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On $*$-Modules Generating the Injectives.

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ABSTRACT - Relations between $*$-modules, quasi-progenerators and other generalizations of progenerators are studied. The $*$-modules generating all injective modules are shown to be finitely generated.

Introduction.

One of the examples showing that category theory is not only a language, but also a useful tool in algebra is the celebrated Morita theorem concerning equivalence of rings. For any ring R, it implies e.g. the important fact that R and the full matrix ring $M_n(R)$ share all the ring theoretic properties which are definable by means of categorical properties of modules.

More specifically, if R and S are rings, the Morita theorem ([AF, Corollary 22.4]) says that mod-R and mod-S are equivalent categories iff there exists a *progenerator* (= a finitely generated projective generator) P such that $S \simeq \mathrm{End}(P_R)$.

In [F], Fuller generalized the theorem as follows: mod-S is equivalent to a full subcategory C of mod-R such that C is closed with respect to submodules, direct sums and quotients iff there exists a *quasi-progenerator* (= a finitely generated quasi-projective module generating all its submodules) P such that $S \simeq \mathrm{End}(P_R)$ and $C = \mathrm{Gen}(P_R)$.

In both cases, the pair (F, G) of functors realizing the equivalence is *represented* by P, i.e. F and G are naturally equivalent to $-\otimes_S P$ and $\mathrm{Hom}_R(P, -)$, respectively.

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In [MO, Theorem 3.1], Menini and Orsatti presented a further generalization: if B and C are equivalent categories, where $B \subseteq$ mod-S is such that $S \in B$ and B is closed with respect to submodules, and $C \subseteq$ mod-R is closed with respect to direct sums and factors, then there is a module P such that $S \simeq \mathrm{End}(P_R)$, $C = \mathrm{Gen}(P_R)$ and $B = \mathrm{Cog}(K_S)$, where $K = \mathrm{Hom}_R(P, Q)$ and Q is an injective cogenerator of mod-R. Moreover, the equivalence is represented by P.

In [MO, 3.5], a question was raised of characterizing the modules P that induce an equivalence between $\mathrm{Gen}(P_R)$ and $\mathrm{Cog}((\mathrm{Hom}_R(P, Q))_S)$ with $S \simeq \mathrm{End}(P_R)$. Since the question was denoted by ($*$), such modules are called *$*$-modules* ([C], [DH]).

For a ring R, denote by *PG*, *QPG* and *STAR* the class of all progenerators, quasi-progenerators and $*$-modules, respectively. Clearly, $PG \subseteq QPG \subseteq STAR$. Surprisingly, there is another important class of $*$-modules, not connected with quasi-progenerators: a module P is a *W-tilting* module if P is finitely presented, proj dim $(P) \leq 1$, $\mathrm{Ext}_R(P, P) = 0$, and there is an exact sequence $0 \to R \to P' \to P'' \to 0$ such that P' and P'' are direct sums of summands of P. For example, the W-tilting modules over finite dimensional algebras coincide with the tilting modules, introduced in [HR]. Denote by *WTILT* the class of all W-tilting modules and by *ISTAR* the class of all $*$-modules such that $I(R) \in \mathrm{Gen}(P_R)$, $I(R)$ being the injective hull of R. The surprise is that $WTILT \subseteq ISTAR$ ([MO, Theorem 4.3]).

The class *ISTAR* was studied in more detail by Colpi and Menini in [C] and [CM]. By [CM, Proposition 1.5], if $P \in ISTAR$ then $\mathrm{Gen}(P_R) = \{M \mid \mathrm{Ext}_R(P, M) = 0\}$. Moreover, [CM, Proposition 1.5] implies that *ISTAR* coincides with the class of all $*$-modules P such that $\mathrm{Gen}(P_R) \supseteq \mathfrak{J}$, where $\mathfrak{J}$ is the class of all injective modules. In [CM, Theorem 3.3], a complete characterization of the rings R for which there is a $*$-module P with $\mathrm{Gen}(P_R) = \mathfrak{J}$ was given.

The main result of our paper is Theorem 1.3 showing that *ISTAR* is very close to the class *WTILT*. In particular, all elements of *ISTAR* are finitely presented. Thus, for this case, we confirm the conjecture of Colpi and Menini (cp. [CM, Propositions 1.8 and 1.9]), D'Este and Happel ([DH, Remark 4]), and Zanardo ([Z, Remark 4]), which claims that every $*$-module is finitely generated. In Propositions 1.6 and 1.7, the structure if *ISTAR* over semiperfect rings is described in greater detail.

Then we turn to applications to particular classes of rings. We show that $ISTAR = PG$ provided R is either a commutative or a local or a von Neumann regular ring (Theorem 1.9). Finally, Theorem 1.10 shows that properties concerning $P^{\perp}$ which are slightly weaker than the ones induced by $*$-modules, can be independent of *ZFC*.

In the following, all rings are associative with unit. Let R be a ring. The category of (unitary right R-) modules is denoted by mod-R. Homomorphisms in mod-R are written as acting on the left. If $M \in$ mod-R, then $\mathrm{Rad}(M)$ denotes the Jacobson radical of M. Further, R is *completely reducible* provided R is a finite ring direct sum of full matrix rings over skew fields. R is *semiperfect* provided $R/\mathrm{Rad}(R)$ is completely reducible and idempotents lift modulo $\mathrm{Rad}(R)$.

Let M be a module. Then $\mathrm{gen}(M)$ denotes the minimal cardinality of an R-generating subset of M and $I(M)$ the injective hull of M. The category of all modules generated by P is denoted by $\mathrm{Gen}(P_R)$, and $\overline{\mathrm{Gen}}(P_R)$ is the category consisting of all submodules of elements of $\mathrm{Gen}(P_R)$. M is said to be *small* provided for every sequence of modules $(N_\alpha \mid \alpha \in A)$ and every homomorphism $h \in \mathrm{Hom}_R(M, \oplus \sum_{\alpha \in A} N_\alpha)$ there is a finite set $F \subseteq A$ such that $\mathrm{Im}(h) \subseteq \oplus \sum_{\alpha \in F} N_\alpha$. The module M is *finitely presented* provided there is an exact sequence $0 \to G \to F \to M \to 0$ in mod-R such that F is projective, and F and G are finitely generated. Further, $\mathrm{proj\,dim}(M)$ denotes the projective dimension of M, and $M^\perp$ the cotorsion class generated by M, i.e. $M^\perp = \{N \in \text{mod-}R \mid \mathrm{Ext}_R(M, N) = 0\}$ (see [S] or [T, § 1]). For further concepts and notation, the reader is referred to [AF] and [EM].

1. The structure of *ISTAR*.

LEMMA 1.1. *Let R be a ring and P a small module. Then either P is finitely generated or* $\mathrm{gen}(P) \geq \aleph_1$.

PROOF. An easy modification of the proof of [CM, Proposition 1.9].

LEMMA 1.2. *Let R be a ring and P a module. Then the following conditions are equivalent:*

(1) *P is small and $P^\perp$ is closed with respect to direct sums and factors,*

(2) *P is finitely presented and* $\mathrm{proj\,dim}(P) \leq 1$.

PROOF. Assume (1). Clearly, $P = R^{(\kappa)}/Q$ for a cardinal κ and a submodule Q of $R^{(\kappa)}$. First we observe that Q is projective. Take an arbitrary $N \in$ mod-R. Since the sequence $0 \to Q \to R^{(\kappa)} \to P \to 0$ is exact and $0 = \mathrm{Ext}_R(R^{(\kappa)}, N) = \mathrm{Ext}^2_R(R^{(\kappa)}, N)$, the abelian groups $\mathrm{Ext}_R(Q, N)$ and $\mathrm{Ext}^2_R(P, N)$ are isomorphic. Since the sequence $0 \to N \to I(N) \to I(N)/N \to 0$ is exact and $0 = \mathrm{Ext}_R(P, I(N)) = \mathrm{Ext}^2_R(P, I(N))$, the abelian groups $\mathrm{Ext}_R(P, I(N)/N)$ and $\mathrm{Ext}^2_R(P, N)$ are isomorphic. Now,

$I(N) \in P^\perp$ and $P^\perp$ is closed with respect to factors, and $\mathrm{Ext}_R(Q, N) \simeq \simeq \mathrm{Ext}_R(P, I(N)/N) = 0$, whence Q is projective. Thus, proj dim $(P) \leqslant 1$. By [AF, Corollary 26.2], the projective module Q is a direct sum of countably generated modules, $Q = \oplus \sum_{\alpha<\lambda} Q_\alpha$. Put $D = \oplus \sum_{\alpha<\lambda} I(Q_\alpha)$. Since $P^\perp$ is closed with respect to direct sums, we have $\mathrm{Ext}_R(P, D) = 0$. In particular, the inclusion $i \in \mathrm{Hom}_R(Q, D)$ has a prolongation $g \in \in \mathrm{Hom}_R(R^{(\kappa)}, D), g|Q = i$. For $\alpha < \lambda$, denote by π_α and by ρ_α the α-th projection of D onto $I(Q_\alpha)$ and of $I(Q_\alpha)$ onto $I(Q_\alpha)/Q_\alpha$, respectively. For $\alpha < \lambda$, put $g_\alpha = \rho_\alpha \pi_\alpha g$.

If $h = \oplus \sum_{\alpha<\lambda} g_\alpha \in \mathrm{Hom}_R(R^{(\kappa)}, \oplus \sum_{\alpha<\lambda} I(Q_\alpha)/Q_\alpha)$ then $Q \subseteq \mathrm{Ker}(h)$ and h induces a homomorphism $\bar{h} \in \mathrm{Hom}_R(P, \oplus \sum_{\alpha<\lambda} I(Q_\alpha)/Q_\alpha)$. Since P is small, there is a finite subset $F \subseteq \lambda$ such that $\mathrm{Im}(\bar{h}) \subseteq \oplus \sum_{\alpha\in F} I(Q_\alpha)/Q_\alpha$. Thus $\mathrm{Im}(g) \subseteq \oplus \sum_{\alpha \in F} I(Q_\alpha) + \oplus \sum_{\alpha<\lambda} Q_\alpha$. Denote by π the projection od D onto $\oplus \sum_{\alpha<\lambda, \alpha\notin F} I(Q_\alpha)$. Put $\bar{g} = \pi g$. Then $\bar{g} \in \mathrm{Hom}_R(R^{(\kappa)}, \bar{Q})$, where $\bar{Q} = = \oplus \sum_{\alpha<\lambda, \alpha\notin F} Q_\alpha$. Since $\bar{g}|\bar{Q} = \mathrm{id}$, we have $R^{(\kappa)} = \mathrm{Ker}(\bar{g}) \oplus \bar{Q}$. Put $A = = \mathrm{Ker}(\bar{g}) \cap Q = \oplus \sum_{\alpha\in F} Q_\alpha$. Then $P = R^{(\kappa)}/Q = (\mathrm{Ker}(\bar{g}) + Q)/Q \simeq \simeq \mathrm{Ker}(\bar{g})/A$. Since $\mathrm{Ker}(\bar{g})$ is projective, [AF, Corollary 26.2], implies it is a direct sum of countably generated projective modules. Since A is countably generated, we infer that P is a direct sum of a countably genrated module C and a projective module B. Since P is small, B is countably generated. Hence, P is a countably generated small module, and 1.1 implies P is finitely generated.

Now, if P is finitely generated and proj dim $(P) \leqslant 1$, there is an exact sequence $0 \to L \to R^{(n)} \to P \to 0$ with L projective, i.e. L a summand of some $R^{(X)}$. Since $P^\perp$ is closed with respect to direct sums, we have $I(R)^{(X)} \in P^\perp$ and the same argument as in the second part of the proof of [CM, Proposition 1.7] shows that L is finitely generated. Hence, P is finitely presented.

Assume (2). Clearly, P is a small module. Since proj dim $(P) \leqslant 1$, $P^\perp$ is closed with respect to factors. Moreover, $P = X/Y$, where X is a projective module and Y is a finitely generated module. Hence every homomorphism of Y into a direct sum of modules actually maps into a finite direct sub-sum. Therefore, as $P^\perp$ is closed with respect to finite direct sums, it is closed with respect to the arbitrary ones.

THEOREM 1.3. *Let R be a ring and P a module.*

(i) *If $P \in ISTAR$, then P is finitely presented,* proj dim $(P) \leqslant 1$,

$\mathrm{Ext}_R(P, P) = 0$, *and there is an exact sequence* $0 \to R \to P' \to P'' \to 0$ *such that* P' *is a finite direct power of* P.

(ii) $P \in ISTAR$ *iff* P *is finitely generated and* $\mathrm{Gen}(P_R) = P^{\perp}$.

PROOF. (i) By [CM, Propositions 1.5 and 1.8], P is a small module with $\mathrm{Gen}(P_R) = P^{\perp}$. In particular, $\mathrm{Ext}_R(P, P) = 0$. By 1.2, P is finitely presented and $\mathrm{proj\,dim}(P) \leq 1$. Finally, [CM, Proposition 1.5] implies R embeds into a finite direct power of P.

(ii) By 1.2 and [CM, Proposition 1.5].

By 1.3(i), the classes *ISTAR* and *WTILT* are quite close to each other. Moreover,

PROPOSITION 1.4. *Let* R *be a finite dimensional algebra over a field. Then* $ISTAR = WTILT$.

PROOF. By [MO, Theorem 4.3], $WTILT \subseteq ISTAR$. On the other hand, every $P \in ISTAR$ is finitely generated by 1.3(ii) and it is faithful by [CM, Proposition 1.5]. Thus [DH, Theorem 1] implies $P \in WTILT$.

Now, 1.4 and [CM, Theorem 3.3] suggest the following.

PROBLEM 1.5. *Characterize the rings* R *such that* $WTILT = ISTAR$.

PROPOSITION 1.6. *Let* R *be a semiperfect ring and* B *a basic set of idempotents of* R. *Let* $P \in ISTAR$. *Then there exist a non-empty subset* C *of* B, *a positive integer* n, *and, for each* $i < n$, *modules* F_i *and* G_i *such that*

(1) F_i *is a non-zero direct sum of direct powers of the modules* eR, $e \in C$,

(2) G_i *is a superfluous submodule of* F_i,

(3) G_i *is isomorphic to a direct sum of direct powers of the modules* eR, $e \in B \setminus C$,

(4) *for every* $e \in C$ *and* $e \in B \setminus C$, *the module* eR *appears as a summand of* F_i *and* G_i, *respectively, for some* $i < n$,

(5) *the module* F_i/G_i *is indecomposable,*

(6) $P \simeq \bigoplus \sum_{i<n} F_i/G_i$.

PROOF. By ([AF, Corollary 15.8]), we have $\mathrm{Rad}(P) = P.\ \mathrm{Rad}(R)$ and, by 1.3(ii), $P/\mathrm{Rad}(P)$ is a non-zero finitely generated completely reducible module. Hence P is a direct sum of indecomposable modules, $P = \oplus \sum_{i<n} P_i$, for a positive integer n. Of course, each P_i has a projective cover, and $\mathrm{proj\,dim}(P_i) \leq 1$ by 1.3(i). By [AF, Theorem 27.11], there exist modules F_i and G_i such that $P_i \simeq F_i/G_i$, where F_i is a non-zero direct sum of non-zero direct powers of the modules eR, $e \in C_i$, for some $C_i \subseteq B$, G_i is a superfluous submodule of F_i, and either $G_i = 0$ and $D_i = \emptyset$, or G_i is isomorphic to a non-zero direct sum of nonzero direct powers of the modules eR, $e \in D_i$, for some $D_i \subseteq B$. Put $C = \bigcup_{i<n} C_i$ and $D = \bigcup_{i<n} D_i$. It remains to prove that $C \cap D = \emptyset$ and $C \cup D = B$. Assume $e \in C_i \cap D_j$. By [AF, Proposition 27.10], there is maximal submodule H of G_j such that $G_j/H \simeq eR/\mathrm{Rad}(eR)$ and G_j/H is isomorphic to a summand of the completely reducible module $F_i/\mathrm{Rad}(F_i)$.

Let $\phi \in \mathrm{Hom}_R(G_j, F_i/\mathrm{Rad}(F_i))$ be the composition of these isomorphisms and of the projection of G_j onto G_j/H. Assume there is some $\varphi \in \mathrm{Hom}_R(F_j, F_i/\mathrm{Rad}(F_i))$ such that $\phi = \varphi\nu$, ν being the inclusion of G_j into F_j. Then $\mathrm{Ker}(\varphi)$ is a maximal submodule of F_j, whence $G_j \subseteq \mathrm{Rad}(F_j) \subseteq \mathrm{Ker}(\varphi)$ and $\varphi\nu = 0$, a contradiction. Therefore, $\mathrm{Ext}_R(P_j, F_i/\mathrm{Rad}(F_i)) \neq 0$. But $F_i/\mathrm{Rad}(F_i)$ is a factor-module of P_i and $\mathrm{proj\,dim}(P_i) \leq 1$, whence $\mathrm{Ext}_R(P_i, P_j) \neq 0$, a contradiction with [CM, Proposition 1.5].

Assume there is some $e \in B \setminus (C \cup D)$. Then [AF, Proposition 27.10] implies $\mathrm{Hom}_R(\oplus \sum_{i<n} G_i, M) = 0$, where $M = eR/\mathrm{Rad}(eR)$ is a simple module. Hence $\mathrm{Ext}_R(P, M) = 0$. By [CM, Proposition 1.5], $M \in \mathrm{Gen}(P_R) \subseteq \mathrm{Gen}((\oplus \sum_{i<n} F_i)_R)$, in contradiction with [AF, 27.13].

PROPOSITION 1.7. *Let R be a semiperfect ring and $P \in ISTAR$.*

(i) *Put $G = \oplus \sum_{i<n} G_i$ and $F = \oplus \sum_{i<n} F_i$ (see* 1.6 *for the notation). Consider the following two conditions:*

(1) $N \in \mathrm{Gen}(P_R)$,

(2) *The completely reducible modules $N/\mathrm{Rad}(N)$ and $G/\mathrm{Rad}(G)$ have no isomorphic direct summands.*

Then (1) *implies* (2) *for any $N \in$ mod-R. If N is completely reducible, then* (1) *is equivalent to* (2). *Moreover,* (1) *is equivalent to* (2) *for all finitely generated modules N iff every homomorphism of G into* $\mathrm{Rad}(F)$ *can be prolonged into an endomorphism of F iff* $\mathrm{Gen}(P_R) = \mathrm{Gen}(F_R)$.

(ii) *$P \in PG$ iff* $\mathrm{Gen}(P_R)$ *contains all simple modules.*

PROOF. (i) Assume (1). Then $\mathrm{Ext}_R(P, N) = 0$, by [CM, Proposition 1.5]. Suppose (2) does not hold. Then there exist a homomorphism $\xi \in \in \mathrm{Hom}_R(G/\mathrm{Rad}(G), N/\mathrm{Rad}(N))$ such that $\mathrm{Im}(\xi)$ is a simple module. Put $\phi = \xi\pi$, where $\pi: G \to G/\mathrm{Rad}(G)$ is the projection. Then $\phi \in \in \mathrm{Hom}_R(G, N/\mathrm{Rad}(N))$ and by 1.3 (ii), there is $\varphi \in \mathrm{Hom}_R(F, N/\mathrm{Rad}(N))$ such that $\phi = \varphi\nu$, ν being the inclusion of G into F. In particular, $\mathrm{Ker}(\varphi)$ is a maximal submodule of F, $G \subseteq \mathrm{Rad}(F) \subseteq \mathrm{Ker}(\varphi)$, a contradiction.

If N is completely reducible and (2) holds, then $\mathrm{Hom}_R(G/\mathrm{Rad}(G), N) = 0$. Hence $\mathrm{Hom}_R(G, N) = 0$, $\mathrm{Ext}_R(P, N) = 0$, and [CM, Proposition 1.5] implies $N \in \mathrm{Gen}(P_R)$.

Assume (2) implies (1) for all finitely generated modules N. By 1.6, $\mathrm{Hom}_R(F/\mathrm{Rad}(F), G/\mathrm{Rad}(G)) = 0$. For $N = F$, we get $\mathrm{Gen}(P_R) = = \mathrm{Gen}(F_R)$.

Assume $\mathrm{Gen}(P_R) = \mathrm{Gen}(F_R)$. Then [CM, Proposition 1.5] implies $\mathrm{Ext}_R(P, F) = 0$. Thus, even every homomorphism of G into F has the desired prolongation.

Assume the prolongations exist and let N be a finitely generated module satisfying (2). By [AF, Theorem 27.6], there are a finitely generated projective module A and a superfluous submodule B of A such that $N = A/B$. In particular, $\mathrm{Rad}(N) = \mathrm{Rad}(A)/B$. By (2), 1.6 and [AF, Theorem 27.11], there exist positive integers p and q such that $A^{(p)}$ is a summand of $F^{(q)}$. Let $\phi \in \mathrm{Hom}_R(G, N^{(p)})$. By (2), $\rho\phi = 0$, where $\rho: N^{(p)} \to N^{(p)}/\mathrm{Rad}(N^{(p)})$ is the projection. Hence, $\mathrm{Im}(\phi) \subseteq \subseteq \mathrm{Rad}(N^{(p)}) = \mathrm{Rad}(A^{(p)})/B^{(p)}$. Since G is projective, there exists $\theta \in \in \mathrm{Hom}_R(G, \mathrm{Rad}(A^{(p)}))$ such that $\sigma\theta = \phi$, where σ is the projection of $A^{(p)}$ onto $A^{(p)}/B^{(p)}$. Using the premise, it is easy to see that θ has a prolongation into a $\varphi \in \mathrm{Hom}_R(F, A^{(p)})$. Thus, $\sigma\varphi\nu = \sigma\theta = \phi$, where ν is the inclusion of G into F. This implies $\mathrm{Ext}_R(P, N^{(p)}) = 0$, and [CM, Proposition 1.5] gives (1).

(ii) If $\mathrm{Gen}(P_R)$ contains all simple modules, then (i) implies $G = 0$ and $P \in PG$.

Clearly, for any ring R, we have $PG \subseteq QPG \subseteq STAR$ and $PG \subseteq \subseteq ISTAR \subseteq STAR$. Moreover, [C, Proposition 4.5 and Theorem 4.7] imply $QPG \cap ISTAR = PG$.

PROPOSITION 1.8. *$QPG = PG$ iff R is a simple completely reducible ring.*

PROOF. Assume $QPG = PG$. Denote by $\mathcal{S}$ the class of all simple modules. Clearly $\mathcal{S} \subseteq QPG$, whence every simple module is projective and R is completely reducible. Moreover, since every element of $\mathcal{S}$ is a generator, R is simple. The opposite implication is obvious.

We turn to applications to particular classes of rings:

THEOREM 1.9. *ISTAR = PG provided one of the following conditions is true:*

(i) *R is a commutative ring,*

(ii) *R is a local ring,*

(iii) *R is a von Neumann regular ring.*

PROOF. (i) Let $P \in ISTAR$. By [CM, Proposition 1.5], P is a faithful $*$-module. By 1.3(ii), P is finitely generated and [CM, Theorem 2.3] shows that $P \in PG$.

(ii) By 1.6, since $\operatorname{card}(B) = 1$.

(iii) By 1.3(i), since every finitely presented module is projective.

In view of 1.3 and 1.9(iii), it is surprising that even for von Neumann regular hereditary rings, the question whether the class $M^{\perp}$ is closed with respect to countable direct powers for a non-projective module M, can be quite difficult to answer.

THEOREM 1.10. *Let R be a simple right hereditary non-completely reducible von Neumann regular ring with* $\operatorname{card}(R) \leq \aleph_1$ *(e.g. R can be any simple countable non-completely reducible von Neumann regular ring). Then, for every module M, the class $M^{\perp}$ is closed with respect to factors, but the assertion*

«$N^{(\aleph_0)} \notin M^{\perp}$ whenever $M, N \in$ mod-R are such that M is non-projective and $0 \neq N \in M^{\perp}$»

is independent of ZFC + GCH.

PROOF. Since R is right hereditary, $M^{\perp}$ is closed with respect to factors for any module M.

Since R is not right perfect, [ES, Corollary 2.2] implies that it is consistent with $ZFC + GCH$ that for every uncountable cardinal κ such that $\operatorname{card}(R) \leq \kappa$ and $cf(\kappa) = \aleph_0$ there is a non-projective module M such that $\operatorname{card}(M) = \kappa^{+}$ and $\operatorname{Ext}_R(M, N) = 0$ for all modules N with $\operatorname{card}(N) < \kappa$. In particular, the negation of our assertion is consistent. On the other hand, assume the axiom of constructibility ($V = L$). We prove the assertion by induction on $\operatorname{gen}(M) = \lambda$.

If $\lambda < \aleph_0$, then $M \simeq R^{(\lambda)}/I$ for an infinitely generated module I. Since I is projective, there exists a countable R-independent set $\{x_n \mid n < \aleph_0\}$ generating a summand of I. Let e_n be the non-zero idempotent of R such that $\operatorname{Ann}(x_n) = (1 - e_n)R$. Since R is simple, there is some $0 \neq y_n \in Ne_n$ for all $n < \aleph_0$. Define $h \in \operatorname{Hom}_R(I, N^{(\aleph_0)})$ by $hx_n = \pi_n y_n$, where π_n is the n-th inclusion of N into $N^{(\aleph_0)}$. Then h does not extend into an element of $\operatorname{Hom}_R(R^{(\lambda)}, N^{(\aleph_0)})$, whence $\operatorname{Ext}_R(M, N^{(\aleph_0)}) \neq 0$.

If $\lambda = \aleph_0$, [T, Lemma 10.3] shows there exists a non-projective finitely generated submodule F of M such that $\operatorname{Ext}_R(F, N) = 0$ and the induction works.

If λ is a regular uncountable cardinal, then [T, Lemma 10.7] shows there is a λ-filtration $(C_\alpha \mid \alpha < \lambda)$ of M such that $\operatorname{Ext}_R(C_{\alpha+1}/C_\alpha, N) = 0$ for all $\alpha < \lambda$, and the set $E = \{\alpha < \lambda \mid C_{\alpha+1}/C_\alpha$ is non-projective$\}$ is stationary in λ. By the induction premise, $\operatorname{Ext}_R(C_{\alpha+1}/C_\alpha, N^{(\aleph_0)}) \neq 0$ for all $\alpha \in E$. By [T, Lemma 10.6], this implies $\operatorname{Ext}_R(M, N^{(\aleph_0)}) \neq 0$.

If λ is singular, then the general compactness theorem [EM, Theorem IV.3.7] implies there is a non-projective submodule U of M such that $\operatorname{gen}(U) < \lambda$ and $\operatorname{Ext}_R(U, N) = 0$, and the induction works.

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