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FC-**nilpotent groups and a Frattini-like subgroup**

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FC-Nilpotent Groups and a Frattini-like Subgroup.

M. J. TOMKINSON (*)

1. Introduction.

In [6], we introduced a subgroup $\mu(G)$ which was a variation on the Frattini subgroup and we showed that, for various classes of infinite groups, $\mu(G)$ had many of the properties of the Frattini subgroup. For example, we showed that $\mu(G)$ satisfied nilpotency conditions and that if $G/\mu(G)$ satisfies a nilpotency condition then so does G. Supersolubility conditions and finiteness conditions on $G/\mu(G)$ were considered in subsequent papers [7, 8].

In all of these earlier papers the main results were obtained within the class of nilpotent-by-finite groups and the class of S_0-groups.

Our aim here is to extend the applicability of our results by considering the class of soluble-by-finite *FC*-nilpotent groups. This is a very considerable generalization of the class of nilpotent-by-finite groups and requires methods closer to those used for S_0-groups.

We recall the definitions of $\mu(G)$. Let U be a subgroup of the group G and consider the properly ascending chains

$$U = U_0 < U_1 < \ldots < U_\alpha = G$$

from U to G. We define $m(U)$ to be the least upper bound of the types α of all such chains. Thus $m(U)$ is a measure of how far the subgroup U is from G. A proper subgroup M of G is said to be a *major subgroup* if $m(U) = m(M)$ whenever $M \leqslant U < G$. Then we define $\mu(G)$ to be the intersection of all major subgroups of G. If G is finitely generated then the major subgroups are just the maximal subgroups of G and so $\mu(G)$ coincides with the Frattini subgroup $\varphi(G)$.

One basic result [6, Lemma 2.3] enables us to use the Frattini argument.

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PROPOSITION 1.1. *Every proper subgroup of a group G is contained in a major subgroup of G.*

When we considered nilpotent-by-finite groups in [6,7], we first considered abelian-by-finite groups and reduced to the case in which the groups are also periodic and so have a good Sylow theory. This enabled us to use the Frattini argument in the usual way.

The *FC-centre* of a group G, denoted $FC(G)$, is the subgroup consisting of all elements having only finitely many conjugates in G. A group G is *FC-nilpotent* if it has a finite normal series

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

such that each factor G_i/G_{i-1} is FC-central in G; that is, $G_i/G_{i-1} \leqslant FC(G/G_{i-1})$. It is clear that every nilpotent-by-finite group is FC-nilpotent and is also soluble-by-finite. However there are many constructions of soluble FC-nilpotent groups which are not nilpotent-by-finite; for example, soluble FC-groups and also direct products of (nilpotent of class c)-by-(finite abelian) groups.

For these groups we will not normally be able to make use of Sylow theory and instead we aim at obtaining splitting theorems in abelian-by-hypercentral or abelian-by-hypercyclic groups. The point is that, by Proposition 1.1, a complement must be contained in a major subgroup and so, in particular, $\mu(G)$ can not have a complement. The splitting theorems that we obtain are rather weak requiring very strong conditions on the groups considered. These appear as Lemma 2.3 and Lemma 3.1. Our main results are Theorems 2.4 and 3.2 and can be summarized as.

THEOREM. *Let G be a soluble-by-finite FC-nilpotent group and let $N \lhd G$.*

   (a) *If $N/N \cap \mu(G)$ is a hypercentral group, then so is N.*

   (b) *If $N/N \cap \mu(G)$ is a hypercyclic group, then so is N.*

Putting $N = \mu(G)$ gives the

COROLLARY. *If G is a soluble-by-finite FC-nilpotent group then $\mu(G)$ is a hypercentral group.*

2. Hypercentral groups.

We begin with a decomposition result for locally finite modules. The significance of this result is that if A is a normal periodic abelian sub-

group contained in the *FC*-centre of a group G then each finite set of elements of A is contained in a finite normal subgroup of G; thus A can be considered as a locally finite $\mathbb{Z}(G/A)$-module.

LEMMA 2.1.   *Let G be a locally nilpotent (locally supersoluble) group and let A be a locally finite $\mathbb{Z}G$-module. Then A has a decomposition*

$$A = A^+ \oplus A^-,$$

where every irreducible $\mathbb{Z}G$-factor of A^+ is trivial (cyclic) and no irreducible $\mathbb{Z}G$-factor of A^+ is trivial (cyclic). This decomposition is unique.

PROOF.   Let F_i, $i \in I$, be the finite submodules of A so that $A = \sum_{i \in I} F_i$. Since $G/C_G(F_i)$ is a finite nilpotent (supersoluble) group, the module F_i has a unique decomposition $F_i = F_i^+ \oplus F_i^-$, where the irreducible $\mathbb{Z}G$-factors of F_i^+ are trivial (cyclic) and no irreducible $\mathbb{Z}G$-factor of F_i^- is trivial (cyclic) [2]. It is clear that if $F_i \leqslant F_j$, then $F_i^+ \leqslant F_j^+$ and $F_i^- \leqslant F_j^-$. Thus the submodules $A^+ = \sum_{i \in I} F_i^+$ and $A^- = \sum_{i \in I} F_i^-$ have the required properties.

Recall that if A is a normal subgroup of E, then E is said to *split conjugately* over A if A has a complement K in E (that is, $AK = E$ and $A \cap K = 1$) and any two complements to A in E are conjugate in E.

Before proving our splitting theorem for abelian-by-hypercentral groups it is useful to record the following useful result which appears as Lemma 3 in [5].

PROPOSITION 2.2.   *Let E be a group containing subgroups $A \leqslant H$ such that A is a normal abelian subgroup and H is an ascendant subgroup. If $C_A(H) = 1$ and H splits conjugately over A, then each complement to A in H is contained in a unique complement to A in E, and E splits conjugately over A.*

We have only been able to obtain a splitting theorem for extensions of a locally finite $\mathbb{Z}G$-module A by a hypercentral group G under very strong restrictions, but the following result will be sufficient for our main purpose.

LEMMA 2.3.   *Let G be a hypercentral group and A a locally finite $\mathbb{Z}G$-module which is also a p-group. Suppose that G has a normal subgroup H of finite index prime to p such that each irreducible $\mathbb{Z}G$-factor of A is H-trivial but not G-trivial.*

Then any extension E of A by G splits conjugately over A.

PROOF. We prove the result by induction on $|G/H|$ and suppose first that G/H is cyclic of prime order $q \neq p$.

In the extension E of A by G, let M be the extension of A by H. Then $E = M\langle x \rangle$ and $x \in E$ can be chosen either to have infinite order or to have order q^n.

If V is some irreducible $\mathbb{Z}G$-factor of A then $[V, G] = V$ but $[V, H] = 0$. Since $E = M\langle x \rangle$, it follows that

$$V = [V, M\langle x \rangle] = [V, \langle x \rangle] = [V, x].$$

Since A is a locally finite module, we see that $[A, x] = A$ and $C_A(x) = 0$.

Let $L = A\langle x \rangle \leqslant E$; since x either has infinite order or has order q^n with $q \neq p$, it is clear that $\langle x \rangle$ is a complement to A in L. Let $\langle y \rangle$ be a second complement to A in L; then $\langle y \rangle = \langle xa \rangle$, for some $a \in A$. But there is an element $b \in A$ such that $a^{-1} = [b, x]$ and so $xa = b^{-1}xb$. Thus $\langle y \rangle = b^{-1}\langle x \rangle b$ and L splits conjugately over A.

Since E/A is hypercentral, L is ascendant in E and so, by Proposition 2.2, E splits conjugately over A.

Now suppose that $E/M \cong G/H$ is not of prime order and choose a subgroup X/M of prime order in the centre of E/M. Since X/A is a hypercentral group, the $\mathbb{Z}X$-module A has a decomposition $A = A_X^+ \oplus A_X^-$ such that every irreducible $\mathbb{Z}X$-factor of A_X^+ is X-trivial and no irreducible $\mathbb{Z}X$-factor of A_X^- is $\mathbb{Z}X$-trivial. Since $X \lhd E$, A_X^+ and A_X^- are $\mathbb{Z}G$-submodules of A.

First consider $\overline{X} = X/A_X^+$. Each irreducible $\mathbb{Z}X$-factor of $\overline{A} = A/A_X^+$ is nontrivial, but each irreducible $\mathbb{Z}M$-factor of $\overline{A}$ is M-trivial. If V is an irreducible $\mathbb{Z}X$-factor of $\overline{A}$, then $C_V(M) \neq 0$; but $C_V(M)$ is a $\mathbb{Z}X$-submodule and so is equal to V. Therefore each irreducible $\mathbb{Z}X$-factor of $\overline{A}$ is M-trivial. Since $|X/M| < |G/H|$, it follows by induction that $\overline{X}$ splits conjugately over $\overline{A}$. Also $\overline{X} \lhd \overline{E}$ and $C_{\overline{A}}(\overline{X}) = 0$ so, by Proposition 2.2, $\overline{E}$ splits conjugately over $\overline{A}$. That is, E contains a subgroup K such that $AK = E$ and $A \cap K = A_X^+$ and all such subgroups are conjugate.

Now K is an extension of A_X^+ by G. Each irreducible $\mathbb{Z}K$-factor of A_X^+ is nontrivial. Each irreducible $\mathbb{Z}X$-factor of A_X^+ is X-trivial and hence each irreducible $\mathbb{Z}K$-factor of A_X^+ is $(K \cap X)$-trivial. Since $|K/K \cap X| < |G/H|$, it follows by induction that K splits conjugately over A_X^+ and it now follows easily that E splits conjugately over A.

THEOREM 2.4. *Let G be a soluble-by-finite FC-nilpotent group and let $N \lhd G$. Then N is a hypercentral group if and only if $N/N \cap \cap \mu(G)$ is a hypercentral group.*

PROOF. The hypotheses on G ensure that it has a finite normal series

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n \lhd G$$

such that G/G_n is finite and, for $i = 1, \ldots, n$, G_i/G_{i-1} is an abelian normal subgroup contained in $FC(G/G_{i-1})$.

We suppose that $N/N \cap \mu(G)$ is a hypercentral group and prove that N is hypercentral. Since $N \cap \mu(G)/G_n \cap N \cap \mu(G)$ is finite, the usual Frattini argument shows that it is nilpotent. Thus there is a finite series of normal subgroups of G

$$1 = A_0 \lhd A_1 \lhd \ldots \lhd A_m = N \cap \mu(G)$$

such that A_j/A_{j-1} is abelian and is contained in $FC(G/A_{j-1})$. By induction on the length of this series we may assume that G has an abelian normal subgroup $A \leqslant N \cap \mu(G)$ such that N/A is hypercentral and $A \leqslant FC(G)$.

Note first that it is sufficient to prove that each G-chief factor of A is central in N. For, since $A \leqslant FC(G)$, A has an ascending series of G-admissible subgroups each factor of which is finitely generated. This can be refined so that the factors are either finite or torsion-free. Any finite factor is hypercentrally embedded in N. Also a free abelian G-admissible subgroup F of rank n is contained in the n-th term of the upper central series of N, since $F/F^p \leqslant Z_n(N/F^p)$, for each prime p, and so $[F, N, \ldots, N] \leqslant \cap F^p = 1$.

Suppose then that there is a G-chief factor U/V with $U \leqslant A$ and U/V not central in N. Then U/V is an elementary abelian p-group for some prime p. Let M be maximal with respect to $M \lhd G$, $M \cap U = V$. Then MU/M is the unique minimal normal subgroup of G/M. We replace G by G/M, U by UM/M and A by $AM/M \leqslant NM \cap \mu(G)M/M \leqslant$ $\leqslant nM/M \cap \mu(G/M)$. Thus U is contained in every nontrivial normal subgroup of G. We claim that A is a p-group. It is clear that the torsion subgroup of A is a p-group and if A is not torsion then it contains an element x of infinite order. Since $A \leqslant FC(G)$, the normal closure $F = \langle x \rangle^G$ is an infinite finitely generate abelian group. There is an integer k such that F^k is torsion-free. Thus $F^k \cap U = 1$ and $F^k \lhd G$ contrary to our condition on U. Hence A is a p-group, as claimed.

In particular, A is a locally finite $\mathbb{Z}G$-module and also a $\mathbb{Z}N$-module. By Lemma 2.1, A has a decomposition $A = A_N^+ \oplus A_N^-$ where A_N^+ and A_N^- are $\mathbb{Z}N$-submodules, each irreducible $\mathbb{Z}N$-factor of A_N^+ is N-trivial and no irreducible $\mathbb{Z}N$-factor of A_N^- is N-trivial. Since $N \lhd G$, A_N^+ and A_N^- are $\mathbb{Z}G$-submodules and, since U is contained in every nonzero $\mathbb{Z}G$-submodule of A, one of them must be zero. But U is not central in N and so

$C_U(N) = 0$ and hence $U \nleqslant A_N^+$. Therefore $A = A_N^-$ and no irreducible $\mathbb{Z}N$-factor of A is N-trivial.

Now let $C = C_N(U)$; since $O_p(G/C_G(U))$ is trivial it follows that N/C is a finite nilpotent p'-group. Let F be any nontrivial finite normal subgroup of G contained in A; then $N/C_N(F)$ is finite and nilpotent. If $C/C_N(F)$ is not a p-group, then there is a finite normal p'-subgroup $Q/C_N(F)$ of $G/C_N(F)$ such that

$$Q/C_N(F) \leqslant Z(N/C_N(F)) \cap C/C_N(F).$$

By Fitting's Lemma [3, Theorem 5.2.3], $F = [F, Q] \times C_F(Q)$. Since $Q \lhd G$, both $[F, Q]$ and $C_F(Q)$ are normal in G and, by the condition on U, one of them must be the identity subgroup. But $Q \leqslant C = C_N(U)$, so $U \leqslant C_F(Q)$. Hence $[F, Q] = 1$ and $Q = C_N(F)$. Thus $C/C_N(F)$ is a finite p-group.

It follows that F is hypercentrally embedded in C and so every N-chief factor of F is centralized by C. This is true for all F and so every irreducible $\mathbb{Z}N$-factor of the module A is C-trivial. Thus A satisfies the conditions of Lemma 2.3 (for $G = N/A$ and $H = C/A$) and so N splits conjugately over A.

Now $N \lhd G$ and since every irreducible $\mathbb{Z}N$-factor of A is nontrivial, Proposition 2.2 shows that G splits conjugately over A. But this is contrary to $A \leqslant \mu(G)$ and the proof is complete.

3. Hypercyclic groups.

We begin with a splitting theorem for extensions of a locally finite $\mathbb{Z}G$-module A by a hypercyclic group G which will play a similar role to Lemma 2.3 for the hypercentral case. We need even stronger restrictions for this result but again these conditions arise in a natural way in the proof of our main theorem.

Recall that a $\mathbb{Z}H$-module V is *H-polycyclic* if it has a finite series of $\mathbb{Z}H$-submodules such that each factor is a cyclic group.

LEMMA 3.1. *Let G be a hypercyclic group and A a locally finite $\mathbb{Z}G$-module which is also a p-group. Suppose that G has a normal subgroup H of finite index prime to p containing G' and suppose that each irreducible $\mathbb{Z}G$-factor of A is G'-trivial and H-polycyclic but not cyclic.*

Then any extension E of A by G splits conjugately over A.

PROOF. We prove the result by induction on $|G/H|$ and suppose first that G/H is cyclic of prime order $q \neq p$.

In the extension E of A by G, let M be the extension of A by H so that $|E/M| = |G/H| = q \neq p$. If $p \neq 2$, let $M^* = E'AM^{p-1}$ and $E^* = E'AE^{p-1}$ so that M/M^* and E/E^* are the largest factor groups of exponent dividing $p - 1$ of $M/E'A$ and $E/E'A$, respectively. If $p = 2$ we take $M^* = E'AM^2$ and $E^* = E'AE^2$. In both cases $E^* \leqslant E'AE^2$. Since each irreducible $\mathbb{Z}G$-factor of A is H-polycyclic and G'-trivial it is M^*-trivial, since it is not cyclic it is not E^*-trivial; hence $E^* > M^*$. In $\overline{E} = E/M^*$ there is an element $\overline{x}$ of q-power order such that $\overline{E} = \overline{M}\langle\overline{x}\rangle$. Thus $\overline{E}/\langle\overline{x}\rangle$ has exponent dividing $p - 1$ (or 2, if $p = 2$) and so $\langle\overline{x}\rangle \geqslant \overline{E}^*$. Thus E^*/M^* is nontrivial cyclic of q-power order. There is an element $x \in E^*$ such that $E^* = M^*\langle x\rangle$ and x either has infinite order or has order a power of q.

If V is an irreducible $\mathbb{Z}G$-factor of A, then

$$V = [V, E^*] = [V, M^*\langle x\rangle] = [V, x]$$

and it follows that $A = [A, x]$ and $C_A(x) = 0$. It is clear that $\langle x\rangle$ is a complement to A in $A\langle x\rangle$. If $\langle y\rangle$ is a second complement then $\langle y\rangle = \langle xa\rangle$ for some $a \in A$. There is a $b \in A$ such that $a^{-1} = [b, x]$ and so $xa = b^{-1}xb$ and $\langle y\rangle = b^{-1}\langle x\rangle b$. Therefore $A\langle x\rangle$ splits conjugately over A.

By taking an ascending normal cyclic series for E/A and intersecting with $E'A/A$ we obtain an ascending series

$$A = E_0 \triangleleft E_1 \triangleleft \ldots \triangleleft E_\alpha \triangleleft E_{\alpha+1} \triangleleft \ldots \triangleleft E_\rho = AE'$$

with $E_\alpha \triangleleft E$, $E_{\alpha+1}/E_\alpha$ either infinite cyclic or cyclic of prime order and $C_E(E_{\alpha+1}/E_\alpha) \geqslant AE'$. We prove by induction on α that $\langle x\rangle$ is contained in a unique complement K_α to A in $E_\alpha\langle x\rangle$ and that $E_\alpha\langle x\rangle$ splits conjugately over A.

Case (i): $\alpha - 1$ exists.

Then $E_{\alpha-1}\langle x\rangle$ has a unique complement $K_{\alpha-1}$ containing $\langle x\rangle$. If $E_\alpha/E_{\alpha-1}$ is infinite cyclic or cyclic of order p, then $E/C_E(E_\alpha/E_{\alpha-1})$ is cyclic of order dividing 2 or $p - 1$ and so $E^* \leqslant C_E(E_\alpha/E_{\alpha-1})$. Since $x \in E^*$, $E_{\alpha-1}\langle x\rangle \triangleleft E_\alpha\langle x\rangle$. Also $C_A(K_\alpha) \leqslant C_A(x) = 1$ and so, by Proposition 2.2, there is a unique complement K_α to A in $E_\alpha\langle x\rangle$ containing $K_{\alpha-1}$ and $E_\alpha\langle x\rangle$ splits conjugately over A.

If $E_\alpha/E_{\alpha-1}$ has prime order $r \neq p$, then $|E_\alpha\langle x\rangle : E_{\alpha-1}\langle x\rangle| = r$ and it follows from a result of Gaschütz [4, Hauptsatz I.17.4] that $E_\alpha\langle x\rangle$ splits conjugately iver A and there is a unique complement to A in $E_\alpha\langle x\rangle$ containing $\langle x\rangle$.

Case (ii): α a limit ordinal.

For each $\beta < \alpha$, let K_β be the unique complement to A in $E_\beta \langle x \rangle$ which contains x. Let $K_\alpha = \underset{\beta < \alpha}{\cup} K_\beta$; then

$$AK_\alpha = \cup \, AK_\beta = \cup \, E_\beta \langle x \rangle = E_\alpha \langle x \rangle$$

and $A \cap K_\alpha = \cup \, (A \cap K_\beta) = 1$, so that K_α is a complement to A in $E_\alpha \langle x \rangle$ and clearly K_α contains x. If L_α is a second such complement then $L_\alpha \cap$ $\cap \, E_\beta \langle x \rangle$ is a complement to A in $E_\beta \langle x \rangle$ and so $L \cap E_\beta \langle x \rangle = K_\beta$. Therefore $L_\alpha = \cup \, (L_\alpha \cap E_\beta \langle x \rangle) = \cup \, K_\beta = K_\alpha$, as required.

This completes the inductive step and we have shown, in particular, that $E'A\langle x \rangle$ splits conjugately over A. But $E'A\langle x \rangle \lhd E$ and $C_A(x) = 1$ so, by Proposition 2.2, E splits conjugately over A.

Now suppose that $E/M \cong G/H$ is not of prime order and choose a subgroup X/M of prime order. Since X/A is hypercyclic, A has a decomposition $A = A_X^+ \oplus A_X^-$ such that every irreducible $\mathbb{Z}X$-factor of A_X^+ is cyclic but no irreducible $\mathbb{Z}X$-factor of A_X^- is cyclic. Since $X \lhd G$, A_X^+ and A_X^- are $\mathbb{Z}G$-submodules of A.

First consider $\overline{X} = X/A_X^+$. Each irreducible $\mathbb{Z}X$-factor of $\overline{A} = A/A_X^+$ is noncyclic but since the irreducible $\mathbb{Z}G$-factors are G'-trivial so are the irreducible $\mathbb{Z}X$-factors. Since $X' \leqslant E'A$, the irreducible $\mathbb{Z}X$-factors of $\overline{A}$ are X'-trivial. Also, since every irreducible $\mathbb{Z}H$-factor of A is cyclic, the irreducible $\mathbb{Z}X$-factors of $\overline{A}$ are M-polycyclic. Since $|X/M| < |G/H|$, it follows that $\overline{X}$ splits conjugately over $\overline{A}$. Since $\overline{A}$ has no X-trivial factors, $C_A(\overline{X}) = 1$ and $\overline{X} \lhd \overline{E} = E/A_X^+$, it follows from Proposition 2.2 that $\overline{E}$ splits conjugately over $\overline{A}$. Let $\overline{K} = K/A_X^+$ be a complement so that $AK = E$ and $A \cap K = A_X^+$ and K is an extension of A_X^+ by G.

Each irreducible $\mathbb{Z}X$-factor of A_X^+ is cyclic and so the irreducible $\mathbb{Z}K$-factors of A_X^+ are $(K \cap X)$-polycyclic. Since $|K/K \cap X| < |G/H|$ it follows that K splits conjugately over A_X^+ and it is easily deduced that E splits conjugately over A.

THEOREM 3.2. *Let G be a soluble-by-finite FC-nilpotent group and let $N \lhd G$. Then N is hypercyclic if and only if $N/N \cap \mu(G)$ is hypercyclic.*

PROOF. As in the proof of Theorem 2.4, we may assume that G has an abelian normal subgroup A such that $A \leqslant N \cap \mu(G)$, N/A is hypercyclic and $A \leqslant FC(G)$. We show that N is hypercyclic.

Again we note that it is sufficient to consider the G-chief factors of A, this time showing that they are N-polycyclic. For, A has an ascending series of G-admissible subgroups in which the factors are finite or

free abelian of finite rank. Each finite factor can be replaced by a finite series of N-admissible subgroups in which the factors are cyclic. If F is free abelian of finite rank with $F \lhd G$, then $G/C_G(F)$ is finite and we consider the split extension H of F by $\overline{G} = G/C_G(F)$. For any positive integer n, H/F^n is finite and each H-chief factor of F/F^n is S-polycyclic, where $S \leqslant H$ is the extension of F by $\overline{N} = NC_G(F)/C_G(F)$. Thus S/F^n is supersoluble. Hence all finite homomorphic images of the finitely generated abelian-by-finite group S are supersoluble. It follows from [1] that S is supersoluble and so F has a finite series of N-admissible subgroups with cyclic factors. Hence A has an ascending series of N-admissible subgroups with cyclic factors and so N is hypercyclic, as required.

Suppose then that there is a G-chief factor U/V with $U \leqslant A$ and U/V not N-polycyclic. Then U/V is an elementary abelian p-group for some prime p. As in Theorem 2.4, we let M be maximal with respect to $M \lhd G$, $M \cap U = V$ and consider G/M. Then we may assume that U is contained in every nontrivial normal subgroup of G and A is a p-group.

In particular A is a locally finite $\mathbb{Z}G$-module and also a $\mathbb{Z}N$-module. Since N/A is hypercyclic, A has a decomposition $A = A_N^+ \oplus A_N^-$ where A_N^+, A_N^- are $\mathbb{Z}N$-submodules, each irreducible $\mathbb{Z}N$-factor of A_N^+ is cyclic and no irreducible $\mathbb{Z}N$-factor of A_N^- is cyclic. Since $N \lhd G$, A_N^+ and A_N^- are $\mathbb{Z}G$-submodules and, since U is contained in every $\mathbb{Z}G$-submodule of A, one of them must be zero. But U is not N-polycyclic and so $U \nleqslant A_N^+$. Therefore $A = A_N^-$ and no irreducible $\mathbb{Z}N$-factor of A is cyclic.

The subgroup $[U, N']$ is normal in G and so is either 1 or U. Consider first the case in which $[U, N'] = U$. Since $N'A/A$ is a hypercentral group, when considered as a $\mathbb{Z}N'$-module, A has a decomposition $A = = A^+ \oplus A^-$, where every irreducible $\mathbb{Z}N'$-factor of A^+ is N'-trivial and no irreducible $\mathbb{Z}N'$-factor of A^+ is N'-trivial. Since $N' \lhd G$, A^+ and A^- are $\mathbb{Z}G$-submodules and one of them must be zero. Since $[U, N'] = U$, there are nontrivial irreducible $\mathbb{Z}N'$-factors in U and so $A^- \neq 0$. Therefore $A = A^-$; that is, no irreducible $\mathbb{Z}N'$-factor of A is N'-trivial.

Let $H = C_{AN'}(U)$; then, as a $\mathbb{Z}(H/A)$-module, A has a decomposition $A = A_H^+ \oplus A_H^-$, where every irreducible $\mathbb{Z}H$-factor of A_H^+ is H-trivial and no irreducible $\mathbb{Z}H$-factor of A_H^- is H-trivial. Again since $H \lhd G$, A_H^+ and A_H^- are $\mathbb{Z}G$-submodules and, since $U \leqslant A_H^+$, it follows that $A = = A_H^+$. That is, every irreducible $\mathbb{Z}H$-factor of A is H-trivial. If V is some irreducible $\mathbb{Z}(AN')$-factor of A, then $C_V(H) \neq 0$ and so $C_V(H) = V$. That is, V is H-trivial. Thus the $\mathbb{Z}(AN'/A)$-module A satisfies the conditions of Lemma 2.3 and so AN' splits conjugately over A. Since $AN' \lhd G$ and $C_A(N') = 1$, G splits conjugately over A, contrary to $A \leqslant \mu(G)$.

Therefore we may assume that $[U, N'] = 1$. By considering the decomposition of A as a $\mathbb{Z}N'$-module, $A = A^+ \oplus A^-$ in which each irreducible $\mathbb{Z}N'$-factor of A^+ is N'-trivial and no irreducible $\mathbb{Z}N'$-factor of A^- is N'-trivial we see that $U \leqslant A^+$ and so $A = A^+$. That is, every irreducible $\mathbb{Z}N'$-factor of A is N'-trivial.

Let $C = C_N(U)$; then $C \geqslant N'$ and so N/C is a finite abelian group. Also $O_p(G/C_G(U))$ is trivial and so $O_p(N/C)$ is trivial. Thus N/C is a finite abelian p'-group. Consider A as a $\mathbb{Z}C$-module. Since C/A is hypercyclic, A has a decomposition $A = A_C^+ \oplus A_C^-$, where every irreducible $\mathbb{Z}C$-factor of A_C^+ is cyclic and no irreducible $\mathbb{Z}C$-factor of A_C^- is cyclic. Again C is normal in G and so A_C^+ and A_C^- are $\mathbb{Z}G$-submodules of A. Since $U \leqslant A_C^+$, it follows that $A = A_C^+$ and every irreducible $\mathbb{Z}C$-factor of A is cyclic. Hence every irreducible $\mathbb{Z}N$-factor of A is C-polycyclic. Also if V is an irreducible $\mathbb{Z}N$-factor of A then $C_V(N') \neq 0$ and so V is N'-trivial. Therefore the $\mathbb{Z}(N/A)$-module A satisfies the conditions of Lemma 3.1 and so N splits conjugately over A. Since $N \lhd G$ and $C_A(N) = 1$, we see that G splits conjugately over A, contrary to $A \leqslant \mu(G)$.

This completes the proof.

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