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Some Properties of Direct Sums of Uniserial Modules Over Valuation Domains.

SILVANA BAZZONI(*)

ABSTRACT - We characterize the direct sums of uniserial modules over a valuation domain satisfying the following property:

(*) every element is contained in a pure uniserial submodule.

We reduce the problem to the direct sums of two uniserial modules, and we consider three different cases determined by all the possible relations between the types of the uniserials involved. In each case we give necessary and sufficient conditions on the uniserials in order that their direct sum satisfies (*). Moreover we observe that, for the class of direct sums of uniserials satisfying (*), we can solve the problem of introducing a «good» definition of height of an element, namely a notion which gives all the informations about the divisibility of the module generated by that element.

Introduction.

R will always denote a valuation domain, Q its quotient field. An R -module is uniserial if its submodules are linearly ordered by inclusion. All the uniserial modules considered will be torsion modules. A uniserial module is said to be standard if it is isomorphic to a submodule of Q/I for some ideal I in R , otherwise it is called non standard.

The knowledge of the class of direct sums of uniserial modules over a valuation domain is far from being complete. Some results are available for direct sums of standard uniserial modules. Fuchs and Salce, in [FS1], define numerical invariants which characterize the direct sums of standard uniserials; moreover they prove that every module has a submodule, which is called prebasic, with the same invariants as

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the module and which is a direct sum of standard uniserial modules, pure and maximal with respect to these properties.

In this paper we consider direct sums of arbitrary uniserials and solve the problem of characterizing those satisfying the following property:

(*) every element is contained in a pure uniserial submodule.

In Section 2 we give informations about the pure uniserial submodules of a direct sum of uniserials, and we prove that to solve our problem it is enough to consider a direct sum of two uniserial modules.

We introduce the notion of type of an element x in an R -module: it is the ordered pair consisting of the height ideal of x (the submodule of Q which measures the divisibility of x) and the annihilator ideal of x .

We prove that the direct sum of two uniserial modules U and V satisfies (*) if and only if for any two elements u, v (in U and V respectively) there exists a homomorphism from one of the two uniserial modules to the other sending one of the two elements into the other; hence in particular we have that the types of the two elements are comparable.

Our investigation splits then into three cases determined by three different kinds of relations between the types of the two uniserials considered.

A complete summary of the results that will be proved afterwards, and that answer completely to our question, will be presented in Theorem 5.1, Section 5; here we outline the situation without entering into detailed conditions.

We notice that, if U and V are two uniserial modules, then one of the following possibilities occurs:

(A) there are two non zero elements $u \in U$ and $v \in V$ with the same height ideal.

(B) there are two non zero elements $u \in U$ and $v \in V$ with the same annihilator ideal.

(C) the isomorphy class of the height ideals of elements of U is different from the corresponding class for V and the same holds for the isomorphy class of the annihilator ideals.

We prove that, in case (A), $U \oplus V$ satisfies (*) if and only if one of the two uniserials is a suitable epimorphic image of the other.

In case (B), $U \oplus V$ satisfies (*) if and only if one of the two uniserials has a suitable embedding into the other.

In case (C), we have to consider three different possibilities distin-

guished by which of the two uniserials U and V is standard or non standard; in these cases the conditions on U and V equivalent to the fact that $U \oplus V$ satisfies (*) are technical and can be formulated only after the preliminaries are settled.

In Section 5 we give also various examples of direct sums of uniserial modules satisfying (*).

In Section 6 we observe that, for the direct sums of uniserial modules satisfying (*), the type of an element gives all the informations needed about the divisibility of all the multiple of that element; thus it can be assumed to be the «indicator» of the element. In this context we prove that a direct sum M of uniserial modules satisfying (*) is both fully transitive and transitive, in the sense that for any two elements x and y in M such that the type of x does not exceed the type of y (the types of x and y are equal), there is an endomorphism (automorphism) of M sending x to y .

1. Preliminaries.

We collect in this section some definitions and results that will be used in the following; for references see [FS1], [BS1], [BS2], [BS3] and [BFS].

If I is a non zero submodule of Q , $I^\#$ denotes the set of all elements r in R such that rI is properly contained in I .

We recall that $I^\#$ is the union of the proper ideals of R isomorphic to I and that it is a prime ideal of R .

For a uniserial module U the ideals $U^\#$ and $U_\#$ are defined by:

$$U^\# = \{r \in R \mid rU < U\},$$

and

$$U_\# = \{r \in R \mid \exists 0 \neq u \in U, ru = 0\}.$$

It is easy to see that these are also prime ideals of R .

Following [BFS] we say that a uniserial module U is *finitely annihilated* if there is an element $u \in U$ such that $\text{Ann } u = \text{Ann } U$. It is shown in [SL] (see also [FS1, VII, § 2]) that if U is finitely annihilated then $U_\# \leq U^\#$ while if U is non finitely annihilated then $U^\# \leq U_\#$. In case $U_\# = U^\#$ the two different possibilities are determined by U being principal or not over $R_{U^\#}$.

If U is non standard then it is non finitely annihilated.

For every uniserial R module U we defined the *threshold submodule* U^c of U as $U[U^\#] = \{u \in U \mid \text{Ann } u \geq U^\#\}$. This has been firstly de-

finied in [BS1] for a non standard uniserial module and it turned out to be crucial in determining the non standard quotients of a non standard uniserial.

Let x be a non zero element of a module M . The *height ideal* of x in M , denoted by $H_M(x)$, is defined as $H_M(x) = \{r^{-1} \in Q \mid x \in rM\}$, while the *annihilator ideal* of x in M is $\text{Ann } x = \{r \in R \mid rx = 0\}$. We will drop the subscript index M in $H_M(x)$, whenever this will not yield to ambiguity.

Let U be a uniserial module and u a non zero element of U . Assume $H_U(u) = J$ and $\text{Ann } u = I$, then the *type* $t(U)$ of U is defined as $[J/I]$, the isomorphic class of the standard uniserial J/I . We remark that in the sequel the type of a uniserial module will always be represented by J/I obtained as illustrated above. (Hence we will have $R \leq J$ and $I < R$). It is easy to see that $U^\# = J^\#$ and $U_\# = I^\#$.

The *level* of a uniserial module U , denoted by $\text{Lev } U$, is the union of the height ideals of the non zero elements of U . This notion has been introduced in [BFS] and it turned out to be a useful tool in studying the algebraic structure of the set of the isomorphism classes of non finitely annihilated uniserial R -modules.

In [BS1] and [BS2], the class of non standard uniserial modules has been partitioned into six classes denoted by \mathcal{U}_i ($i = 1, 2, \dots, 6$) determined by which quotients of their modules are non standard, and in [BS3] the classes \mathcal{U}_5 and \mathcal{U}_6 have been divided into subclasses. For a complete description of the different classes and subclasses we refer to [BS2]; in Section 4, where we will have to make use to these results, we will give a summary of the characterizations and properties of the various classes \mathcal{U}_i .

We introduce the notion of the *type* of an element in the following way.

For every element x of the R module M , the type of x , denoted by $t_M(x)$, is defined to be the ordered pair $(H_M(x), \text{Ann}_M(x))$. In the set \mathcal{C} of all the ordered pairs consisting of submodules of Q containing R and ideals of R , we consider the partial order defined componentwise by the inclusion. Hence if x and y are two elements of the modules M and N respectively, then $t_M(x) \leq t_N(y)$ if and only if $H_M(x) \leq H_N(y)$ and $\text{Ann}_M(x) \leq \text{Ann}_N(y)$.

It is easy to see that the partial order set (\mathcal{C}, \leq) has inf and sup; \mathcal{C} has (Q, R) as the maximum element and it is the type, in any module, of the zero element; while the minimum is given by $(R, 0)$ which is the type of the identity in R .

We notice that the types of two elements belonging to the same uniserial module are always comparable.

We consider also the class \mathcal{C} of all the pairs of the form (U, u) where

U is a uniserial R module and u is an element of U . If (U, u) and (V, v) are two elements of \mathfrak{C} we set $(U, u) \leq (V, v)$ if there exists a homomorphism ϕ of U into V such that $\phi(u) = v$. Clearly \leq is a preorder; consider the equivalence relation defined by $(U, u) \sim (V, v)$ if and only if there exists an isomorphism ϕ of U into V such that $\phi(u) = v$, and denote by $[U, u]$ the equivalence class determined by (U, u) . It is easy to check that \leq induces a partial order on the set \mathfrak{C}/\sim .

Notice that $[U, u] \leq [V, v]$ implies $t(u) \leq t(v)$, and that $[U, u] = [U, ru]$ for every element $r \notin U^\# \cup U_\#$, since the multiplication by such an r induces an automorphism of U .

REMARK. If U and V are isomorphic uniserial modules and $u \in U$, $v \in V$ then $[U, u]$ and $[V, v]$ are comparable.

We prove now some technical Lemmas that will be needed later.

If I and L are submodules of Q , $I:L$ denotes the set of all elements $q \in Q$ such that $qL \leq I$.

It is easy to see that, for each $0 \neq r \in R$, $rI:L = r(I:L) = I:r^{-1}L$.

LEMMA 1.1. *Let I, L be non zero fractional ideals, then $(IL)^\# = I^\# \wedge L^\#$.*

PROOF. See [BFS] Lemma 2.7. ■

LEMMA 1.2. *Let I, L be a non zero fractional ideals, then $(I:L)^\# = I^\# \wedge L^\#$.*

PROOF. It is clear that $(I:L)^\# \leq I^\# \wedge L^\#$. Let first $I^\# \geq L^\#$ and assume, by way of contradiction, that $(I:L)^\# < L^\#$. Consider $r \in L^\#$ such that $r(I:L) = I:L = I:r^{-1}L$. Since $r \in L^\#$, there is an element x with $L < xR \leq r^{-1}L$, hence $I:L \geq I:xR \geq I:r^{-1}L = I:L$. But $I:xR = x^{-1}I$ yields $(I:L)^\# = I^\#$ a contradiction. Let now $I^\# < L^\#$ and assume that $(I:L)^\# < I^\#$. Then there exists $r \in I^\#$ such that $r(I:L) = I:L = rI:L$. Take $a \in I \setminus rI$, then $rI < aR \leq I$ gives $I:L = rI:L = aR:L = a(R:L)$ hence $(I:L)^\# = (R:L)^\#$ and the wanted contradiction follows if we prove that $(R:L)^\# = L^\#$. But this has already been proved in the case considered above since $L^\# \leq R^\# = P$. ■

Next Lemma is essentially Lemma 1.1 in [BFS].

LEMMA 1.3. *Let I be a fractional ideal. Then:*

- (i) $R_{I^*}I = I$.
- (ii) $I^*I = I$ if and only if $I \not\cong R_{I^*}$.
- (iii) If $I \cong R_{I^*}$, then $(R_{I^*}:I)I = R_{I^*}$.
- (iv) If $I \not\cong R_{I^*}$, then $R:I = R_{I^*}:I$ and $(R_{I^*}:I)I = I^*$.

PROOF. (iii) is obvious. (i), (ii) and the first part of (iv) are proved in [BFS], Lemma 1.1. Thus only the second part of (iv) has to be proved.

Since $I \not\cong R_{I^*}$, then $R_{I^*}:I = \{t^{-1} \in I \mid t \notin I\}$, hence $(R:I)I = I^*$, since I^* is the union of the proper ideals of R isomorphic to I . ■

LEMMA 1.4. *Let I, L be non zero fractional ideals. Then:*

- (i) If $I^* < L^*$, then there exists an element $r \notin L$ such that $r^{-1}L > I^*$ and $I:L = r^{-1}I$.
- (ii) If $I^* > L^*$, then there exists an element $r \notin I$ such that $r^{-1}I > L^*$ and $I:L = r\{t^{-1} \mid t \notin L\} = r(R:L)$.

PROOF. (i) Recall that L^* is the union of the proper ideals of R isomorphic to L , hence of the ideals of the form $r^{-1}L$ for an element $r \in Q \setminus L$. Thus there is an element $r \notin L$ such that $r^{-1}L > I^*$. Clearly $I:L = r^{-1}(I:r^{-1}L)$ and we prove now that $I:r^{-1}L$ is equal to I . Obviously $I:r^{-1}L \geq I$; let $x \notin I$ and assume $xr^{-1}L \leq I$, then $r^{-1}L \leq x^{-1}I \leq I^*$ which is a contradiction.

(ii) The first part of the claim is analogous to the one in (i). For the second part we notice that $I:L = r(r^{-1}I:L)$; if $x \in r^{-1}I:L$ then $xL \leq r^{-1}I \leq R$ hence $x^{-1} \notin L$ (L cannot be principal, since $L^* < I^* \leq P$). Conversely if $x \notin L$, then $x^{-1}L \leq L^*$ and by the first part $L^* < r^{-1}I$; notice that $\{t^{-1} \mid t \notin L\}$ coincides with $R:L$ since L is not principal, thus the claim follows. ■

LEMMA 1.5. *Let I, L be non zero fractional ideals with $I^* = L^*$.*

- (i) If $I \cong L$ then $I:L \cong R_{I^*}$.
- (ii) If $I \not\cong L$ then $I:L = I(R_{L^*}:L)$.

PROOF. (i) Obvious.

(ii) Assume first that L is principal over R_{L^*} . Then $L = rR_{L^*}$ and $R_{L^*}:L = r^{-1}R_{L^*}$. Now $I:L = r^{-1}(I:R_{L^*}) = r^{-1}I = I(r^{-1}R_{L^*})$. If L is not principal over R_{L^*} then $L(R_{L^*}:L) = L^*$. To prove that $I:L \geq$

$\geq I(R_{L^*}:L)$ it is enough to observe that $I(R_{L^*}:L)L = II^* \leq I$. We prove now that $I:L \leq I(R_{L^*}:L)$. Let $xL \leq I$, since $I \not\leq L$ we have that $xL < I$, hence there exists $a \in I$ with $xL < aR_{L^*}$. This implies $a^{-1}x \in (R_{L^*}:L)$, hence $x \in I(R_{L^*}:L)$. ■

LEMMA 1.6. *Let I, L be non zero fractional ideals. Then:*

(i) *If $I \geq L^*$, then $I:L^* = R_{L^*}$.*

(ii) *If $I:L \cong R_{I^*}$, then either $I \cong L$ or $I \cong R_{I^*}$.*

PROOF. (i) Obvious if $I = L^*$. If $I > L^*$, by Lemma 1.4 we obtain $I:L^* = R:L^*$, hence $I:L^* = R_{L^*}$.

(ii) Lemma 1.2 implies that $I^* \leq L^*$.

Assume first that $I^* < L^*$, then Lemma 1.4 yields $I:L \cong I$, hence $I \cong R_{I^*}$.

If $I^* = L^*$ and $I \not\leq L$, then, by Lemma 1.5, we obtain $I:L = I(R_{L^*}:L) = I(R_{I^*}:L)$, where I and $R_{I^*}:L$ are clearly R_{I^*} -fractional ideals. Hence, the hypothesis $I(R_{I^*}:L) \cong R_{I^*}$ implies that I is invertible or equivalently principal over R_{I^*} . ■

2. Reduction to the case of two summands.

R will always denote a valuation domain. Let M be a direct sum of uniserial modules. We want to find necessary and sufficient condition in order to have that M satisfies the following property:

(*) every element of M is contained in a pure uniserial submodule of M .

LEMMA 2.1. *Let M be the direct sum $\bigoplus_{i \in I} U_i$, where U_i is a uniserial module for each $i \in I$.*

Let $x \in M$ and V a pure uniserial submodule of M containing x . If $x \in \bigoplus_{i \in F} U_i$ where F is finite, then the projection $\pi_F(V)$ of V is a pure submodule of M containing x and isomorphic to V .

PROOF. The claim is clear by the fact that π_F is the identity on the submodule generated by x and by the fact that V is a pure uniserial submodule of M . ■

PROPOSITION 2.2. *Let M be the direct sum $\bigoplus_{i \in I} U_i$, where the U_i 's are uniserial modules and let W be a uniserial submodule of M . Assume one of the following conditions is satisfied:*

(i) W is non standard.

(ii) M is a finite direct sum of uniserials and W is pure in M .

Then there exists $i_0 \in I$ such that π_{i_0} induces an isomorphism between W and U_{i_0} , hence W is a summand of M .

PROOF. By the uniseriality of W we obtain that there exists $i_0 \in I$ such that π_{i_0} restricted to W is injective. Now if W is non standard then π_{i_0} has to be surjective; if W is pure in M , and the direct sum is finite, then the conclusion follows by [FS1] (IX, Theorem 5.6). ■

LEMMA 2.3. Let x be an element of the direct sum $M = \bigoplus_{i \in I} U_i$, where the U_i 's are uniserial modules. Let L and J be respectively the annihilator ideal and the height ideal of x in M . Assume V and W are uniserial submodules of M containing x and of type $[J/L]$. Then V and W are isomorphic.

PROOF. Obviously it is enough to prove the assertion in case at least one between V and W is non standard. Assume W is non standard; by Proposition 2.2, W is a summand of M , hence $M = W \oplus A$. We prove now that $A \cap V$ is 0; in fact if y is a non zero element of $A \cap V$, then $y \notin W$, hence $x \in W \cap V < Ry \leq A \cap V$ which gives the contradiction $x \in A$. Thus the projection π_W of M onto W is injective when restricted to V ; moreover $\pi_W(V)$ is W since $\pi_W(x) = x$ and the heights ideal of x in V and W are both J . ■

REMARK. Owing to Lemma 2.1, a direct sum of uniserial modules satisfies (*) if and only if every finite direct sum contained in it satisfies (*). Thus from now on we will consider finite direct sums of uniserial modules.

Using the terminology introduced in § 1 we formulate the following result which will play a fundamental role in characterizing the direct sum of uniserial modules satisfying (*).

PROPOSITION 2.4. Assume U and V are uniserial modules, $u \in U$ and $v \in V$. The following are equivalent:

(i) $u + v$ is contained in a pure uniserial submodule of $U \oplus V$.

(ii) $t(u)$ and $t(v)$ are comparable and if $t(u) \leq t(v)$, then also $[U, u] \leq [V, v]$.

(iii) $t(u)$ and $t(v)$ are comparable. Assume $t(u) \leq t(v)$; if $H(u) = H(v)$, there exists an epimorphism ϕ of U into V such that $\phi(u) = v$; if

$H(u) < H(v)$, $U/(\text{Ann } v)u$ is standard and there is an homomorphism ϕ of U into V such that $\phi(u) = v$.

In (i) and (ii) everything holds symmetrically changing the roles of u and v .

PROOF. (i) \Rightarrow (ii) Let W be a pure uniserial submodule of $U \oplus V$ containing $u + v$. By Proposition 2.2, W is isomorphic to U or to V via the corresponding projection. This implies $t(u + v) = t(u)$ or $t(u + v) = t(v)$ hence $t(u)$ and $t(v)$ are comparable. Assume $t(u) \leq t(v)$, then the projection π_U is an isomorphism of W into U . Thus $\pi_{U^{-1}}$ followed by π_V is a homomorphism ϕ of U into V such that $\phi(u) = v$.

(ii) \Rightarrow (iii) If $H(u) = H(v)$, then obviously the homomorphism ϕ considered in (ii) is surjective. If $H(u) < H(v)$ then $\phi(U)$ is properly contained in V hence the claim follows since clearly $\phi(U)$ is isomorphic to $U/(\text{Ann } v)u$.

(iii) \Rightarrow (i) Let $H(u) = \bigcup_{\sigma < \kappa} r_\sigma^{-1}R$ with $r_0 = 1$. Then U is generated by elements u_σ , $\sigma < \kappa$, such that $u_0 = u$ and $r_\tau r_\sigma^{-1}u_\sigma = e_\sigma^{-1}u_\tau$, $\sigma < \tau < \kappa$, where e_σ^{-1} are units of R subject to the relations $e_\sigma^{-1}e_\tau^{-1} - e_\tau^{-1}e_\sigma^{-1} \in r_\sigma \text{Ann } u$, $\sigma < \tau < \kappa$ (see [BS1] § 1).

Assume $H(u) = H(v)$ and ϕ is an epimorphism of U into V such that $\phi(u) = v$; then V is generated by elements v_σ , $\sigma < \kappa$ such that $v_0 = v$ and $r_\tau r_\sigma^{-1}v_\sigma = e_\sigma^{-1}v_\tau$, $\sigma < \tau$. Thus we can consider the submodule W of $U \oplus V$ generated by the elements $u_\sigma + v_\sigma$. It is easy to check that W is isomorphic to U via the projection π_U , hence it is a pure uniserial submodule of $U \oplus V$ containing $u + v$.

In case $H(u) < H(v)$, since $U/(\text{Ann } v)u$ is standard, the test Lemma (see [BS1] Lemma 1.3) guarantees the existence of a family $\{c_\sigma\}_{\sigma < \kappa}$ of unit of R such that $c_\tau - c_\sigma e_\sigma^{-1} \in r_\sigma \text{Ann } v$ for every $\sigma < \tau < \kappa$. Moreover, there is a homomorphism of $H(u)$ in V sending 1 to v . Hence we can choose in V elements v'_σ for every $\sigma < \kappa$, such that $v'_0 = v$ and $r_\tau r_\sigma^{-1}v'_\sigma = v'_\tau$. Now it is easy to show that the submodule of $U \oplus V$ generated by the elements $\{u_\sigma + c_\sigma^{-1}c_\tau v'_\tau, \sigma < \kappa\}$, is uniserial, it contains $u + v$ and it is pure in $U \oplus V$. ■

With the following result we will obtain a further reduction in the study of our question; namely we will be allowed to restrict the problem to the direct sum of exactly two uniserial modules.

PROPOSITION 2.5. Let M be the direct sum $\bigoplus_{i=1}^n U_i$, where the U_i 's are uniserial modules. Then M satisfies (*) is and only if $U_i \oplus U_j$ satisfies (*) for every pair $i \neq j$.

PROOF. Necessity is clear.

For the sufficiency, let $x = u_1 + u_2 + \dots + u_n$; we first notice that, by Proposition 2.4, the types $t(u_i)$ are pairwise comparable, thus, without loss of generality, we can assume $t(u_1) \leq t(u_2) \leq \dots \leq t(u_n)$. Using the same notations as in the proof of Proposition 2.4, we let $H(u_1) = \bigcup_{\sigma < \kappa} r_\sigma^{-1}R$ and U_1 generated by elements u_σ , $\sigma < \kappa$ satisfying the relations as above. Let W_j be a pure uniserial submodule of $U_1 \oplus U_j$ containing $u_1 + u_j$, for $j = 2, \dots, n$. By Proposition 2.4 there exists a homomorphism ϕ from U_1 to U_j with $\phi_j(u_1) = u_j$. Let E be the submodule of the direct sum $\bigoplus_{i=1}^n U_i$ generated by the elements $u_\sigma + \phi_2(u_\sigma) + \dots + \phi_n(u_\sigma)$ for every $\sigma < \kappa$. It is routine to check that W is uniserial, W contains x and moreover it is pure in the direct sum $\bigoplus_{i=1}^n U_i$ since it is isomorphic to U_1 via the projection π_1 . ■

Combining the two preceding Propositions and the remark after Lemma 2.3, it is easy to see that the following holds.

PROPOSITION 2.6. *Let M be a direct sum $\bigoplus_{i \in I} U_i$, with the U_i 's uniserials, satisfying $(*)$. Let $x = u_1 + u_2 + \dots + u_n$ be an element of M ; then the types $t(u_i)$ ($i \in I$) are pairwise comparable and the type $t(x)$ of x is the minimum of the types $t(u_i)$. ■*

3. Direct sum of two comparable uniserials.

As shown by Proposition 2.5 we can restrict the investigation of our problem to the case of a direct sum of two uniserial modules.

We start by considering a pair of uniserial modules subject to particular relations. We first need a definition.

DEFINITION. If U is a uniserial module we define $H(U)$ and $A(U)$ to be, respectively, the set of the height ideals and of the annihilator ideals of non zero elements of U namely:

$$H(U) = \{H_U(u) \mid 0 \neq u \in U\},$$

$$A(U) = \{\text{Ann } u \mid 0 \neq u \in U\}.$$

Two uniserial modules U and V satisfying either

$$(1) H(U) \cap H(V) \neq \emptyset \quad \text{or} \quad (2) A(U) \cap A(V) \neq \emptyset,$$

are said to be *comparable*.

In this section we will consider the case of two comparable uniserials; Theorem 3.3 will characterize the direct sums $U \oplus V$ satisfying (*) when U and V are as in the case (1), and Theorem 3.6 will settle case (2).

LEMMA 3.1. *Let U and V be uniserial modules of types $[J/I]$ and $[H/L]$ respectively. Then $H(U) \cap H(V) \neq \emptyset$ if and only if $J \cong H$ and $A(U) \cap A(V) \neq \emptyset$ if and only if $I \cong L$.*

PROOF. It is enough to observe that $J = rH$ implies that r^{-1} belongs to H and thus rH is the height ideal of a non zero element of V ; moreover $I = rL$ implies that r does not belong to I hence $r^{-1}I$ is the annihilator of a non zero element of U . ■

REMARK. The height ideals (annihilator ideals) of the elements in a uniserial module are all isomorphic, hence we can define the isomorphism class of the height ideals (annihilator ideals) of a uniserial module, Lemma 3.1 says that if U and V are comparable uniserial modules, then the isomorphism class of the height ideals (annihilator ideals) of U and V are the same.

We prove now a general result on uniserial modules.

LEMMA 3.2. *Assume U, V are uniserial modules and ϕ is a non zero epimorphism of U onto V . Then, for every $u \in U, v \in V, t(u)$ and $t(v)$ are comparable if and only if $\text{Ker } \phi \leq U^c$.*

PROOF. Assume $\text{Ker } \phi > U^c = \bigcap_{r \in U^\#} U[r]$; then there exists $r \in U^\#$ such that $\text{Ker } \phi > U[r]$. Take an element u of U not in $\text{Ker } \phi$; then $ru \neq 0$ and $H_U(ru) > H_U(u)$ moreover, since ϕ is epic and $\phi(u) \neq 0$, we have that $H_U(u) = H_V(\phi(u))$. Now the annihilator ideal of ru is $r^{-1} \text{Ann } u$ and we claim that it is properly contained in the annihilator ideal of $\phi(u)$. In fact it is easy to see that $U[r] = (r^{-1} \text{Ann } u)u$ and that $\text{Ker } \phi = (\text{Ann } \phi(u))u$ hence the claim follows. This leads to the contradiction that the types of ru and of $\phi(u)$ are not comparable.

For the converse, assume $\text{Ker } \phi \leq U^c$ and $u \in U, v \in V$ be non zero elements; we prove that their types are comparable.

Let $u' \in U$ be such that $\phi(u') = v$. If $t(u) \leq t(u')$, then obviously $t(u) \leq t(v)$. If $t(u') < t(u)$, then $H(u') \leq H(u)$ and there is $r \in R$ such that $ru' = u$; but $H(u') = H(v)$ since ϕ is epic. We have thus to prove

that, if $H(v) < H(u)$, then also $\text{Ann } v \leq \text{Ann } u$. Assume $H(v) < H(u)$, then $r \in U^\#$ and, by hypothesis, $\text{Ker } \phi \leq U[r]$. But, as noticed above, $\text{Ker } \phi = (\text{Ann } v)u'$ and $U[r] = (r^{-1}\text{Ann } u')u'$ hence the conclusion follows. ■

We will consider now the case of two uniserial modules with the same isomorphy class of height ideals.

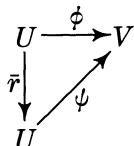
THEOREM 3.3. *Let U, V be uniserial modules of types $[J/I]$ and $[H/L]$ respectively with $J \cong H$. Then $U \oplus V$ satisfies $(*)$ if and only if there is an epimorphism ϕ of U into V such that $\text{Ker } \phi \leq U^c$ (or the same holds changing the roles of U and V).*

PROOF. First we prove the necessary condition.

By the proof of Lemma 3.1, there exist non zero elements $u \in U$ and $v \in V$ with the same height ideal. By Proposition 2.4 (ii), the types of u and v are comparable and assuming $\text{Ann } u \leq \text{Ann } v$ there is a homomorphism from U to V sending u to v . By the assumption on the height ideals of the elements, this homomorphism has to be surjective; thus by Lemma 3.2, the conclusion follows.

We prove now the sufficient condition.

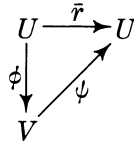
Let $u \in U$ and $v \in V$, we have to show that $u + v$ is contained in a pure uniserial submodule of $U \oplus V$. Clearly we can assume that u and v are non zero elements and that ϕ is non zero. By Proposition 2.4 we have to show that $[U, u]$ and $[V, v]$ are comparable. Notice that, by Lemma 3.2, the types of u and v are comparable. Assume first $t(u) \leq t(v)$, and consider $u' \in U$ such that $\phi(u') = v$, then $H(u') = H(v)$. If $t(u) \leq t(u')$, there is $r \in R$ such that $ru = u'$. Hence the multiplication by r followed by ϕ is a homomorphism ψ sending u to v . We have to consider now the case $t(u') < t(u)$; since $t(u') < t(u) \leq t(v)$ we have $H(u') = H(u) = H(v)$ and $\text{Ann } u' < \text{Ann } u \leq \text{Ann } v$. Thus there is $r \in U_\# \setminus U^\#$ such that $ru = u$. Hence \bar{r} , the multiplication by r , is surjective on U and we can consider the following diagram;



where $\text{Ker } \bar{r} = (\text{Ann } v)u'$ is contained in $\text{Ker } \phi = (\text{Ann } v)u'$. Thus, there is ψ making the diagram commutative. Hence $\psi(u) = \psi(ru') = \phi(u') = v$.

It remains to consider the case $t(v) < t(u)$; let again $\phi(u') = v$,

then $t(u') \leq t(v) \leq t(u)$. Hence there is $r \in R$ such that $u = ru'$. We can consider the following diagram:



Since ϕ is surjective there exists ψ making the diagram commutative if and only if $\text{Ker } \phi \leq \text{Ker } \bar{r}$. But this is true since $\text{Ker } \phi = (\text{Ann } v)u'$, $\text{Ker } \bar{r} = (\text{Ann } u)u'$ and $t(v) < t(u)$. Thus $\psi(v) = \psi(\phi(u')) = ru' = u$. ■

By means of the characterizations of the non standard quotients of non standard uniserial modules proved in [BS1] it is clear that the following holds.

COROLLARY 3.4. *Let U, V be non standard uniserial modules of types $[J/I]$ and $[H/L]$ respectively with $J \cong H$. Then $U \oplus V$ satisfies (*) if and only if there is an epimorphism ϕ of U into V (or viceversa).*

We prove now a result dual to Lemma 3.2.

LEMMA 3.5. *Let U, V be uniserial modules and ϕ a non zero embedding of U into V . For every $u \in U, v \in V, t(u)$ and $t(v)$ are comparable if and only if $\text{Im } \phi \geq V_{\#}V$.*

PROOF. Let $\text{Im } \phi \geq V_{\#}V$ and let $u \in U, v \in V$; we prove that the types of u and v are comparable. Consider $\phi(u) = v'$ then, if $t(v') \leq t(v)$ we obviously obtain $t(u) \leq t(v)$. Thus assume $t(v') > t(v)$; we can write $rv = v'$ for an element $r \in R$, hence $\text{Ann } v = r \text{Ann } v'$ and, since ϕ is injective, $\text{Ann } v = r \text{Ann } u$. Thus in order to prove that $t(u)$ and $t(v)$ are comparable we have to show that, if $r \in V_{\#}$ then $H_V(v) \leq H_U(u)$. Let $sx = v$ be solvable in V ; by hypothesis we know that $rx \in \text{Im } \phi$, hence $rx = \phi(y)$ for an element y in U . Now $s\phi(y) = srx = rv = \phi(u)$ and by the injectivity of ϕ we obtain $sy = u$, hence s divides u in U .

For the converse take $v \in V, r \in V_{\#}$ and assume, by way of contradiction, that $rv \notin \text{Im } \phi$. Fix $0 \neq u_0 \in U$; then there exists $s \in R$ such that $srv = \phi(u_0)$. Now the annihilator of sv is $r \text{Ann } \phi(u_0)$ which is equal to $r \text{Ann } u_0$ since ϕ is injective; moreover $\text{Ann } sv$ is properly contained in the annihilator of u_0 since $r \in V_{\#}$. Thus by hypothesis, the height ideal of sv in V is contained in the height ideal of u_0 in U ; this implies that there is u_1 in U such that $su_1 = u_0$. Since $rv \notin \text{Im } \phi$, there is a non unit

$t \in R$ such that $trv = \phi(u_1)$ and multiplying this equality by s we obtain $t\phi(u_0) = \phi(su_1) = \phi(u_0)$ hence $\phi(u_0) = 0$ contrary to the assumption $u_0 \neq 0$. ■

We consider now the case of two uniserial modules with the same isomorphism class of annihilator ideals.

THEOREM 3.6. *Let U, V be uniserial modules of types $[J/I]$ and $[H/L]$ respectively with $I \cong L$. $U \oplus V$ satisfies $(*)$ if and only if there is an embedding ϕ of U into V such that $\text{Im } \phi \geq V_{\#} V$ (or the same holds changing the roles of U and V).*

PROOF. First the necessary condition.

By the proof of Lemma 3.1, there exist non zero elements $u \in U$ and $v \in V$ with the same annihilator ideal. By Proposition 2.4 (ii), the types of u and v are comparable and assuming $H(u) \leq H(v)$ there is a homomorphism from U to V sending u to v . Clearly this homomorphism is injective (since the elements have the same annihilator ideal); thus by Lemma 3.5, the conclusion follows.

We prove now the sufficient condition.

Let $u \in U$ and $v \in V$, by Lemma 3.5 the types of u and v are comparable. In case ϕ is an isomorphism, then condition (ii) of Proposition 2.4 is satisfied, hence the claim follows.

Thus we can assume V is standard; in fact assume V is non standard then $V_{\#} V = V$ (if $V^{\#} < V_{\#}$ this is clear; if $V^{\#} = V_{\#}$, then V is equiannihilated, see [BS1]) and thus ϕ is an isomorphism.

Since V is standard so is U and thus, by Proposition 2.4 (iii) we have only to consider the case in which $H_U(u) = H_V(v)$. If $H_U(u) = H_V(\phi(u))$ then ϕ is an isomorphism and we are done. Thus we can assume $H_V(\phi(u)) > H_U(u) = H_V(v)$. In this case $\phi(u) = rv$ for an element $r \in R$ and $\text{Ann } v = r \text{Ann } \phi(u) = r \text{Ann } u$. Now since U and V are standard we have $U \cong H(u)/\text{Ann } u$ and $V \cong H(v)/\text{Ann } v$; hence there exists an epimorphism of V into U sending v to u and thus the conclusion follows. ■

In case the uniserials considered are non standard, the situation becomes simpler, as it is shown by the following.

COROLLARY 3.7. *Let U, V be non standard uniserial modules of types $[J/I]$ and $[H/L]$ respectively with $I \cong L$. $U \oplus V$ satisfies $(*)$ if and only if U and V are isomorphic.*

PROOF. Obvious since the embedding ϕ of Corollary 3.9 has to be surjective. ■

If we consider two uniserial modules with the same isomorphy class both of annihilators and of height ideals, we have:

PROPOSITION 3.8. *Let U and V be uniserial modules of the same type and assume $U \oplus V$ satisfies (*). Then U and V are isomorphic.*

PROOF. By definition of the type of a uniserial module, there are elements $u \in U$ and $v \in V$ such that $t(u) = t(v)$. By Proposition 2.4 (ii), there exists a homomorphism from U to V sending u to v ; but, this homomorphism is necessarily bijective since $t(u) = t(v)$. ■

4. Direct sums of two non comparable uniserials.

We fix the hypothesis that will be assumed throughout this section.

U and V will be uniserial modules of type $[J/I]$ and $[H/L]$ respectively, with $J \not\cong H$ and $I \not\cong L$, i.e., by Lemma 3.1, $H(U) \cap H(V) = \emptyset$ and $A(U) \cap A(V) = \emptyset$.

Two uniserial modules U and V satisfying the above conditions, are said to be *non comparable*.

LEMMA 4.1. *Let U and V be non comparable uniserial modules of types $[J/I]$ and $[H/L]$ respectively. The following are equivalent:*

- (i) $L : I = H : J$.
- (ii) $I : L = J : H$.
- (iii) *For every $u \in U, v \in V, t(u)$ and $t(v)$ are comparable.*

PROOF. (i) \Rightarrow (ii) Let $x \in I : L$, then $xL \leq I$; since $I \not\cong L$ we have that $xL < I$, hence x^{-1} does not belong to $L : I = H : J$. This means $x^{-1}J > H$, thus $xH < J$ and $x \in J : H$. The converse inclusion is proved in a similar way by using $J \not\cong H$.

(ii) \Rightarrow (iii) Notice that if $u \in U$, then $\text{Ann } u = qI$ and $H_U(u) = qJ$ for an element q such that $q^{-1} \in J \setminus I$, and analogously if $v \in V, \text{Ann } v = pL$ and $H_V(v) = pH$ for an element p such that $p^{-1} \in H \setminus L$.

Since $I \not\cong L, qI \leq pL$ if and only if $qI < pL$ and this is equivalent to $qI : pL < R$; but, by (ii) $qI : pL = qJ : pH$, hence the types of u and v are comparable.

(iii) \Rightarrow (i) We first prove that $L : I \leq H : J$. Let $qI \leq L$, then $qI < L$; if $q^{-1} \in J \setminus I$ then, by the above remark, qI is the annihilator of an element u of U and qJ is the height ideal of u . Let $v_0 \in V$ be of type

(H, L) , then $t(u) < t(v_0)$, hence $qJ < H$. If $q^{-1} \notin J$ then $qJ < R$ hence again $qJ < H$; the possibility $q^{-1} \in I$ cannot occur since in that case we would have $R \leq qI < L$ which is absurd.

We prove now that $H : J \leq L : I$. Let $qJ < H$, if $q^{-1} \in J \setminus I$ then we argue as above. If $q^{-1} \notin J$ but $q \notin I$ then $q^{-1}I$ is the annihilator of an element u of U ; if it were $L < q^{-1}I$ then $H < q^{-1}J$, since (H, L) is the type of the element v_0 of V defined above, hence it must be $pL < I$. If $q \in I$, then clearly $qL < I$. ■

REMARK. If U and V are as above and satisfy one of the equivalent condition of Lemma 4.1, then none of them can be divisible. In fact, assume V is divisible, then condition (i) yields $L : I = Q$ which is impossible since L is the annihilator of an element of V . If U is divisible we argue in the same way using condition (ii).

PROPOSITION 4.2. *Let U and V be non comparable standard uniserial modules of types $[J/I]$ and $[H/L]$ respectively. $U \oplus V$ satisfies $(*)$ if and only if one of the equivalent conditions of Lemma 4.1 is satisfied.*

PROOF. The necessary condition is clear by Proposition 2.4. For the sufficiency we observe that condition (iii) of Proposition 2.4 is satisfied for every pair of elements $u \in U$ and $v \in V$ since U and V are standard. ■

We will consider now a mixed case, namely the case in which one of the two modules is standard and the other is non standard.

We will have to use some of the results proved in [BS1] and [BS2] to characterize the various classes of non standard uniserial modules.

For convenience, we summarize here the definitions of the classes \mathcal{U}_i ($i = 1, 2, \dots, 6$) and their properties.

Let U be a non standard uniserial module of type $[J/I]$, the following hold:

(A) $U \in \mathcal{U}_1$ if and only if U is divisible.

Then necessarily $0 = \text{Ann } U = U^\# < U_\#$ and $U^c = U$. U is strongly non standard i.e. every non zero quotient of U is non standard. (Notice that, by the remark after Lemma 4.1, the uniserial modules we are dealing with in this Section cannot be divisible).

(B) $U \in \mathcal{U}_2$ if and only if $0 < \text{Ann } U = U^\# < U_\#$.

Then necessarily $U^c = U$ and $UU^\# < U$, i.e. $JJ^\# < J$, hence J is principal over $R_{U^\#}$ (see 1.3). U is again strongly non standard.

- (C) $U \in \mathcal{u}_3$ if and only if $0 < \text{Ann } U < U^\# < U_\#$ and U/U^c is standard.
Necessarily $0 < U^c < U$ and $UU^\# < U$, i.e. J is principal over $R_{U^\#}$.
- (D) $U \in \mathcal{u}_4$ if and only if $0 < \text{Ann } U < U^\# < U_\#$ and U/U^c is non standard. Then $0 < U^c < U$ and $UU^\# = U$, hence J is not principal over $R_{U^\#}$.
- (E) $U \in \mathcal{u}_5$ if and only if $0 < \text{Ann } U < U^\# = U_\#$ and $0 < U^c$.
In this case we have: U/U^c is non standard, $UU^\# = U$ (hence J is not principal over $R_{U^\#}$) and $I \cong U^\#$.
- (F) $U \in \mathcal{u}_6$ if and only if $0 < \text{Ann } U < U^\# = U_\#$ and $U^c = 0$.
In this case we have: $UU^\# = U$ (hence J is not principal over $R_{U^\#}$) and $I \not\cong U^\#$. U is barely non standard, i.e. every proper quotient of U is standard.
- (G) $U \in \mathcal{u}_{66}$ if and only if $U \in \mathcal{u}_6$ and $I \not\cong R_{U^\#}$.

We will use also the following facts:

- (H) If $U^c < K \leq U$, then U/K is standard. ([BS1], Proposition 2.1).
If $K < U^c \leq U$, then U/K is non standard. ([BS1], Proposition 2.1 and 2.8).

PROPOSITION 4.3. *Let U and V be non comparable uniserial modules of types $[J/I]$ and $[H/L]$ respectively with U non standard. Assume $U \oplus V$ satisfies $(*)$. The following conditions hold:*

- (i) $I:L = J:H$.
- (ii) If $V_\# > U^\#$, then $U \in \mathcal{u}_{66}$.
- (iii) If $V_\# = U^\#$ and the annihilator ideals of the elements of V are principal over $R_{V_\#}$, then $U \in \mathcal{u}_2 \cup \mathcal{u}_3 \cup \mathcal{u}_{66}$.

PROOF. (i) Obvious from Proposition 2.4 and Lemma 4.1.

(ii) First we prove that $U \in \mathcal{u}_6$, i.e. that $U^c = U[U^\#] = 0$ (see (F)). Assume, by way of contradiction that $U^c \neq 0$, then we claim that $U^c = U^\#$. In fact this is clear if $U^c = U$ (by (A) and (B)); if $U^c < U$, then it is easy to see that U^c is isomorphic to $I:U^\#/I$ (see [BS3] Remark after Lemma 1.5). Letting I be the annihilator of an element of U^c , we can assume $I \geq U^\#$, hence by Lemma 1.6 (i), we obtain $I:U^\# = R_{U^\#}$ and $\text{Ann } U^c = \text{Ann } R_{U^\#}/I = I:R_{U^\#} = U^\#$, as claimed.

Now $V_{\#}$ is the union of the annihilator ideals of the non zero elements of V , hence the hypothesis $V_{\#} > U^{\#}$ implies the existence of elements $v_0 \in V$ and $u_0 \in U^c$ such that if $L = \text{Ann } v_0$ and $I = \text{Ann } u_0$, then $L > I \geq U^{\#}$. By Proposition 2.4 ($t(v_0) > t(u_0)$), hence $H(v_0) > H(u_0)$ (remember that $H(U) \cap H(V) = \emptyset$). Thus U/Lu_0 is a non zero standard quotient of U (by 2.4 (iii)). But $Lu_0 < Ru_0 \leq U^c$, hence, by (H), U/Lu_0 is non standard, a contradiction.

It remains now to prove that $U \in \mathcal{U}_{66}$, i.e. that, if I is the annihilator ideal of an element u_0 of U then I is not isomorphic to $R_{I^{\#}} = R_{U^{\#}}$ (see (G)).

Since $L^{\#} = V_{\#} > U^{\#} = U_{\#} = I^{\#}$, we can choose $v_0 \in V$ such that $\text{Ann } v_0 = L > U^{\#}$; hence, by Lemma 1.4 (i), $I:L = I$. By (i) we have then that $I = J:H$ where J and H are the height ideals of u_0 and v_0 respectively.

Assume $I \cong R_{U^{\#}}$, then $J:H \cong R_{U^{\#}} = R_{J^{\#}}$; Lemma 1.6 (ii) yields $J \cong R_{U^{\#}}$ (since $J \not\cong H$), which is impossible by (F). Hence $U \in \mathcal{U}_{66}$.

(iii) By the remark after Lemma 4.1 U cannot belong to \mathcal{U}_1 and by (B) and (C) above, if $U \in \mathcal{U}_2 \cup \mathcal{U}_3$, then J is principal over $R_{U^{\#}}$.

Thus we must prove that if J is not principal over $R_{U^{\#}}$, then $U \in \mathcal{U}_{66}$. But this amounts to show that $U \in \mathcal{U}_6$; in fact by hypothesis $I \not\cong L$ and L is principal over $R_{V_{\#}} = R_{U^{\#}}$ (see (G)).

Assume by way of contradiction that J is not principal over $R_{U^{\#}}$ and $U \notin \mathcal{U}_6$, then $U^c \neq 0$. Let u_0 be a non zero element of U^c , we can choose $I = \text{Ann } u_0$, hence $I \geq U^{\#} = J^{\#}$. Now $t(V) = [H/L]$ where $L = sR_{U^{\#}}$ with $s \in U^{\#}$ and $s^{-1}H > R_{U^{\#}}$.

Since J is not principal over $R_{U^{\#}}$, we certainly have that $J > R_{U^{\#}}$ and, since $s^{-1}H > R_{U^{\#}}$, there exists an element $r \in U^{\#}$ such that $r^{-1} \in s^{-1}H \cap J$. This means that $r^{-1} \in J$ and $r^{-1}s \in H \setminus L$, hence we can find $u_1 \in U$ and $v_1 \in V$ with $\text{Ann } u_1 = rI$ and $\text{Ann } v_1 = rR_{U^{\#}}$. Clearly $rI < rR_{U^{\#}}$, hence by Proposition 2.4 (iii) $U/rR_{U^{\#}}u_1$ is standard.

We claim that $rR_{U^{\#}}u_1 = U^c$.

In fact, $U^c < Ru_1$ since $\text{Ann } u_1 = rI$ is properly contained in $U^{\#}$ ($r \in U^{\#}$); hence $U^c = \{tu_1 \mid \text{Ann } tu_1 = t^{-1}rI \geq U^{\#}\} = (rI:U^{\#})u_1$. Applying Lemma 1.6 (i) we obtain that $rI:U^{\#} = rR_{U^{\#}}$, since $I \geq U^{\#}$.

Thus we have that U/U^c is standard which is impossible, by the hypothesis J not principal over $R_{U^{\#}}$ (see (D), (E) and (F)). ■

REMARK. Let U and V be as in Proposition 4.3 (iii) with $U \in \mathcal{U}_2 \cup \mathcal{U}_3$. Then $U^{\#}$ is principal over $R_{U^{\#}}$.

In fact, by (B) and (C), J is principal over $R_{U^{\#}}$ and $I^{\#} > U^{\#}$; moreover, by Proposition 4.3 (i), $I:L = J:H$.

Using Lemma 1.4(ii) we obtain $I:L \cong U^\#$, hence $J:H \cong U^\#$ and, by Lemma 1.2, $U^\# \leq H^\#$.

Assume first that $U^\# < H^\#$, then by Lemma 1.4(i), $J:H \cong J \cong R_{U^\#}$, hence $U^\# \cong R_{U^\#}$.

If $U^\# = H^\#$, then by Lemma 1.5, $J:H \cong R_{U^\#}(R_{U^\#}:H) = R_{U^\#}:H$. Thus we have that $R_{U^\#}:H \cong U^\#$; multiplying by H and using Lemma 1.3 we obtain $H \cong U^\#$ (remember that H is not principal over $R_{U^\#}$ since $J \not\cong H$). But then $J:H \cong R_{U^\#}:U^\# \cong R_{U^\#}$; hence again we conclude that $U^\# \cong R_{U^\#}$.

THEOREM 4.4. *Let U and V be a non comparable uniserial modules of types $[J/I]$ and $[H/L]$ respectively with U non standard and V standard. $U \oplus V$ satisfies $(*)$ if and only the conditions (i), (ii) and (iii) of Proposition 4.3 are satisfied.*

PROOF. The necessary condition follows by Proposition 4.3.

To prove the sufficient condition we show that for each pair of elements $u \in U$ and $v \in V$ condition (iii) of Proposition 2.4 is satisfied.

Observe that the condition $I:L = J:H$ guarantees, by Lemma 4.1, that the types of elements of U and V are comparable.

Let $u \in U$, $v \in V$ and $L = \text{Ann } v$, $I = \text{Ann } u$.

If $L < I$ then condition (iii) of Proposition 2.4 is satisfied since V is standard. Assume $L > I$, we have to prove that U/Lu is standard. We proceed by steps.

(1) If $V_\# > U^\#$, then $U \in \mathcal{U}_{66}$, hence the conclusion follows. (U is barely non standard).

(2) If $L^\# = V_\# < U^\#$ or $V_\# = U^\#$ and L is not principal over $R_{V_\#}$, then $LU^\# = L$ (see Lemma 1.3), hence $LU^\#u \neq 0$ namely $Lu > U^c$. This yields U/Lu standard, by (H).

(3) It remains to consider the case $L^\# = V_\# = U^\#$ and L principal over $R_{V_\#}$.

By hypothesis $U \in \mathcal{U}_2 \cup \mathcal{U}_3$ or $U \in \mathcal{U}_{66}$. If $U \in \mathcal{U}_{66}$ the conclusion follows.

Notice that the possibility $U \in \mathcal{U}_2$ cannot occur. In fact, if $U \in \mathcal{U}_2$, then $U^\# = \text{Ann } U$ (by (B)), hence for every $u \in U$, $\text{Ann } u > U^\#$. But $L \leq L^\# = U^\#$, thus $L \leq \text{Ann } u$, contrary to our assumption.

Thus we have to prove that, if $U \in \mathcal{U}_3$ and $L = \text{Ann } v > I = \text{Ann } u$, then U/Lu is standard, i.e., by (C) and (H) that $Lu \geq U^c$.

We have $I < L \leq U^\#$, hence $U^c = \{tu \mid t^{-1}I \geq U^\#\} = (I:U^\#)u$.

We show that, if $t \in I:U^\#$, then $t \in L$. Notice that $tU^\# \leq I$ implies $t^{-1}I > U^\# = L^\#$ (since $U^\# < U_\# = I^\#$, by (C)); by the proof of Lemma

1.4 (i) we infer that $L:I = t^{-1}L$. But $L:I$ contains R , since $I < L$; hence $t \in L$, otherwise $t^{-1}L < R$. ■

We consider now the case of two non standard uniserial modules and we will see that the conditions in Proposition 4.3 show a certain degree of symmetry.

THEOREM 4.5. *Let U and V be non comparable uniserial modules of types $[J/I]$ and $[H/L]$ respectively with U and V non standard. $U \oplus V$ satisfies $(*)$ if and only if the following are satisfied:*

(i) $I:L = J:H$.

(ii) *At least one between U and V belongs to \mathcal{U}_{66} ; if U belongs to \mathcal{U}_{66} and V does not then $U_{\#} \leq V^{\#}$.*

PROOF. First we prove the necessary condition.

(i) Is satisfied by Proposition 4.3.

(ii) Assume none between U and V belong to \mathcal{U}_{66} . Proposition 4.3 yields $U_{\#} \leq V^{\#}$ and $V_{\#} \leq U^{\#}$. But $U_{\#} \geq U^{\#}$ and $V_{\#} \geq V^{\#}$, since U and V are non standard, hence $U_{\#} = V^{\#} = V_{\#} = U^{\#}$ and thus $U, V \in \mathcal{U}_5 \cup \mathcal{U}_6$ (see (E) and (F)).

By Proposition 4.3 (iii) and our assumption, the annihilator ideals of the elements in U and V are not principal over $R_{U_{\#}}$ or $R_{V_{\#}}$ respectively, otherwise U or V belong to \mathcal{U}_{66} ; hence if U (or V) belongs to \mathcal{U}_6 , then it belongs to \mathcal{U}_{66} , a contradiction. Thus the only possibility is that both U and V belong to \mathcal{U}_5 . But this means that I and L are isomorphic to $U_{\#} = V_{\#}$ (by (E)) contradicting the hypothesis $I \neq L$.

The second part of (ii) follows by Proposition 4.3 (ii).

To prove sufficiency we argue as in the prove of the sufficient condition of Theorem 4.4, i.e. we use Proposition 2.4 (iii).

Clearly we have to consider only the case in which one between U or V does not belong to \mathcal{U}_{66} . Assume V does not belong to \mathcal{U}_{66} , but U does (hence $V^{\#} \geq U_{\#}$) and let $u \in U$, $v \in V$ be such that $L = \text{Ann } v$, $I = \text{Ann } u$.

We must show that, if $L < I$, then V/Iv is standard. We first notice that V cannot be strongly non standard, since otherwise we would have (by (A) and (B)) $\text{Ann } V = V^{\#} \geq U_{\#} = I^{\#}$, and thus for every pair of elements $u \in U$, $v \in V$, $\text{Ann } u < I^{\#} \leq \text{Ann } v$. By the hypothesis $I^{\#} = U_{\#} \leq V^{\#}$ and by the fact that I is not isomorphic to $R_{U_{\#}}$ we obtain that $IV^{\#} = I$ (using Lemma 1.3 in case $I^{\#} = V^{\#}$) which amounts to $(IV^{\#})v \neq 0$ or equivalently to $Iv > V^c$, hence the conclusion follows by (H). ■

5. Summary of the preceding results and examples.

The results proved up to now answer completely to our mean question (see the introduction), but we preceded by steps considering different cases, hence we think it convenient to summarize in a Theorem all the results obtained.

THEOREM 5.1. *Let M be the direct sum $\bigoplus_{i \in I} U_i$, where U_i is a uniserial module for each $i \in I$.*

M has the property that every element is contained in a pure uniserial submodule if and only if for every pair $i, j \in I$ the summands U_i, U_j satisfy one of the following conditions:

(A) *There is an epimorphism of U_i onto U_j whose kernel is contained in U_i^c (or the same holds changing the roles of i and j).*

(B) *There is an embedding of U_i onto U_j whose image contains $(U_j)_\# U_j$ (or the same holds changing the roles of i and j).*

(C) *If $t(U_i) = [J/I]$, $t(U_j) = [H/L]$ then $I \not\cong L$, $J \not\cong H$ and the following are satisfied:*

(i) $I:L = J:H$.

(ii) *If U_i is non standard and U_j is standard, then:*

(a) *if $(U_j)_\# > (U_i)_\#$, U_i belongs to \mathcal{U}_{66} ,*

(b) *if $(U_j)_\# = (U_i)_\#$ and the annihilator ideals of the elements of U_j are principal over R_L^* , U_i belongs to $\mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_{66}$.*

(iii) *If U_i and U_j are non standard, then:*

(a) *at least one between U_i and U_j belongs to \mathcal{U}_{66} , and*

(b) *if U_i belongs to \mathcal{U}_{66} and U_j does not, then $(U_i)_\# \leq (U_j)_\#$.*

Moreover if one of the preceding conditions holds, then the type of an element in M is the minimum of the types of its components.

We give now examples of direct sum of uniserial modules satisfying the preceding Theorem.

We emphasize that, by Proposition 2.5, the direct sum $\bigoplus_{i=1}^n U_i$ satisfies (*) if and only if $U_i \oplus U_j$ does, for every pair $i \neq j$.

Thus we give here examples of uniserial modules U and V such that $U \oplus V$ satisfies (*). Theorems 3.3 and 3.6 give already satisfactory relations between U and V in case they are comparable uniserials. Hence the interesting examples concern the remaining case, i.e. case (C) in Theorem 5.1.

To give examples of non comparable uniserial modules we have to

find fractional ideals I, L, J, H satisfying $I:L = J:H$ and we would like to find equivalent formulations for this condition. We can prove the following.

LEMMA 5.2. *Let U and V be non comparable uniserial modules of types $[J/I]$ and $[H/L]$ respectively and assume they are non finitely annihilated. The following are equivalent:*

- (i) $I:L = J:H$.
- (ii) $JL = IH$.

PROOF. Without loss off generality we can assume $I^\# \leq L^\#$ (use Lemma 4.1).

(i) \Rightarrow (ii) Assume first $I^\# < L^\#$; then there is an element $s \notin L$ such that $s^{-1}L > I^\#$, hence we can assume $L > I^\#$. By Lemma 1.4 we have $I = I:L = J:H$. Thus $HI \leq J$; moreover, if $r \notin I$ then $rH > J$, hence $H > r^{-1}J$ and $H \geq \text{Lev } U$. By [BFS], Lemma 2.2 we obtain $HI \geq (\text{Lev } U)I = J$. We have so proved that $HI = J$, but $JL = J$ since $L > I^\# \geq J^\#$ hence $HL = JL$.

Let now $I^\# = L^\#$; by Lemma 1.2, $I^\# = (I:L)^\# = (J:H)^\# = J^\# \wedge H^\#$. But U and V are non finitely annihilated, hence $I^\# \geq J^\#$ and $L^\# \geq H^\#$; thus we have that $I^\# = J^\# = H^\# = L^\#$ and we can assume $I^\# = P$. Using Lemma 1.5(ii) we have $I:L = I(R:L)$ and $J:H = J(R:H)$. (since $I \neq L$ and $J \neq H$).

Multiplying $I(R:L)$ by LH we obtain IH since $(R:L)L$ is R or P , by Lemma 1.3, and $HP = H$ otherwise V is cyclic; multiplying $J(R:H)$ by HL we obtain JL , since J and H cannot be principal and thus $(R:H)H = P$ and $JP = J$.

(ii) \Rightarrow (i) First assume $I^\# < L^\#$ and let, as above, $L > I^\#$ then $I:L = I$.

We prove that $J:H = I$. Notice that $J = JL$, since $L > I^\# \geq J^\#$; hence $J = HI$, by hypothesis, consequently we have $I \leq J:H$.

For the converse inclusion, we show that, if $r \notin I$, then $r \notin J:H$. Assume, by way of contradiction that $rH \leq J$, then $J = IH \leq rH \leq J$ implies $rH = J$, contrary to the hypothesis $J \neq H$.

Let now $I^\# = L^\#$. The equality $JL = HI$ together with Lemma 1.1 and the hypothesis U and V non finitely annihilated, give $J^\# = H^\#$ ($\leq I^\# = L^\#$). Without loss of generality we can assume $I^\# = L^\# = P$.

Multiplying JL by $R:L$, we obtain J . In fact $(R:L)L$ is P or R , by Lemma 1.3. Now $JP = J$, since $P = I^\# \geq J^\#$ (see [BFS], Lemma 1.2), thus we have $J = HI(R:L)$, hence $J:H \geq I(R:L) = I:L$ (by Lemma 1.5(ii)).

It remains to prove that $J:H \leq I:L = I(R:L)$. Assume, by way of contradiction that there is $r \in J:H \setminus I(R:L)$; then $J \geq rH \geq I(R:L)H = J$ but this contradicts the hypothesis $J \not\equiv H$. ■

We list now some examples.

EXAMPLE 1. Let U and V be non standard uniserial modules of types $[J/I]$ and $[H/L]$ respectively with $I \not\equiv R_{I^*}$, $I \not\equiv I^*$.

If $I = sJ$ and $L > I^*$, $s^{-1}R > H > s^{-1}J^*$, then $U \oplus V$ satisfies (*).

In fact in our hypothesis $L^* > I^*$ and $J^* > H^*$, hence $I \not\equiv L$ and $J \not\equiv H$. It is clear that $JL = J$ and $IH = s^{-1}I$ hence, by Lemma 5.2, condition (i) of Theorem 4.5 is satisfied. By our assumptions on I and J , $U \in \mathcal{U}_{66}$ and $U_{\#} = I^* < V^* = H^*$, hence also condition (ii) of the same Theorem is satisfied.

EXAMPLE 2. Let U and V be non standard uniserial modules as in Example 1. If $L > I^*$, $I^* = H^*$ and $J = IH$, then $U \oplus V$ satisfies (*).

In fact, clearly $I \not\equiv L$, since $L^* > I^*$. Assume $J = IH = rH$; multiplying both sides of the equality by $R_{H^*}:H$ it is easy to see that we obtain the contradiction $I \cong R_{I^*}$ in case $H \cong R_{H^*}$ or $I \cong I^*$ in case $H \not\equiv R_{H^*}$. Hence $J \not\equiv H$.

Now $J^* = I^*$ hence $U \in \mathcal{U}_{66}$ (see (E), (F) and (G) in Section 4), $U_{\#} = I^* = H^* = V^*$ and $JL = J = IH$. Thus the conditions of Theorem 4.5 are all satisfied.

EXAMPLE 3. Let U and V be uniserial modules of types $[J/I]$ and $[H/L]$ respectively with U non standard, V standard.

If $L^* < I$, $s^{-1}R > J > s^{-1}L^*$ and $H = s^{-1}L$, then $U \oplus V$ satisfies (*).

In fact in this situation we have: $H^* = L^* < J^* \leq I^*$, hence $I \not\equiv L$ and $J \not\equiv H$. Moreover $L:I = L = H:J$ (by Lemma 1.4) and equivalently $I:L = J:H$. We also have $V_{\#} = L^* < U_{\#} = J^*$ thus, applying Theorem 4.4 the claim is proved.

We notice that the relations between the fractional ideals defined in Examples 1 and 3 are symmetric.

EXAMPLE 4. Let U and V be uniserial modules of types $[J/I]$ and $[H/L]$ respectively with U non standard, V standard. Let I, L, J, H satisfy the same hypothesis as in Example 1 or in Example 2. The $U \oplus V$ satisfies (*).

In fact, in both situations we have $I \not\equiv L$ and $J \not\equiv H$, as shown

in Examples 1 and 2. It is routine to check that $I:L = J:H$; moreover $U \in \mathcal{U}_{66}$, hence Theorem 4.4 yields the conclusion.

EXAMPLE 5. Let U and V be standard uniserial modules of types $[J/I]$ and $[H/L]$ respectively with I, L, J, H satisfying the hypothesis of one of the preceding examples, then $U \oplus V$ satisfies $(*)$.

In fact, by Theorem 4.2, the only thing to be verified is that $I:L = J:H$; but this true as already noticed above.

6. Transitivity and full transitivity.

Using the notations of [F], we recall that if G is a p group and $x \in G$, then $h^*(x)$ denotes the generalized height of x , namely $h^*(x) = \sigma$ if $x \in p^\sigma G \setminus p^{\sigma+1} G$; the indicator $H(x)$ of an element x of order p^n denotes the sequence $(h^*(x), h^*(px), \dots, h^*(p^n x) = \infty)$. In [K], Kaplansky introduced the notion of fully transitive and of transitive group as follows. A reduced p group G is fully transitive if for any two elements x and y in G with $H(x) \leq H(y)$ there is an endomorphism of G sending x to y . G is transitive if whenever the indicators of the above elements are equal there is an automorphism of G sending one element to the other (see [F]).

We would like to generalize the notion of fully transitivity and the one of transitivity to modules over valuation domains. Obviously a suitable definition of the indicator of an element is needed.

The analogous question for torsion free abelian groups has been studied by Metelli. In [M] an appropriate definition of height of an element in a torsion free group is considered and it is proved that the class of separable torsion free groups is fully transitive with respect to this height.

In [FS1] the height and the indicator of an element are defined: in particular given an element x in a module M with height ideal J , the height of x is J/R if x is contained in a standard uniserial module of type $[J/\text{Ann } x]$, otherwise it is denoted by $(J/R)^-$ and it is called non limit height. Thus if the element x is contained in some non standard uniserial submodule of M of type $[J/\text{Ann } x]$, the above notion of height does not distinguish between non isomorphic uniserials of the same type. Hence we think that a finer definition is needed, but the problem presents difficult aspects and we leave it to future investigations.

We can observe that the situation becomes much simpler for the direct sums of uniserial modules that we are considering in this paper. In fact in these direct sums every elements is contained in a pure uniserial submodule which is unique up to isomorphism and is a summand, as it

has been showed in Section 2. Moreover any two summands of the same type are isomorphic, as shown by Proposition 3.8.

Hence in order to measure the divisibility of an element, it is enough to consider its height and its annihilator, namely its type.

Thus the notion of fully transitivity or the one of transitivity can be easily generalized as follows.

DEFINITION. Let M be a direct sum of uniserial modules satisfying $(*)$. M is *fully transitive* (*transitive*) if and only if for any two elements x and y in M with $t(x) \leq t(y)$ ($t(x) = t(y)$) there is an endomorphism (automorphism) of M sending x to y .

We can thus prove.

THEOREM 6.1. Let $M = \bigoplus_{i \in I} U_i$ be a direct sum of uniserial modules satisfying $(*)$. Then M is both fully transitive and transitive.

PROOF. Let $x = x_1 + x_2 + \dots + x_n$ and $y = y_1 + y_2 + \dots + y_n$ be two elements of M such that $t(x) \leq t(y)$ (or $t(x) = t(y)$). We have to find an endomorphism (or an automorphism) of M sending x to y . By hypothesis there exist pure uniserial submodules U and V of M containing x and y respectively; moreover, by Lemmas 2.1 and 2.2, we may assume that U and V are isomorphic to the summands U_1 and U_j respectively, via the corresponding projections π_1 and π_j .

Then we have that $t(x) = t(x_1)$, $t(y) = t(y_j)$ and $t(x_1) \leq t(y_j)$. If $1 = j$ then obviously, there is an endomorphism φ of U_1 sending x_1 to y_j ; if $i \neq j$, such a φ exists by Proposition 2.4. Hence $\phi = \pi_{j-1} \circ \varphi \circ \pi_1$ is a homomorphism from U to V sending x to y and obviously ϕ is extendible to an endomorphism of M , since U is a summand. If $t(x_1) = t(y_j)$, then clearly φ and ϕ are isomorphisms; moreover it is easy to show that ϕ is extendible to an automorphism of M since U and V are summands of M isomorphic, via projections, to the summands U_1 and U_j . ■

We consider now a direct sum of uniserial modules satisfying the condition:

(a) for any two elements x and y in M with $t(x) \leq t(y)$ there is an endomorphism of M sending x to y .

Theorem 6.1 states that $(*)$ implies condition (a), and we show now, with an example, that they are not equivalent.

EXAMPLE. Let R be the discrete valuation ring \mathbb{Z}_p and $M = \mathbb{Z}(p) \oplus \mathbb{Z}(p^3)$. M does not satisfy $(*)$. In fact it is the sum of two uniserials of type $[R/pR]$ and $[R/p^3R]$ respectively, and every epimorphism from

$\mathbb{Z}(p^3)$ onto $\mathbb{Z}(p)$ has kernel $\mathbb{Z}(p^2)$, hence M does not satisfy Theorem 3.3, since $(\mathbb{Z}(p^3))^e = \mathbb{Z}(p)$.

An easy computation shows that, given two element x and y in M such that $t(x) \leq t(y)$, the indicator of x is less or equal to the indicator of y . Thus M satisfies (a) since it is separable (see [F]).

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