{R}^n} g(x, \nu(x))\, d|\beta| = \int_{\boldsymbol{R}^n \times S^{n-1}} g(x, y)\, d\mu.$$

Now we end the proof of statement ii) of the lemma. Assume by contradiction there is a function $g \in C_0^0(\boldsymbol{R}^n \times S^{n-1})$ such that

$\int_{\boldsymbol{R}^n \times S^{n-1}} g(x, y)\, d\mu_j$ does not converge to $\int_{\boldsymbol{R}^n \times S^{n-1}} g(x, y)\, d\mu$. Then there are a number ε and a subsequence $\{\mu_{j_k}\}_{k \in \boldsymbol{N}}$ such that

$$\left| \int_{\boldsymbol{R}^n \times S^{n-1}} g(x, y)\, d\mu_{j_k} - \int_{\boldsymbol{R}^n \times S^{n-1}} g(x, y)\, d\mu \right| > \varepsilon \quad \forall k. \tag{10}$$

On the other hand one has $\mu_{j_k}(\boldsymbol{R}^n \times S^{n-1}) \leq C\ \forall k$. Hence there is a further subsequence $\{\mu_{j_{k_h}}\}_{h \in \boldsymbol{N}}$ that converges weakly to some measure, which, by the second step, must be μ, and this contradicts (10). ■

PROOF OF THEOREM 2. Let $\{K_j\}_{j \in \boldsymbol{N}}$ be the sequence introduced in the Lemma 2. From the weak convergence of the varifolds μ_j we have

$$\lim_{j \to +\infty} \int_{\partial K_j} \operatorname{div}_{\partial K_j} X\, dH^{n-1} = \int_{\partial K} \operatorname{div}_{\partial K} X\, dH^{n-1},$$

for each vector field $X \in C_0^1(\boldsymbol{R}^n)$.

Hence, for the definition of generalized mean curvature, we have also

$$\overline{H}_j(x) H^{n-1} \rightharpoonup \overline{H}, \tag{11}$$

where $\overline{H}_j$ is the mean curvature vector of the surfaces ∂K_j.

Now if we let E be a Borel set of $\boldsymbol{R}^n$ such that $\overline{H}(\partial E) = 0$ and $\phi_{n-2}(K, \partial E) = 0$, we have [SCH2]

$$\Theta_{n-2}(K_j) \llcorner E \times S^{n-1} \rightharpoonup \Theta_{n-2}(K) \llcorner E \times S^{n-1},$$

which implies in particular

$$\lim_{j \to +\infty} \int_{\partial K_j \times S^{n-1}} \mathcal{X}_{(E \times S^{n-1})}\, y\, d\Theta_{n-2}(K_j) = \int_{\partial K \times S^{n-1}} \mathcal{X}_{(E \times S^{n-1})}\, y\, d\Theta_{n-2}(K) =$$

$$= \int_{\partial K \cap E} \int_{S^{n-1}} y\, d\lambda_x\, d\phi_{n-2} = \int_{\partial K \cap E} b_{\lambda_x}\, d\phi_{n-2}(K).$$

But ∂K_j is a regular surface, thus we have

$$\int_{\partial K_j \times S^{n-1}} \mathcal{X}_{(E \times S^{n-1})}\, y\, d\Theta_{n-2}(K_j) = \frac{n-1}{2\pi} \int_{\partial K_j \cap E} \nu_j \|\overline{H}_j\|\, dH^{n-1} =$$

$$= -\frac{n-1}{2\pi} \int_{\partial K_j \cap E} \overline{H}_j\, dH^{n-1},$$

moreover, being $\overline{H}(\partial E) = 0$, convergence (11) implies

$$\lim_{j \to +\infty} \int_{\partial K_j \cap E} \overline{H}_j\, dH^{n-1} = \overline{H}(E).$$

Hence, for each Borel set $E \subset \boldsymbol{R}^n$ such that $\overline{H}(\partial E) = 0$ and $\phi_{n-2}(K, \partial E) = 0$, we get

$$\frac{n-1}{2\pi}\overline{H}(E) = -\int_{\partial K \cap E} \left(\int_{S^{n-1}} y\, d\lambda_x \right) d\phi_{n-2}(K) = -\int_{\partial K \cap E} b_{\lambda_x}\, d\phi_{n-2}(K)$$

and this equality holds also for every Borel set $A \subset \boldsymbol{R}^n$. ■

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