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MICOL AMAR
VIRGINIA DE CICCO

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The Uniqueness as a Generic Property for Some One-Dimensional Segmentation Problems.

Micol Amar - Virginia De Cicco (*)

Abstract - We give a uniqueness result concerning the minimizers of the functional proposed by Mumford and Shah in order to study the problem of image segmentation in Computer Vision Theory. Our result concerns the model case in dimension one. It is easy to see that the uniqueness of this minimum problem does not hold, but we state that it is a «generic property» in the sense that for «almost all» the grey-level functions and the parameters of the problem, the minimum point is unique.

1. Introduction.

Given a function $g \in L^2(\Omega)$, with Ω an open bounded subset of $\boldsymbol{R}^n$, and three real numbers $\alpha, \beta, \gamma \in (0, +\infty]$, let us consider the functional

$$F^g_{\alpha, \beta, \gamma}(u) = \alpha \int_\Omega |\nabla u|^2 dx + \beta \int_\Omega |u - g|^2 dx + \gamma H^{n-1}(S_u) \tag{1.1}$$

where S_u is the jumping set of the function u and H^{n-1} is the $n-1$ Hausdorff measure on $\boldsymbol{R}^n$.

We can associate to $F^g_{\alpha, \beta, \gamma}$ the following minimizing problem

$$\min_u F^g_{\alpha, \beta, \gamma}(u), \tag{1.2}$$

where the minimum is taken on a suitable class of functions.

In the case $n = 2$, the functional defined in (1.1) was proposed by Mumford and Shah in [11], in order to give a mathematical description to a problem of image segmentation in Computer Vision Theory.

(*) Indirizzo degli AA.: S.I.S.S.A., Strada Costiera 11, 34014 Trieste Italia.

In [11] and [12], Mumford and Shah conjectured that $F^g_{\alpha,\beta,\gamma}$ has minimizers, whose discontinuity set S_u is piecewise smooth. In [2] Ambrosio proved the existence of the solution of (1.2) in the general case of the space dimension $n \geqslant 1$. Some results about the regularity of S_u can be found in [7].

In [6] Dal Maso, Morel and Solimini studied the particular case $n = 2$, giving a constructive proof of the existence.

Further results about this problem can be found in [1], [3] and [7].

Moreover, we recall also that in [14], the one-dimensional case has been considered; in particular the smoothing properties given by the formulation (1.1) of the segmentation problem have been studied.

It is possible also to consider the functional

$$\tilde{F}^g_\gamma(u) = \int_\Omega |u - g|^2 dx + \gamma H^{n-1}(S_u) \tag{1.3}$$

and the associated problem

$$\min_u \tilde{F}^g_\gamma(u). \tag{1.4}$$

We point out that (1.3) can be considered a particular case of (1.1), in which we restrict our attention to the piecewise constant functions or equivalently in which we put $\alpha = +\infty$ and $\beta = 1$. In the case of $n = 2$, a constructive method provides the existence of minimizers for problem (1.4), as proved in [9] and [10]. The general case $n \geqslant 1$ is studied in [5] by Congedo and Tamanini.

However it is not possible, in general, to say that the minimizers for these problems are unique. To this pourpose, let us consider the simple case $n = 1$, $\Omega = [0,1]$ and the function $g: [0, 1] \to \boldsymbol{R}$, $g \in L^\infty([0, 1])$ defined by $g(x) = \chi_{[1/2, 1]}(x)$, where χ_E is the characteristic function of the set E; then the minimum problem (1.4) for that g has, as unique solution, the function $u_1 = \chi_{[1/2, 1]}$ for $0 < \gamma < 1/4$ and the function $u_2 = 1/2$ for $\gamma > 1/4$, but for $\gamma = 1/4$ both functions u_1 and u_2 are solutions.

From these arguments, one could expect that given a function $g \in L^2(\Omega)$, there is uniqueness for these minimum problems except for a «small» set (possibly countable) of values of the parameter γ.

Unfortunately, this is not the case in general, as the following counterexample shows.

Let $g(x) = \chi_{[1/3, 2/3]} + 2\chi_{(2/3, 1]}$ and consider again the problem (1.4) associated to this function g; then it is easy to prove that for $\gamma > 1/2$ the unique solution is $u_1 = 1$ and for $0 < \gamma < 1/6$ the unique solution is $u_2 = g$, but for all the interval $1/6 < \gamma < 1/2$ we have two solutions $u_3 =$

$= (3/2)\chi_{[1/3,1]}$ and $u_4 = (1/2)\chi_{[0,2/3]} + 2\chi_{(2/3,1]}$ and finally for $\gamma = 1/6$ the functions u_2, u_3 and u_4 are solutions and for $\gamma = 1/2$ the functions u_1, u_3 and u_4 are solutions.

Actually, we will see that for every non constant function $g \in L^2([0,1])$ it will be possible to find $\gamma \in (0, +\infty)$ such that the problems (1.2) and (1.4) have more than one solution.

On the other hand, fixed $\gamma \in (0, +\infty)$, we can find $g \in L^2([0,1])$ such that (1.4) has more minimizers. In fact it is enough to take, for instance, $g = (1 + 2\sqrt{\gamma})\chi_{(0,1/2)} + \chi_{(1/2,1)}$, and to observe that $\tilde{F}^g_\gamma(g) = \tilde{F}^g_\gamma(\bar{g})$, where $\bar{g}$ is the mean value of g on [0, 1]. The same property holds also for problem (1.2) (see Corollary 3.5 and Remark 3.6).

These arguments lead us to observe that the best we can hope is the uniqueness for these minimum problems only if we restrict the functions g or the values of the parameter γ to suitable «large» subsets of $L^2(\Omega)$ and $\boldsymbol{R}^+$ respectively.

The aim of this paper is, indeed, to give a rigorous proof of this fact for problems (1.2) and (1.4), in dimension $n = 1$.

The main result is, in fact, that for every γ belonging to $\boldsymbol{R}^+$ uniqueness for (1.2) and (1.4) is a generic property of $g \in L^2([0,1])$.

Moreover, for a generic g belonging to $L^2([0,1])$, uniqueness for (1.2) and (1.4) is a generic property of $\gamma \in \boldsymbol{R}^+$.

To prove these results, we adapt an argument of G. Vidossich in [15] to our situation, following the outline of Carriero and Pascali in [4].

More precisely, given $\alpha > 0$ and $\beta > 0$, we construct a countable subset $\mathcal{M}^0$ of $L^2([0,1])$, dense in $L^2([0,1])$ and a countable subset Γ of $\boldsymbol{R}^+$, such that for every $g \in \mathcal{M}^0$ and for every $\gamma \in \boldsymbol{R}^+ \setminus \Gamma$, problem (1.2) relative to g has a unique solution. Then, by means of $\mathcal{M}^0$, for every $\gamma \in \boldsymbol{R}^+$, we can construct a dense G_δ-subset $\mathcal{M}^*_\gamma$ of $L^2([0,1])$, such that when the datum g is chosen in $\mathcal{M}^*_\gamma$ the corresponding problem (1.2) has only one minimizer. Really this result can be improved by constructing a dense G_δ-subset which works for all the parameters γ of a countable subset contained in $\boldsymbol{R}^+$. On the other hand, we can construct a dense G_δ-subset of $L^2([0,1])$ such that when g belongs to this set, problem (1.2) is uniquely solvable if γ belongs to the complement of a countable subset Γ^g in $\boldsymbol{R}^+$ depending on g.

Similar arguments are used to obtain analogous results for problem (1.4).

Since the complement of a G_δ-subset of $L^2([0,1])$ is a set of first category and Γ^g is countable it is clear now what we meant by «large» or «generic» in the previous informal discussion. We observe that, from this point of view, our results are in line with the genericity results of [4], [8], [13] and [15].

In particular, the set $\mathcal{M}^0$ will be constructed by means of a suitable class of piecewise constant functions. In order to find this class, we will study in detail the properties of the solution, and in particular its form and its discontinuities, when g is piecewise constant.

The paper is organized as follows: in the second section we reformulate the problem in a suitable way to the one-dimensional case, which permits us to reduce (1.2) and (1.4) to the study of simpler problems, with fixed jump term; in the third section we state some preliminary results about the form of the solutions of (1.2) and (1.4) and their continuous dependence on the datum g; finally section 4 contains the main theorems.

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2. Formulation of the problem.

We will write L^2, L^∞ instead of $L^2([0,1])$, $L^\infty([0,1])$. We will denote by $\boldsymbol{R}^+$ the subset of strictly positive real numbers.

In the following, for $j \in \boldsymbol{N}$, a partition $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ of $[0,1]$ will be identified with a subset $\{b_0, \dots, b_{j+1}\}$ of $[0,1]$ such that $0 = b_0 < b_1 < \dots < b_{j+1} = 1$.

Fixed $j \in \boldsymbol{N}$, we denote by $\mathcal{H}_j^1$ the space of all the functions u on $[0,1]$ such that there exists a partition $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ of $[0,1]$ such that the restriction of u to (b_s, b_{s+1}) belongs to $H^1((b_s, b_{s+1}))$, for every $s = 0, \dots, j$. Therefore, we define $\mathcal{H}^1 = \bigcup_{j \in \boldsymbol{N}} \mathcal{H}_j^1$. For every $j \in \boldsymbol{N}$ we consider also the subset $\mathcal{K}_j^1$ of $\mathcal{H}_j^1$ composed by the functions which have exactly j jumps.

Moreover, we denote by $\mathcal{S}$ the space of all the piecewise constant functions on $[0,1]$. It is easy to see that each function $u \in \mathcal{S}$ can be written in the form $u = \sum_{s=0}^{j} \beta_s \chi_{(b_s, b_{s+1})}$ with $\beta_s \in \boldsymbol{R}$ for $s = 0, \dots, j$ and $j \in \boldsymbol{N}$.

Finally S_u is the set of jump points of the function u belonging to $\mathcal{H}^1$ or $\mathcal{S}$ and $\#$ is the counting measure on $\boldsymbol{R}$.

Given $g \in L^2$ and $\gamma \in \boldsymbol{R}^+$, we consider the following functional $F_\gamma^g: \mathcal{H}^1 \to [0, +\infty]$

$$F_\gamma^g(u) = \sum_{s=0}^{l} \int_{b_s}^{b_{s+1}} (u')^2 dx + \int_0^1 (u-g)^2 dx + \gamma \#(S_u), \tag{2.1}$$

where $l = \#(S_u)$; it is easy to see that the functional depends only on u and not on its representation. Moreover we consider the functional $\tilde{F}_\gamma^g: \mathcal{S} \to [0, +\infty]$

$$\tilde{F}_\gamma^g(u) = \int_0^1 (u-g)^2 dx + \gamma \#(S_u)$$

and the associated problems:

$$\min\{F_\gamma^g(u): u \in \mathcal{H}^1\} \tag{2.2}$$

and

$$\min\{\tilde{F}_\gamma^g(u): u \in \mathcal{S}\}; \tag{2.3}$$

we note that (2.1) is obtained by (1.1) with $\alpha = \beta = 1$.

We note that all the results we are going to prove still hold for finite α and β different from 1, because it is possible to reduce the general functional to our case.

We observe that the existence for this problems will be discussed in the following.

We point out that the results we are going to obtain for problem (2.3) cannot be derived directly from those for problem (2.2), but since the method is the same in both cases, we treat explicitly only problem (2.2), remarking, when it is necessary, the differences and the analogies with problem (2.3).

Given $g \in L^2$, for every $\gamma \in \boldsymbol{R}^+$ we define

$$m^g(\gamma) = \min\{F_\gamma^g(u): u \in \mathcal{H}^1\}; \tag{2.4}$$

we shall see later that the minimum is achieved.

Moreover, we consider the functional $G^g: \mathcal{H}^1 \to [0, +\infty]$ defined by

$$G^g(u) = \sum_{s=0}^{j} \int_{b_s}^{b_{s+1}} (u')^2 dx + \int_0^1 (u-g)^2 dx. \tag{2.5}$$

For every $j \in \boldsymbol{N}$ we consider the problem

$$M_j^g = \min\{G^g(u) \colon u \in \mathcal{H}_j^1\}. \tag{2.6$_j$}$$

The existence for this minimum problem follows by the usual compactness property of the sequences of partitions and by the standard direct methods of calculus of variation applied on each subinterval of [0, 1].

It is clear that

$$M_j^g = \inf\{G^g(u) \colon u \in \mathcal{K}_j^1\},$$

but, in this case, the minimum is not always achieved.

Let us define the non empty subset $\boldsymbol{N}^g$ of $\boldsymbol{N}$ of the integers j for which the value M_j^g is attained on at least a function which has exactly j jumps.

Moreover, it can be easily seen that $j \in \boldsymbol{N}^g$ if and only if the minimum of G on $\mathcal{K}_j^1$ is achieved and, in this case,

$$\min_{\mathcal{H}_j^1} G^g(u) = \min_{\mathcal{K}_j^1} G^g(u). \tag{2.7}$$

For every $j \in \boldsymbol{N}$ and for every $\gamma \in \boldsymbol{R}^+$, let us define now

$$m_j^g(\gamma) = M_j^g + \gamma j. \tag{2.8$_j$}$$

Since $m_j^g(\gamma) \geqslant \gamma j$ and $\gamma > 0$, it follows that for every $\gamma \in \boldsymbol{R}^+$ there exists the $\min_{j \in \boldsymbol{N}} m_j^g(\gamma)$. Moreover we are going to prove that

$$m^g(\gamma) = \min_{j \in \boldsymbol{N}} m_j^g(\gamma). \tag{2.9}$$

In fact, given $u \in \mathcal{H}^1$ with $\#(S_u) = j$, we have

$$F_\gamma^g(u) = G^g(u) + \gamma j \geqslant M_j^g + \gamma j \geqslant \min_{j \in \boldsymbol{N}} m_j^g(\gamma)$$

and taking the infimum with respect to $u \in \mathcal{H}^1$, it follows that

$$\inf_{u \in \mathcal{H}^1} F_\gamma^g(u) \geqslant \min_{j \in \boldsymbol{N}} m_j^g(\gamma).$$

The opposite inequality is trivial.

In order to prove that such an infimum is attained, we fix $\gamma \in \boldsymbol{R}^+$ and we choose $j_0 \in \boldsymbol{N}$ such that $\min_{j \in \boldsymbol{N}} m_j^g(\gamma) = m_{j_0}^g(\gamma)$; this implies that there exists $u_0 \in \mathcal{H}_{j_0}^1$ such that

$$\inf_{u \in \mathcal{H}^1} F_\gamma^g(u) = G^g(u_0) + \gamma j_0 \geqslant F_\gamma^g(u_0), \tag{2.10}$$

hence the infimum in (2.10) is attained on u_0 and actually it is a minimum, moreover (2.9) holds.

We note that $\#(S_{u_0}) = j_0$; in fact if $\#(S_{u_0}) = l < j_0$ we have $F^g_\gamma(u_0) = G^g(u_0) + \gamma l < G^g(u_0) + \gamma j_0$ and this contraddicts (2.10). Therefore $u_0 \in \mathcal{K}^1_{j_0}$ and $j_0 \in \boldsymbol{N}^g$. This proves that

$$m^g(\gamma) = \min_{j \in N^g} m^g_j(\gamma), \tag{2.11}$$

for every $\gamma \in \boldsymbol{R}^+$, and that, if $m^g_{j_0}(\gamma) = \min_{j \in \boldsymbol{N}} m^g_j(\gamma)$ for some $\gamma \in \boldsymbol{R}^+$, then every minimizer u_0 of problem $(2.6)_{j_0}$ has exactly j_0 jumps, i.e. $u \in \mathcal{K}^1_{j_0}$.

This leads us to define the subset J^g of N^g in the following way:

$$J^g = \{j \in N^g : \exists \gamma \in \boldsymbol{R}^+ \text{s.t. } m^g_j(\gamma) = \min_{s \in N^g} m^g_s(\gamma)\}$$

(see figure 1).

REMARK 2.1. It is clear that (2.11) can be rewritten as

$$m^g(\gamma) = \min_{j \in J^g} m^g_j(\gamma). \tag{2.12}$$

Moreover, if $j \in J^g$, every minimizer u of problem $(2.6)_j$ has exactly j jumps, i.e. $u \in \mathcal{K}^1_j$.

We observe that, by $(2.8)_j$, $m^g_j(\gamma)$ has a linear dependence on γ, hence by (2.12) $m^g(\gamma)$ is a concave function (see figure 1).

Finally we point out that the sequence $(M^g_j)_{j \in \boldsymbol{N}}$ is decreasing since $\mathcal{K}^1_j \subseteq \mathcal{K}^1_k$ for $j < k$. In particular, if $j \in J^g$, it is strictly decreasing; in fact if by contradiction $j, k \in J^g$ with $j < k$ and $M^g_j = M^g_k$, then for every $\gamma \in \boldsymbol{R}^+$

$$m^g_j(\gamma) = M^g_j + \gamma j < M^g_k + \gamma k = m^g_k(\gamma),$$

which implies $k \notin J^g$.

We introduce, for every $j \in J^g$, the non empty subsets of $\boldsymbol{R}$

$$\Gamma^g_j = \{\gamma \in \boldsymbol{R}^+ : m^g(\gamma) = m^g_j(\gamma)\}$$

and

$$\Gamma^g = \{\gamma \in \boldsymbol{R}^+ : \exists j, j' \in J^g, \text{ consecutive in } J^g, \text{ s.t. } \gamma \in \Gamma^g_j \cap \Gamma^g_{j'}\} \tag{2.13}$$

(see figure 1).

REMARK 2.2. It is clear that Γ^g_j is a (possible degenerate) interval of $\boldsymbol{R}^+$, since it can be rewritten as

$$\Gamma^g_j = (m^g - m^g_j)^{-1}([0, +\infty))$$

and $m^g - m_j^g$ is a concave function. Moreover the intervals Γ_j^g with $j \in J^g$ are non overlapping, since the angular coefficient of m_j^g is strictly increasing with j, and Γ_0^g is unbounded. Hence for every $i \neq j$ and every γ belonging to the interior of Γ_j^g we have $m^g(\gamma) = m_j^g(\gamma) < m_i^g(\gamma)$. Given two consecutive elements j and j' of J^g, the equality $m_j^g(\gamma) = m_{j'}^g(\gamma)$ is satisfied for at most one $\gamma \in \boldsymbol{R}^+$; finally we note that Γ^g is the set of all the endpoints of the intervals Γ_h^g, hence it is a discrete countable subset of $\boldsymbol{R}^+$ and the only possible accumulation point is the point $\gamma = 0$ (see figure 1).

PROPOSITION 2.3. Fixed $\gamma \in \boldsymbol{R}^+$, we consider the non empty subset of J^g

$$J_\gamma^g = \{j \in J^g : m_j^g(\gamma) = m^g(\gamma)\}.$$

Then $m^g(\gamma)$ is attained on $u \in \mathcal{H}^1$ if and only if there exists $j \in J_\gamma^g$ such that u is a minimum point of the problem which defines M_j^g. In particular, if there exists a unique $j \in J_\gamma^g$ and if the problem $(2.6)_j$ has uniqueness, then also the problem (2.2) has uniqueness.

PROOF. If $j = \#(S_u)$ then $u \in \mathcal{H}_j^1$. Hence, if $m^g(\gamma)$ is attained on u, then

$$m^g(\gamma) = \sum_{s=0}^{j} \int_{b_s}^{b_{s+1}} (u')^2 dx + \int_0^1 (u-g)^2 dx + \gamma j \geq m_j^g(\gamma) \geq m^g(\gamma),$$

which implies that $j \in J_\gamma^g$ and that M_j^g is attained on u.

Viceversa, if $j \in J_\gamma^g$ and M_j^g is attained on u, then $j = \#(S_u)$ (see Remark 2.1). The conclusion follows by

$$m^g(\gamma) = m_j^g(\gamma) = M_j^g + \gamma j = G_\gamma^g(u) + \gamma j = F_\gamma^g(u). \qquad \blacksquare$$

COROLLARY 2.4. (i) If $\bar{\gamma} \in \Gamma^g$, then problem (2.2) has more than one solution.

(ii) If for a $\bar{\gamma}$ belonging to the interior of γ_j^g with $j \in J^g$ problem (2.2) has more than one solution, then for all γ belonging to the interior of Γ_j^g problem (2.2) has not uniqueness.

PROOF. (i) Follows by the definition of Γ^g and Γ_j^g.

(ii) If u_1, u_2 are minimizers of $F_{\bar{\gamma}}^g$, then by Proposition 2.3 they are minimizers of G^g, that is for every $s = 1, 2$ $G^g(u_s) = M_j^g$. Since for every γ belonging to the interior of Γ_j^g we have

$$M_j^g + \gamma j = m^g(\gamma),$$

with j fixed, it follows that u_1, u_2 are minimizers of F_γ^g. ∎

PROPOSITION 2.5. Let $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ be a fixed partition of $[0,1]$ and $\mathcal{H}_{\mathcal{Q}_j}^1$ be the subset of $\mathcal{H}_j^1$ constituted by the functions $u = \sum_{s=0} \beta_s(x)\chi_{(b_s, b_{s+1})}(x)$ with $\beta_s \in H^1((b_s, b_{s+1}))$ for every $s = 0, 1, \ldots, j$. Then the functional G^g defined in (2.5) has exactly one minimizer on $\mathcal{H}_{\mathcal{Q}}^1$.

PROOF. The existence is standard. For the uniqueness it is sufficient to observe that $\mathcal{H}_{\mathcal{Q}}^1$ is a linear subspace of $\mathcal{H}^1$ and the functional G^g is strictly convex on $\mathcal{H}_{\mathcal{Q}}^1$. ∎

PROPOSITION 2.6. Fixed a partition $(a_i)_{i=0}^{k+1}$ of $[0,1]$; if g is a function of the type $g = \sum_{i=0}^{k} \alpha_i \chi_{(a_i, a_{i+1})}$, where $\alpha_i \in \boldsymbol{R}$ for every $i = 0, \ldots, k$, then fixed $j > k$ we have that $m^g(\gamma) < m_j^g(\gamma)$ for every $\gamma \in \boldsymbol{R}^+$, $J^g \subseteq \{1, \ldots, k\}$ and Γ^g is finite.

PROOF. Since for every $j \geq k$ $g \in \mathcal{H}_j^1$, it follows that $M_j^g = 0$. Assume by contradiction that, given $j > k$ there exists $\bar{\gamma} \in \boldsymbol{R}^+$ such that $m^g(\bar{\gamma}) = m_j^g(\bar{\gamma})$, then

$$m^g(\bar{\gamma}) = \bar{\gamma} j > \bar{\gamma} k.$$

But, if we take $u = g$, we have that $F_{\bar{\gamma}}^g(g) \leq \bar{\gamma} k$ which is a value strictly less than $m^g(\bar{\gamma})$ and this is not possible.

This implies also that for every $j > k$ we obtain that $j \notin J^g$; then J^g is finite and is contained in $\{1, \ldots, k\}$ and, by (2.13) and Remark 2.2, Γ^g is the set of points $\gamma \in \boldsymbol{R}^+$ such that $m^g(\gamma) = m_j^g(\gamma) = m_{j'}^g(\gamma)$, where j and j' are two consecutive elements of J^g. Therefore it follows that Γ^g is finite and contains at most k points. ∎

REMARK 2.7. We may analogously define the minimum problem

$$\tilde{m}^g(\gamma) = \min\{\tilde{F}_\gamma^g(u) : u \in \mathcal{S}\};$$

moreover we can consider the functional $\tilde{G}^g : \mathcal{S} \to [0, +\infty]$ defined by

$$\tilde{G}^g(u) = \int_0^1 (u-g)^2\,dx$$

and the associated problem

$$\tilde{M}_j^g = \min\{\tilde{G}^g(u) \colon u \in \mathcal{H}_j^1 \cap \mathcal{S}\}.$$

With suitable modifications, we can also introduce the definitions of $\tilde{m}_j^g$, $\tilde{N}^g$, $\tilde{J}^g$, $\tilde{\Gamma}^g$ and $\tilde{\Gamma}_j^g$ relative to the problem (2.3).

REMARK 2.8. In order to explain better the previous definitions, we give an easy example, relative to the functional F_γ^g, in which we emphasize those concepts.

Let $g(x) = \chi_{(1/3,\,2/3)}(x)$. An easy calculation shows that G^g has one minimizer u_1 on $\mathcal{H}_0^1$ and one minimizer u_4 on $\mathcal{H}_2^1$, and that G^g has two minimizer u_2 and u_3 on $\mathcal{H}_1^1$. Moreover, u_1 is the minimizer of F_γ^g on $\mathcal{H}^1$ for $0 \leqslant \gamma \leqslant \bar{\gamma}$ and u_4 is the minimizer of F_γ^g on $\mathcal{H}^1$ for $\gamma \geqslant \bar{\gamma}$, where $\bar{\gamma} \simeq 0.11$. This permits us to construct the following graph:

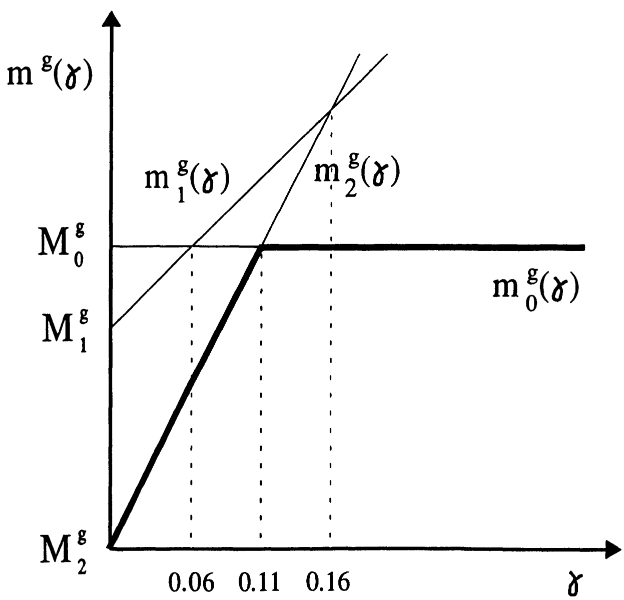

Figure 1.

$$N^g = \{0, 1, 2\},\ J^g = \{0, 2\},$$
$$M_0^g \simeq 0.22,\ M_1^g \simeq 0.16,\ M_2^g = 0,$$
$$\bar{\gamma} \simeq 0.11 \colon \Gamma^g = \{\bar{\gamma}\},\ \Gamma_0^g = [\bar{\gamma}, +\infty),\ \Gamma_2^g = [0, \bar{\gamma}].$$

3. Preliminary results.

In this section we state some results concerning the form of a solution of the minimum problem. In particular, we give an explicit formula, in terms of g, for a minimum point $u \in \mathcal{H}^1$ of the problem $(2.6)_j$, and for a minimum point $\tilde{u} \in \mathcal{S}$ of the analogous problem for the functional without the derivative term, and we study where such a minimum point can jump and the continuous dependence of it on the datum g. Moreover we investigate the non uniqueness: in particular we show how it is possible, fixed $g \in L^2$ (or $\gamma \in \boldsymbol{R}^+$), to construct $\gamma \in \boldsymbol{R}^+$ (or $g \in L^2$ respectively) for whose minimum problems have non uniqueness.

REMARK 3.1. It is easy to see that, when $j \in J^g$, a minimizer u of problem $(2.6)_j$ must be of the form

$$u(x) = \sum_{s=0}^{j} \bar{\beta}_s(x)\, \chi_{(b_s,\, b_{s+1})}(x),$$

where $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ is a partition of $[0,1]$ and for every $s = 0, \ldots, j$ $\bar{\beta}_s$ is the solution of the Euler equation in the subinterval (b_s, b_{s+1}) of $[0,1]$, i.e.

$$\bar{\beta}_s(x) = c_s \cosh(x - b_s) + d_s(x), \tag{3.1}$$

where

$$c_s = \frac{e^{b_s}}{e^{2b_{s+1}} - e^{2b_s}} \left[e^{2b_{s+1}} \int_{b_s}^{b_{s+1}} g(t)\, e^{-t}\, dt + \int_{b_s}^{b_{s+1}} g(t)\, e^{t}\, dt \right],$$

$$d_s(x) = \frac{e^{-x}}{2} \int_{b_s}^{x} g(t)\, e^{t}\, dt - \frac{e^{x}}{2} \int_{b_s}^{x} g(t)\, e^{-t}\, dt$$

and cosh is the hyperbolic cosine. Finally, recalling the definition of J^g and Remark 2.1, for every $s = 0, \ldots, j-1$ $\bar{\beta}_s(b_{s+1}) \neq \bar{\beta}_{s+1}(b_{s+1})$.

REMARK 3.2. In the case of problem (2.3), the situation is even much easier, so that for $j \in \tilde{J}^g$ a minimizer function has the form

$$\tilde{u}(x) = \sum_{s=0}^{j} \bar{\beta}_s\, \chi_{(b_s,\, b_{s+1})}(x),$$

where $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ is a partition of $[0,1]$ and for $s = 0,\dots,j$

$$\bar{\beta}_s = \frac{1}{b_{s+1} - b_s} \int_{b_s}^{b_{s+1}} g(t)\,dt\,;$$

finally we have that $\bar{\beta}_s \neq \bar{\beta}_{s+1}$ for every $s = 0, \dots, j-1$.

REMARK 3.3. Fixed $\gamma \in \boldsymbol{R}^+$, $j \in J_\gamma^g$ and the solution $u(x) = \sum_{s=0}^{j} \bar{\beta}_s(x)\chi_{(b_s, b_{s+1})}(x)$ of problem $(2.6)_j$, we have that for every $s = 0,\dots,j$ and every function $\beta_s \in H^1((b_s, b_{s+1}))$

$$F_\gamma^g\left(\sum_{s=0}^{j} \beta_s(x)\chi_{(b_s, b_{s+1})}(x)\right) > F_\gamma^g\left(\sum_{s=0}^{j} \bar{\beta}_s(x)\chi_{(b_s, b_{s+1})}(x)\right),$$

if for some $s \in \{0, \dots, j\}$ we have that $\beta_s \neq \bar{\beta}_s$.

In fact, by Proposition 2.5 the functional G^g defined in (2.5) is strictly convex on $\mathcal{H}_{\mathcal{Q}}^1$, where $\mathcal{Q} = (b_s)_{s=0}^{j+1}$, hence G^g has a unique minimizer on $\mathcal{H}_{\mathcal{Q}}^1$.

In the following two corollaries we will show that, arbitrarily fixed the non constant datum $g \in L^2$ or the parameter $\gamma \in \boldsymbol{R}^+$, it is possible to choose the parameter $\gamma \in \boldsymbol{R}^+$ or the datum $g \in L^2$ respectively such that problems (2.2) and (2.3) have non uniqueness.

COROLLARY 3.4. For every non constant function $g \in L^2$

(i) there exists $\bar{\gamma} \in \boldsymbol{R}^+$ such that $F_{\bar{\gamma}}^g$ has more than one minimizer;

(ii) there exists $\tilde{\gamma} \in \boldsymbol{R}^+$ such that $\tilde{F}_{\tilde{\gamma}}^g$ has more than one minimizer.

PROOF. Let us fix a non constant function $g \in L^2$.

(i) Let u_0 be the unique solution of the equation

$$-u'' + u = g\,, \tag{3.2}$$

with the Neumann conditions $u'(0) = 0 = u'(1)$; then by definition $M_0^g = \int_0^1 (u_0')^2\,dx + \int_0^1 (u_0 - g)^2\,dx$. By (3.2), since g is not constant, also u_0 is not constant; moreover $u_0 \in C^1([0, 1])$, hence there exists $b \in (0,1)$ such that $u_0'(b) \neq 0$. Let now $v(x) = \beta_0(x)\chi_{(0,b)}(x) + \beta_1(x)\chi_{(b,1)}(x)$, where β_0 and β_1 are the solutions of the equation (3.2) with the Neumann conditions $\beta_0'(0) = 0 = \beta_0'(b)$ and $\beta_1'(b) = 0 = \beta_1'(1)$ respectively.

We observe that clearly u_0 does not satisfy the Neumann conditions in $[0, b]$ and in $[b, 1]$, hence it follows that $G^g(v) < G^g(u_0)$, i.e.

$$M_0^g = G^g(u_0) > G^g(v) \geqslant M_1^g.$$

This implies that, if $\bar{\gamma} = \min \Gamma_0^g$, then $\bar{\gamma} > 0$; since $\bar{\gamma} \in \Gamma^g$, by Corollary 2.4 (i) we obtain that $F_{\bar{\gamma}}^g$ has at least two minimizers.

(ii) Let $u_0 = \int_0^1 g(t)\,dt$, then $\tilde{M}_0^g = \int_0^1 (u_0 - g)^2\,dx$. Let $v(x) = \beta_0 \chi_{(0,b)}(x) + \beta_1 \chi_{(b,1)}(x)$, where $\beta_0 = (1/b)\int_0^b g(t)\,dt$ and $\beta_1 = 1/(1-b)\int_b^1 g(t)\,dt$. Since g is not constant, then for a proper choice of b we have

$$\left(\int_0^1 g(t)\,dt - b\int_0^1 g(t)\,dt\right)^2 > 0,$$

which implies, after some calculation,

$$\tilde{M}_0^g = \tilde{G}^g(u_0) > \tilde{G}^g(v) = \tilde{M}_1^g.$$

Now, the same arguments used in (i) give that $\tilde{\gamma} = \min \tilde{\Gamma}_0^g$ is strictly positive and $\tilde{F}_{\tilde{\gamma}}^g$ has at least two miinimizers. ∎

COROLLARY 3.5. For every $\gamma \in \boldsymbol{R}^+$ there exists $g \in L^2$ such that F_γ^g has more than one minimizer.

PROOF Let us fix $\bar{g} \in L^2$; by Corollary 3.4 (i) there exists $\bar{\gamma} \in \boldsymbol{R}^+$ such that $F_{\bar{\gamma}}^{\bar{g}}$ has more than one minimizer. We can find $\alpha \in \boldsymbol{R}^+$ such that $\bar{\gamma}\alpha = \gamma$; then defining $v = \sqrt{\alpha}\,u$ and $g = \sqrt{\alpha}\,\bar{g}$, it follows that $\alpha F_{\bar{\gamma}}^{\bar{g}}(u) = F_\gamma^g(v)$. This implies that, if $u_1, \ldots, u_l$ are minimizers for $F_{\bar{\gamma}}^{\bar{g}}$, then $\sqrt{\alpha}\,u_1, \ldots, \sqrt{\alpha}\,u_l$ are minimizers for F_γ^g. ∎

REMARK 3.6. It is clear that Corollary 3.5 can be proved with the same rescaling technique also for the functional $\tilde{F}_\gamma^g$.

Moreover the previous proof shows that there exists $g \in C^\infty$ (or g piecewise constant) such that F_γ^g has more than one minimizer.

We want to study now the continuous dependence of the solution u of problem (2.2) on the datum g. We will prove that this dependence holds when problem (2.2) has uniqueness, and in this case it is a direct consequence of the one-dimensional case of the results of compactness and lower semicontinuity of Ambrosio in [1].

LEMMA 3.7. Let (g_n) be a sequence of functions in L^2 such that $g_n \to g$ strongly in L^2.

Fix $\gamma \in \boldsymbol{R}^+$ and assume that problem (2.2) for F_γ^g is uniquely solvable by $\tilde{u} \in \mathcal{H}^1$. Let $(\tilde{u}_n)$ be a sequence of functions in $\mathcal{H}^1$ such that for every $n \in \boldsymbol{N}$ $F_\gamma^{g_n}(\tilde{u}_n)$ takes the minimum value. Then $\tilde{u}_n \to \tilde{u}$ strongly in L^1.

PROOF. By the convergence of g_n to g in L^2, it follows that $\|g_n\|_{L^2} \leq C_1$.

Since $\tilde{u}_n$ is a minimizer, it is easy to verify that $F_\gamma^{g_n}(\tilde{u}_n) \leq F_\gamma^{g_n}(\tilde{u}) \leq \leq C_2$, where C_2 depends on C_1 and the H^1-norm of $\tilde{u}$.

This implies that there exists a constant C_3 depending on C_1 and C_2 such that $\|\tilde{u}_n'\|_{L^2} + \|\tilde{u}_n\|_{L^2} \leq C_3$; moreover $\#(S_{\tilde{u}_n}) \leq C_2$, hence by a compactness result due to Ambrosio (see [1] Theorem 2.1), there exists a subsequence $(n_k)_{k \in \boldsymbol{N}}$ such that $\tilde{u}_{n_k} \to \bar{u}$ strongly in L^1, with $\bar{u} \in \mathcal{H}^1$.

To show that $\bar{u}$ coincide with $\tilde{u}$, we apply again Theorem 2.1 in [1] obtaining, after some calculation,

$$F_\gamma^g(\bar{u}) \leq \liminf_{k \to +\infty} F_\gamma^g(\tilde{u}_{n_k}) \leq$$

$$\leq \liminf_{k \to +\infty} \left[F_\gamma^{g_{n_k}}(\tilde{u}_{n_k}) + \int_0^1 (g_{n_k} - g)^2\, dx + 2 \int_0^1 (\tilde{u}_{n_k} - g_{n_k})(g_{n_k} - g)\, dx \right] \leq$$

$$\leq \lim_{k \to +\infty} F_\gamma^{g_{n_k}}(v) = F_\gamma^g(v) \qquad \forall v \in \mathcal{H}^1.$$

This shows that $\bar{u}$ is a solution of (2.2) for F_γ^g, hence, by uniqueness, $\bar{u} = \tilde{u}$ and all the sequence $\tilde{u}_n$ converges to $\tilde{u}$. ■

To conclude this section, we want to show that, when the datum $g \in \in L^2$ is piecewise constant, a solution of problem $(2.6)_j$ (and hence a solution of problem (2.2)) can jump only where g jumps.

We note that this fact had already appeared in the examples reported in the introduction; in general this kind of behaviour is a feature of the minimum points of F_γ^g, independently of the choice of g, if g is piecewise constant.

Given $k \in \boldsymbol{N}$, for every partition $\mathcal{P} = (a_i)_{i=0}^{k+1}$ of $[0,1]$ we consider the set $\mathcal{M}_{\mathcal{P}}$ of the functions g of the type

$$g(x) = \sum_{i=0}^{k} \alpha_i \chi_{(a_i, a_{i+1})}(x),$$

with $\alpha_i \in \boldsymbol{R}$ for every $i = 0, \dots, k$; we remark that $\mathcal{M}_{\mathcal{P}}$ is a linear subspace of L^2.

LEMMA 3.8. Let $g(x) = \sum_{i=0}^{k} \alpha_i \chi_{(a_i, a_{i+1})}(x)$ be a function belonging to $\mathcal{M}_{\mathcal{P}}$. Fixed $j \in J^g$, and let u be a solution of $(2.6)_j$ of the type

$$u(x) = \sum_{s=0}^{j} \beta_s(x) \chi_{(b_s, b_{s+1})}(x),$$

where $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ is a partition of $[0, 1]$. Then

$$\{b_1, \ldots, b_j\} \subseteq \{a_1, \ldots, a_k\}.$$

PROOF. We recall that by Proposition 2.6 J^g is contained in $\{0, \ldots, k\}$.

We argue by contradiction. Suppose that there exist $s \in \{1, \ldots, j\}$ and $i \in \{0, \ldots, k\}$ such that $b_s \in (a_i, a_{i+1})$. First of all we observe that, since $j \in J^g$, $\beta_{s-1}(b_s) \neq \beta_s(b_s)$, hence we may assume that $\beta_{s-1}(b_s) > > \beta_s(b_s)$ (the other case following by analogous arguments) and we can consider the following two cases:

(1) $\beta_{s-1}(b_s) > \beta_s(b_s) \geq \alpha_i$,

(2) $\beta_{s-1}(b_s) \geq \alpha_i \geq \beta_s(b_s)$.

(The third case $\alpha_i \geq \beta_{s-1}(b_s) > \beta_s(b_s)$ is similar to the first one).

For every $0 < \varepsilon < b_s - a_i$ we define a function $u_\varepsilon : [0, 1] \to \boldsymbol{R}$ by

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if } x \in (0, b_s - \varepsilon) \cup (b_s, 1), \\ \beta_s(b_s) & \text{if } x \in (b_s - \varepsilon, b_s). \end{cases}$$

In the case (1) we note that there exists $\delta > 0$ such that for every x such that $b_s - \delta < x < b_s$ we have

$$\beta_{s-1}(x) > \beta_s(b_s).$$

Hence when $\varepsilon < \delta$ we obtain

$$G^g(u_\varepsilon) - G^g(u) \leq \int_{b_s - \varepsilon}^{b_s} [(\beta_s(b_s) - \alpha_i)^2 - (\beta_{s-1}(x) - \alpha_i)^2]\, dx =$$

$$= \int_{b_s - \varepsilon}^{b_s} [\beta_s(b_s) - \beta_{s-1}(x)][(\beta_s(b_s) - \alpha_i) + (\beta_{s-1}(x) - \alpha_i)]\, dx < 0.$$

This contradicts the hypothesis that u is a minimizer of G^g.

In the case (2) we may consider the following two subcases:

$(2)_a$ $\beta_{s-1}(b_s) - \alpha_i > \alpha_i - \beta_s(b_s)$,

$(2)_b$ $\beta_{s-1}(b_s) - \alpha_i = \alpha_i - \beta_s(b_s) > 0$.

(The last case $\beta_{s-1}(b_s) - \alpha_i < \alpha_i - \beta_s(b_s)$ can be studied similarly to the $(2)_a$).

If $(2)_a$ is satisfied, then there exists $\delta > 0$ such that for every x with $0 < b_s - \delta < x < b_s$ we have $\beta_{s-1}(x) - \alpha_i > a_i - \beta_s(b_s)$ and $\beta_{s-1}(x) > \beta_s(b_s)$. Hence for every $\varepsilon < \delta$ we obtain again $G^g(u_\varepsilon) - G^g(u) < 0$. Now we consider the case where $(2)_b$ is satisfied. First we remark that from (3.1) for every $x \in (b_s, a_{i+1})$ we have

$$\beta_s(x) = (c_s - \alpha_i)\cosh(x - b_s) + \alpha_i$$

and so

$$\beta_s'(x) = (c_s - \alpha_i)\sinh(x - b_s);$$

then since $c_s = \beta_s(b_s)$, we can conclude that $\beta_s'(x) < 0$ for every $x \in (b_s, a_{i+1})$. Therefore β_s is a strictly decreasing function on (b_s, a_{i+1}). Now for every $0 < \eta < a_{i+1} - b_s$ we define a function $v_\eta : [0, 1] \to \boldsymbol{R}$ by

$$v_\eta(x) = \begin{cases} u(x) & \text{if } x \in (0, b_s) \cup (b_s + \eta, 1), \\ \beta_{s-1}(b_s) & \text{if } x \in (b_s, b_s + \eta). \end{cases}$$

We point out that there exists $\delta > 0$ such that for every x with $b_s < x < b_s + \delta$ we have $\beta_{s-1}(b_s) > \beta_s(x)$; hence for every $\eta < \delta$, using $(2)_b$ and the fact that $\beta_s'(x) < 0$ implies that $\beta_s(x) < \beta_s(b_s)$, we obtain

$$G^g(u_\eta) - G^g(u) \leqslant \int_{b_s}^{b_s+\eta} [\beta_{s-1}(b_s) - \beta_s(x)][(\beta_{s-1}(b_s) - \alpha_i) - (\alpha_i - \beta_s(x))]\,dx < 0.$$

This contradiction concludes the proof. ■

COROLLARY 3.9. *Let g, k and j as in Lemma 3.8 and $\gamma \in \boldsymbol{R}^+$. Then the minimizers of the functional F_γ^g with j iumps are at most $\binom{k}{j}$. Moreover the minimizers of F_γ^g are at most 2^k.*

PROOF. By Proposition 2.5, for a fixed partition $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ of $[0, 1]$, G^g has exactly one solution $u \in \mathcal{H}_{\mathcal{Q}}^1$; by the preceeding lemma the partitions $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ corresponding to a minimizer of $(2.6)_j$ must be contained in the partition $\mathcal{P} = (a_i)_{i=0}^{k+1}$ and hence they can be at most $\binom{k}{j}$. The conclusion follows since $J^g \subseteq \{0, \ldots, k\}$ (see Proposition 2.6) and $\sum_{j=0}^{k} \binom{k}{j} = 2^k$. ■

REMARK 3.10. It is clear that if we repeat step by step the arguments used in Remark 3.3, Lemma 3.7 and Lemma 3.8 cancelling out the term with the derivative in the functional F^g_γ, we obtain the same results also for $\tilde{F}^g_\gamma$.

4. Main results.

In Theorem 4.3 we will prove that for «almost all» $g \in \mathcal{M}_\mathcal{P}$ we have uniqueness for problem $(2.6)_j$, for each $0 \leqslant j \leqslant k$; but to obtain this result we need before the following lemmas.

LEMMA 4.1. Assume that $0 \leqslant a_{i_0} < a_{i_1} \leqslant a_{i_2} < a_{i_3} \leqslant 1$ and that $g = \chi_{[a_{i_0}, a_{i_1}]}$. Let us consider for $m = 2, 3$ the following functionals

$$(4.1)_m \qquad \int_{a_{i_0}}^{a_{i_m}} [|u'|^2 + |u(x) - g(x)|^2]\, dx .$$

Assume that for $m = 2,3$ u_m are minimum points for $(4.1)_m$ on $H^1([a_{i_0}, a_{i_m}])$, then

$$(4.2) \qquad \int_{a_{i_0}}^{a_{i_2}} [|u_2'|^2 + |u_2(x) - g(x)|^2]\, dx < \int_{a_{i_0}}^{a_{i_3}} [|u_3'|^2 + |u_3(x) - g(x)|^2]\, dx .$$

PROOF. Since u_2 is a minimum point for $(4.1)_2$, it follows that

$$(4.3) \qquad \int_{a_{i_0}}^{a_{i_2}} [|u_2'|^2 + |u_2(x) - g(x)|^2]\, dx \leqslant \int_{a_{i_0}}^{a_{i_2}} [|u_3'|^2 + |u_3(x) - g(x)|^2]\, dx .$$

Moreover, if we had that

$$\int_{a_{i_0}}^{a_{i_2}} [|u_3'|^2 + |u_3(x) - g(x)|^2]\, dx = \int_{a_{i_0}}^{a_{i_3}} [|u_3'|^2 + |u_3(x) - g(x)|^2]\, dx ,$$

then it should be $u_3 = g$ on (a_{i_2}, a_{i_3}), that means $u_3 \equiv 0$ on (a_{i_2}, a_{i_3}); but by Remark 3.1 it is not possible, since u_3 is a minimum point for $(4.1)_3$. Hence it is clear that

$$\int_{a_{i_0}}^{a_{i_2}} [|u_3'|^2 + |u_3(x) - g(x)|^2]\, dx < \int_{a_{i_0}}^{a_{i_3}} [|u_3'|^2 + |u_3(x) - g(x)|^2]\, dx$$

and this inequality together with (4.3) gives (4.2). This concludes the proof. ■

LEMMA 4.2. Let us fix a partition $\mathcal{P} = (a_i)_{i=0}^{k+1}$ of $[0,1]$ and for any choice of $(\alpha_0, \ldots, \alpha_k) \in \boldsymbol{R}^{k+1}$ let us define a function $g = \sum_{i=0}^{k} \alpha_i \chi_{(a_i, a_{i+1})}$ belonging to $\mathcal{M}_{\mathcal{P}}$. Let $j \in \{0, \ldots, k\}$ and let $\mathcal{Q} = (q_s)_{s=0}^{j+1}$, $\mathcal{R} = (r_s)_{s=0}^{j+1}$ be two different partitions of $[0,1]$ such that $\mathcal{Q}, \mathcal{R} \subseteq \mathcal{P}$. Let us define two functions $Q, R \colon \boldsymbol{R}^{k+1} \to \boldsymbol{R}$ by

$$Q(\alpha_0, \ldots, \alpha_k) = \min_{u \in \mathcal{H}^1_{\mathcal{Q}}} \left[\sum_{s=0}^{j} \int_{q_s}^{q_{s+1}} |u'|^2 dx + \int_0^1 |u - g|^2 dx \right]$$

and

$$R(\alpha_0, \ldots, \alpha_k) = \min_{v \in \mathcal{H}^1_{\mathcal{R}}} \left[\sum_{s=0}^{j} \int_{r_s}^{r_{s+1}} |v'|^2 dx + \int_0^1 |v - g|^2 dx \right],$$

where $\mathcal{H}^1_{\mathcal{Q}}$ and $\mathcal{H}^1_{\mathcal{R}}$ are defined as in Proposition 2.5. Then Q and R are two different polynomial functions and the set

$$\mathcal{A}_{\mathcal{Q}\mathcal{R}} = \{(\alpha_0, \ldots, \alpha_k) \in \boldsymbol{R}^{k+1} \colon Q(\alpha_0, \ldots, \alpha_k) \neq R(\alpha_0, \ldots, \alpha_k)\}, \tag{4.4}$$

is an open set dense in $\boldsymbol{R}^{k+1}$.

PROOF. Clearly Q and R are polynomial functions of degree 2 in the $k+1$ variables $\alpha_0, \ldots, \alpha_k$. The proof is accomplished if we prove that the equality $Q(\alpha_0, \ldots, \alpha_k) = R(\alpha_0, \ldots, \alpha_k)$ is not identically satisfied. Since $\mathcal{Q}$ is different from $\mathcal{R}$ there exists l belonging to $\{0, \ldots, j\}$ such that $q_m = r_m$ for every $m \in \{0, \ldots, l\}$ and $q_{l+1} \neq r_{l+1}$; we suppose that $q_{l+1} < r_{l+1}$. By hypothesis there exist $i_0, i_2, i_3 \in \{0, \ldots, k\}$ such that $q_l = r_l = a_{i_0}$, $q_{l+1} = a_{i_2}$ and $r_{l+1} = a_{i_3}$. Let us take now $\alpha_0, \ldots, \alpha_k$, where $\alpha_{i_0} = 1$ and $\alpha_m = 0$ for $m \neq i_0$. Then by Lemma 4.1 with $a_{i_1} = a_{i_0+1}$ we have that $Q(\alpha_0, \ldots, \alpha_k) < R(\alpha_0, \ldots, \alpha_k)$. This concludes the proof. ■

THEOREM 4.3. Given a partition $\mathcal{P} = (a_i)_{i=0}^{k+1}$ of $[0,1]$, there exists a subset $\mathcal{M}^0_{\mathcal{P}}$ of $\mathcal{M}_{\mathcal{P}}$, which is dense in $\mathcal{M}_{\mathcal{P}}$ with respect to the L^2-topology, and such that for every $g \in \mathcal{M}^0_{\mathcal{P}}$ problem $(2.6)_j$ has a unique solution, for every $j \in J^g$.

PROOF. Let $\mathcal{P} = (a_i)_{i=0}^{k+1}$ be a partition of $[0,1]$ and let $g \in \mathcal{M}_{\mathcal{P}}$. Let $j \in J^g$; by Proposition 2.6 we have that $0 \leq j \leq k$. If $j = k$, then for every $g \in \mathcal{M}_{\mathcal{P}}$ the problem $(2.6)_k$ has the unique solution g.

Now we define

$$\mathcal{A}_{\mathcal{P}} = \bigcap_{0 \leq j < k} \bigcap_{\substack{\mathcal{Q}, \mathcal{R} \subseteq \mathcal{P} \\ \#\mathcal{Q} = \#\mathcal{R} = j+2}} \mathcal{A}_{\mathcal{Q}\mathcal{R}},$$

where the set $\mathcal{A}_{\mathcal{Q}\mathcal{R}}$ is defined by (4.4), and

$$\mathcal{M}^0_{\mathcal{P}} = \left\{ g \in \mathcal{M}_{\mathcal{P}} : g = \sum_{i=0}^{k} \alpha_i \chi_{(a_i, a_{i+1})} \text{ with } (\alpha_0, \ldots, \alpha_k) \in \mathcal{A}_{\mathcal{P}} \right\};$$

by Lemma 4.2 and by Baire's theorem, $\mathcal{A}_{\mathcal{P}}$ is an open set dense in $\boldsymbol{R}^{k+1}$ and hence $\mathcal{M}^0_{\mathcal{P}}$ is dense in $\mathcal{M}_{\mathcal{P}}$ with respect to the L^2-topology. Moreover for every $g \in \mathcal{M}^0_{\mathcal{P}}$ the problem $(2.6)_j$ has uniqueness, for every $j \in J^g \setminus \{k\}$. In fact let $g = \sum_{i=0}^{k} \alpha_i \chi_{(a_i, a_{i+1})}$ be a function belonging to $\mathcal{M}^0_{\mathcal{P}}$. Then $(\alpha_0, \ldots, \alpha_k) \in \mathcal{A}_{\mathcal{Q}\mathcal{R}}$ for every $\mathcal{Q}$, $\mathcal{R} \subseteq \mathcal{P}$, with $\#\mathcal{Q} = \#\mathcal{R} = j+2$ and for every $0 \leq j < k$. We suppose by contradiction that there exist $j_0 \in J^g$ and two different solutions $u, v \in \mathcal{K}^1_{j_0}$ of the problem $(2.6)_{j_0}$ (see Remark 2.1). let $\mathcal{Q}$ and $\mathcal{R}$ the partitions associated to u and v; by Proposition 2.5 $\mathcal{Q}$ and $\mathcal{R}$ are different and by Lemma 3.8 $\mathcal{Q}$ and $\mathcal{R}$ are contained in $\mathcal{P}$. Hence from the definition of $\mathcal{A}_{\mathcal{Q}\mathcal{R}}$ $Q(\alpha_0, \ldots, \alpha_k)$ must be different from $R(\alpha_0, \ldots, \alpha_k)$, where Q and R are defined as in Lemma 4.2. But since u and v are minimizers of the problem $(2.6)_{j_0}$, we have that $Q(\alpha_0, \ldots, \alpha_k) = M^g_{j_0} = R(\alpha_0, \ldots, \alpha_k)$; this contradiction concludes the proof. ■

THEOREM 4.4. There exists a countable set $\mathcal{M}_0$ dense in L^2 and a countable set Γ in $\boldsymbol{R}^+$ such that for every $g \in \mathcal{M}^0$ and $\gamma \in \boldsymbol{R}^+ \setminus \Gamma$ problem (2.2) admits a unique solution.

PROOF. For every $k \in \boldsymbol{N}$ we consider the partition $\mathcal{P}_k = \{0, 1/k, 2/k, \ldots, 1\}$ of $[0, 1]$; hence by Theorem 4.3 there exists a set $\mathcal{M}^0_k$ dense in $\mathcal{M}_{\mathcal{P}_k}$ such that for every $g \in \mathcal{M}^0_k$ problem $(2.6)_j$ has a unique solution for every $j \in J^g$. By the density of characteristic functions in L^2, the set

$$\mathcal{M} = \bigcup_{k \in \boldsymbol{N}} \mathcal{M}^0_k;$$

is dense in L^2. Moreover by the separability of L^2, there exists a countable set $\mathcal{M}^0 \subseteq \mathcal{M}$, which is dense in L^2. Let us consider $\Gamma = \bigcup_{g \in \mathcal{M}^0} \Gamma^g$, where Γ^g is defined by (2.13). Since, by Remark 2.2, Γ^g is a countable set and $\mathcal{M}^0$ is countable, then Γ is a countable set.

Now, fixed $g \in \mathcal{M}^0$ and $\gamma \in \boldsymbol{R}^+ \setminus \Gamma$ the uniqueness for the problem (2.2) follows by the uniqueness for problem $(2.6)_j$, by Proposition 2.3 and by the definition of Γ. ■

In the following theorem we give a «genericity» result: we establish that the uniqueness of the solution to problem (2.2) is a generic property.

THEOREM 4.5. Let us assume that there exists a countable set $\mathcal{M}^0$, which is dense in L^2, and a countable set Γ in $\boldsymbol{R}^+$ such that for every $g \in \mathcal{M}^0$ and for every $\gamma \in \boldsymbol{R}^+ \setminus \Gamma$ problem (2.2) has a unique solution. Then for every $\gamma \in \boldsymbol{R}^+$ there exists a G_δ-set $\mathcal{M}_\gamma^*$ dense in L^2 such that for each $g \in \mathcal{M}_\gamma^*$ the solution of problem (2.2) is unique.

PROOF. Let $\mathcal{M}^0$ be as in the statement of the theorem. We fix $\gamma \in \boldsymbol{R}^+ \setminus \Gamma$ and $g \in L^2$ and define

$$S(g) = \{u \in \mathcal{H}^1 : u \text{ is a solution of } (2.2)\}.$$

We observe that $S(g) \neq \emptyset$, since as we have seen, there exists at least one solution of (2.2).

Let us define $D: L^2 \to [0, +\infty]$ by

$$D(g) = \sup_{u,\, v \in S(g)} \|u - v\|_{L^1}.$$

This definition implies that (2.2) has a unique solution if and only if $D(g) = 0$.

Now, we are going to prove that the function D is continuous in the points of the set $\mathcal{M}^0$.

Let us fix $\bar{f} \in \mathcal{M}^0$ and suppose that there exist $\bar{n} \in \boldsymbol{N}$ and a sequence (f_k) in L^2 such that f_k converges to $\bar{f}$ in the L^2-topology and

$$D(f_k) \geq \frac{1}{\bar{n}}, \qquad \text{for every } k \in \boldsymbol{N}.$$

This implies that there are two sequences (v_k) and (u_k) in $S(f_k)$ such that

$$\|v_k - u_k\|_{L^1} \geq \frac{1}{\bar{n}}, \qquad \text{for every } k \in \boldsymbol{N}. \tag{4.5}$$

On the other hand, since $\bar{f} \in \mathcal{M}^0$, by hypothesis there exists a unique solution $u_{\bar{f}}$ of the problem

$$\min\{F_\gamma^{\bar{f}}(u) : u \in \mathcal{H}^1).$$

Therefore from Lemma 3.7 we can conclude that v_k and u_k converge to u_f strongly in L^1; but this contradicts (4.5). Hence for every $f \in \mathcal{M}^0$ and $n \in \boldsymbol{N}$ there exists an open neighborhood U_f^n of f in the L^2-topology such that $D(g) < 1/n$ for all $g \in U_f^n$.

At this point, if we denote $U^n = \bigcup_{f \in \mathcal{M}^0} U_f^n$, we have that U^n is an open subset of L^2 with respect to the L^2-topology. Then let us define $\mathcal{M}_\gamma^* = \bigcap_{n \in \boldsymbol{N}} U^n$; $\mathcal{M}_\gamma^*$ is a G_δ-set in L^2 and, by construction, contains $\mathcal{M}^0$; so, by hypothesis, $\mathcal{M}_\gamma^*$ is dense in L^2. Moreover we observe that, fixed $g \in \mathcal{M}_\gamma^*$, for each $n \in \boldsymbol{N}$, g belongs to U^n and so $D(g) = 0$.

This proves the theorem when $\gamma \in \boldsymbol{R}^+ \setminus \Gamma$. Let now $\gamma \in \Gamma$ and fix $\gamma_0 \in \boldsymbol{R}^+ \setminus \Gamma$. Then there exists $\alpha > 0$ such that $\alpha\gamma_0 = \gamma$. By the first part of the theorem, we know that for every $g \in \mathcal{M}_{\gamma_0}^*$ the problem

$$\min_{u \in \mathcal{K}^1} \left[\sum_{s=0}^{l} \int_{b_s}^{b_{s+1}} |u'|^2 dx + \int_0^1 |u - g|^2 dx + \gamma_0 \# (S_u) \right],$$

has only one solution. Multiplying this expression by α, defining $v = \sqrt{\alpha}\, u$ and taking into account that $\#(S_u) = \#(S_v)$ we obtain that, if $f \in \sqrt{\alpha}\ \mathcal{M}_{\gamma_0}^* = \mathcal{M}_\gamma^*$, then the problem

$$\min_{v \in \mathcal{K}^1} \left[\sum_{s=0}^{l} \int_{b_s}^{b_{s+1}} |v'|^2 dx + \int_0^1 |v - f|^2 dx + \gamma \# (S_v) \right],$$

has only one minimizer. Since $\sqrt{\alpha}\, \mathcal{M}_{\gamma_0}^*$ is clearly a dense G_δ-set the proof is accomplished. ∎

COROLLARY 4.6. If Γ_0 is a countable subset of $\boldsymbol{R}^+$, then there exists a dense G_δ-set $\mathcal{M}_{\Gamma_0}^*$ such that for every $g \in \mathcal{M}_{\Gamma_0}^*$ and for every $\gamma \in \Gamma_0$ problem (2.2) has uniqueness.

PROOF. It is enough to define $\mathcal{M}_{\Gamma_0}^* = \bigcap_{\gamma \in \Gamma_0} \mathcal{M}_\gamma^*$ and to observe that by Baire's lemma the countable intersection of dense G_δ-set is still a dense G_δ-set. ∎

In the following theorem, we shall construct a dense G_δ-subset of $L^2([0, 1])$ such that when g belongs to this set, problems (1.2) is uniquely solvable if γ belongs to the complement of a countable subset Γ^g in $\boldsymbol{R}^+$ depending on g.

THEOREM 4.7. There exists a G_δ-set $\mathcal{M}^*$ dense in L^2 such that for every $g \in \mathcal{M}^*$ and $\gamma \in \boldsymbol{R}^+ \setminus \Gamma^g$, where Γ^g is countable, problem (2.2) has uniqueness.

PROOF. In the previous corollary we may choose in particular $\Gamma_0 = \boldsymbol{Q}^+$, where $\boldsymbol{Q}^+$ denotes the set of the positive rational numbers and we can define $\mathcal{M}^* := \mathcal{M}^*_{\Gamma_0}$. Let us take now $g \in \mathcal{M}^*$. Since Γ_0 is dense in $\boldsymbol{R}^+$, we have that its intersection with the interior of Γ^g_h is non empty, for every interval Γ^g_h. Moreover, when γ is a rational number belonging to the interior of Γ^g_h, by Corollary 4.6 problem (2.2) relative to g has uniqueness. Hence by Corollary 2.4, we have uniqueness for every γ belonging to the interior of Γ^g_h and this is true for every $h \in J^g$. The proof follows, recalling that $\boldsymbol{R}^+ = \Gamma^g \cup \left(\bigcup_{h \in J^g} \operatorname{int} \Gamma^g_h \right)$, where $\operatorname{int} \Gamma^g_h$ denotes the interior of Γ^g_h. ∎

REMARK 4.8. It is clear that there is nothing difference in the proof if we substitute the functional F^g_γ with $\widetilde{F}^g_\gamma$, hence the preceeding results continue to hold.

REMARK 4.9. We observe that Theorem 4.7 cannot be improved, that is we cannot expect, fixed $g \in L^2$, to have a unique solution for problem (2.2) for every γ belonging to the complement in $\boldsymbol{R}^+$ of a countable set depending on g. In fact, as we saw in the second example of the introduction, there are functions $g \in L^2$ for which we have to remove a whole interval of $\boldsymbol{R}^+$ in order to have uniqueness.

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