

# RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

CARLO PETRONIO

## **Direct sums of Cartan factors**

*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 88 (1992), p. 111-125

[http://www.numdam.org/item?id=RSMUP\\_1992\\_\\_88\\_\\_111\\_0](http://www.numdam.org/item?id=RSMUP_1992__88__111_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1992, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques*  
<http://www.numdam.org/>

## Direct Sums of Cartan Factors.

CARLO PETRONIO(\*)

In [Pe2] we envisaged the orbit of the origin in the unit ball of a direct sum of two complex Banach spaces (endowed with a suitable norm), with respect to the group of holomorphic automorphisms, and we obtained some general results. As a special case we considered the class of  $p$ -norms, and we proved that the most interesting case is when  $p$  equals 2. For  $p = 2$  we succeeded in giving some information about the orbit of the origin when one of the spaces is either a Hilbert space or a commutative  $C^*$ -algebra with identity. In this paper we consider the case when one of the spaces is a Cartan factor. The reason for considering Cartan factors is that, as we proved in [Pe2], only spaces in which the orbit of the origin in the unit ball is non-trivial can give rise to a direct sum in which such an orbit is non-trivial: and the unit ball of a Cartan factor is homogeneous.

Our main result can be expressed in the following way: if  $F$  is a Cartan factor of type I, II, III or IV and  $F$  is not isometric to a Hilbert space, then, given a non-trivial complex Banach space  $G$ , no point in the orbit of the origin in the unit ball of the 2-sum of  $G$  and  $F$  can have non-zero  $F$ -coordinate.

In the last section we shall prove some results concerning duality theory for Cartan factors.

### 1. Preliminaries and notations.

First of all we recall the definition of Cartan factors (see e.g. [Ha]).

If  $H$  and  $K$  are complex Hilbert spaces, we shall denote by  $\mathcal{L}(H, K)$  the Banach space of continuous linear operators from  $H$  to  $K$ , endowed with the usual «sup» norm;  $\mathcal{L}(H, K)$  will be called a *Cartan factor of type I*.

(\*) Indirizzo dell'A.: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italia.

An anti-linear involutive isometry  $\tau$  of  $H$  will be called a conjugation on  $H$  (remark that such a  $\tau$  enjoys  $(\tau x | \tau y) = (y | x) \forall x, y \in H$ ); given  $\tau$  we define the associated transposition on  $\mathcal{L}(H)$  by  ${}^tA = \tau A^* \tau$ .

LEMMA 1. Given a conjugation  $\tau$  on  $H$  there exists an orthonormal basis  $\{\phi_\alpha\}$  of  $H$  such that  $(\tau\phi | \phi_\alpha) = \overline{(\phi | \phi_\alpha)} \forall \phi \in H, \forall \alpha$ .

With respect to this basis  $({}^tA\phi_\alpha | \phi_\beta) = (A\phi_\beta | \phi_\alpha) \forall A \in \mathcal{L}(H), \forall \alpha, \beta$ .

PROOF. For the first assertion it suffices to show that  $\exists\{\phi_\alpha\}$  such that  $\tau\phi_\alpha = \phi_\alpha \forall \alpha$ . Given  $\phi \in H \setminus \{0\}$  set

$$\phi_1 = \begin{cases} i\phi & \text{if } \tau\phi + \phi = 0, \\ \tau\phi + \phi & \text{otherwise;} \end{cases}$$

we have  $\tau\phi_1 = \phi_1, \phi_1 \neq 0$ ; since  $\tau$  preserves orthogonality the conclusion follows at once by a maximality argument.

The second assertion is a direct consequence of the first one. ■

Given  $\tau$ , the space  $\{A \in \mathcal{L}(H): {}^tA = A\}$  will be called a *Cartan factor of type II*, and the space  $\{A \in \mathcal{L}(H): {}^tA + A \doteq 0\}$  will be called a *Cartan factor of type III*; it is easily checked that they are closed subspaces of  $\mathcal{L}(H)$ , and hence they are naturally endowed with a Banach space structure.

According to Lemma 1, a Cartan factor of type II (resp. III) is the space  $\mathcal{L}_{\{\phi_\alpha\}}^{(s)}(H)$  (resp.  $\mathcal{L}_{\{\phi_\alpha\}}^{(a)}(H)$ ) of symmetric (resp. skew-symmetric) operators with respect to some fixed orthonormal basis  $\{\phi_\alpha\}$  of  $H$ . Since different choices of the basis give rise to isomorphic Banach spaces, the subscript will be omitted.

A closed subspace  $\mathcal{U}$  of  $\mathcal{L}(H)$  will be called a *Cartan factor of type IV* if for any  $A$  in  $\mathcal{U}$  the square of  $A$  is a scalar multiple of the identity operator and  $A^*$  belongs to  $\mathcal{U}$ . As well-known (see [Ha]), a Cartan factor of type IV is linearly and topologically isomorphic to a Hilbert space  $K$ , and there exists a conjugation  $\tau$  on  $K$  such that the norm of  $\psi \in K$  as a point of the Cartan factor is given by

$$\|\psi\|_{\mathcal{U}}^2 = (\psi | \psi) + ((\psi | \psi)^2 - |(\psi | \tau\psi)|^2)^{1/2}.$$

As above, let  $H$  and  $K$  be complex Hilbert spaces. We shall denote by  $\mathcal{L}_0(H, K)$  the closed subspace of  $\mathcal{L}(H, K)$  consisting of compact operators; for  $\phi \in H$  and  $\psi \in K$ , an element  $\psi \otimes \bar{\phi}$  of  $\mathcal{L}_0(H, K)$  is defined by

$$(\psi \otimes \bar{\phi})(\phi_1) = (\phi_1 | \phi) \cdot \psi, \quad \phi_1 \in H.$$

In [Pe1] (extending Schatten's works [Sc1] and [Sc2] from the case of operators on one Hilbert space to the case of operators between two possibly different Hilbert spaces) we defined a subspaces  $\mathcal{L}_1(H, K)$  of  $\mathcal{L}_0(H, K)$  as the set of those operators  $A$  such that  $\|A\|_1 \equiv \text{tr}((A^*A)^{1/2})$  is finite, and we proved that  $\mathcal{L}_1(H, K)$  is a Banach space with respect to the norm  $\|\cdot\|_1$ . Moreover, we checked that for  $A \in \mathcal{L}(H, K)$  and  $B \in \mathcal{L}_1(K, H)$  the trace  $\text{tr}(AB)$  of  $AB$  can be defined, and we established the following.

**THEOREM 1.** The following isometrical isomorphisms hold:

$$\mathcal{L}_0(H, K)^* \cong \mathcal{L}_1(K, H) \quad \mathcal{L}_1(H, K)^* \cong \mathcal{L}(K, H)$$

the value of  $A$  on  $B$  being defined in any case by  $\text{tr}(AB)$ .

If  $F$  and  $G$  are complex Banach spaces, and  $1 \leq p \leq \infty$ , we shall denote by  $F \oplus_p G$  the direct sum of  $F$  and  $G$  endowed with the so-called  $p$ -norm

$$\|(f, g)\|_p = \begin{cases} (\|f\|^p + \|g\|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{\|f\|, \|g\|\} & \text{if } p = \infty. \end{cases}$$

## 2. A few technical lemmas.

According to a theorem proved by Stachó in [St], for any complex Banach space  $F$ , the orbit of the origin with respect to the group of holomorphic automorphisms of the open unit ball  $B$  of  $F$  is given by  $B \cap \cap F_0$ , where  $F_0$  is a closed linear subspace of  $F$ . Moreover the elements of  $F_0$  can be characterized as those points  $c$  of  $F$  for which there exists a continuous homogeneous polynomial  $Q_c: F \rightarrow F$  of degree 2 such that

$$\phi(Q_c(a)) = \|a\|^2 \cdot \overline{\phi(c)}$$

whenever  $a \in F$ ,  $\phi \in F^*$  and  $\phi(a) = \|a\| \cdot \|\phi\|$ ;  $Q_c$  is uniquely determined by this condition, and it will be referred to as «the polynomial relative to  $c$ ».

As in [Pe2], all our results will deal with the subspace  $F_0$  and not with the orbit of the origin itself.

**LEMMA 2.** Let  $E$  and  $F$  be non-trivial complex Banach spaces and assume there exist a linear isometry  $i: E \rightarrow F$  and a surjective linear projection  $p: F \rightarrow i(E)$  such that  $\|p\| = 1$ . Then  $p(F_0) \subseteq i(E_0)$ .

PROOF. Let  $c \in F_0$  and let  $Q_c: F \rightarrow F$  be the polynomial relative to  $c$ . Set  $c_1 = (i^{-1} \circ p)(c)$  and  $Q_1 = i^{-1} \circ p \circ Q_c \circ i: E \rightarrow E$ . We will prove that

$$\phi(Q_1(a)) = \|a\|^2 \cdot \overline{\phi(c_1)}$$

whenever  $a \in E$ ,  $\phi \in E^*$  and  $\phi(a) = \|a\| \cdot \|\phi\|$ , which implies  $c_1 \in E_0$  and then the conclusion.

Let us define a linear mapping  $j: E^* \rightarrow F^*$  by the formula

$$j(\phi)(b) = \phi((i^{-1} \circ p)(b)) \quad \forall \phi \in E^*, b \in F.$$

It follows from  $\|p\| = 1$  that  $j$  is an isometry. Now, let  $a \in E$ ,  $\phi \in E^*$  be such that  $\phi(a) = \|a\| \cdot \|\phi\|$ ; using the definition of  $j$  and the fact that both  $i$  and  $j$  are isometries, we obtain

$$j(\phi)(i(a)) = \|i(a)\| \cdot \|j(\phi)\|$$

and hence, by the definition of  $Q_c$ ,

$$\begin{aligned} j(\phi)(Q_c(i(a))) &= \|i(a)\|^2 \cdot \overline{j(\phi)(c)} \Rightarrow \\ &= \phi((i^{-1} \circ p \circ Q_c \circ i)(a)) = \|a\|^2 \cdot \overline{\phi((i^{-1} \circ p)(c))} \end{aligned}$$

*i.e.*

$$\phi(Q_1(a)) = \|a\|^2 \cdot \overline{\phi(c_1)}$$

and the lemma is proved. ■

We recall a result we proved in [Pe2], which is the basis for all our further investigations.

LEMMA 3. If  $E$  is the Banach space  $\mathbb{C} \oplus_2(\mathbb{C} \oplus_\infty \mathbb{C})$ , then  $E_0 = \mathbb{C} \times \{0\} \times \{0\}$ .

Combining Lemmas 2 and 3 we obtain the following.

LEMMA 4. Let  $L$  be a non-trivial complex Banach space and assume that there exist a linear isometry  $i: \mathbb{C} \oplus_\infty \mathbb{C} \rightarrow L$  and a linear projection  $p$  of  $L$  onto  $i(\mathbb{C} \oplus_\infty \mathbb{C})$  such that  $\|p\| = 1$ .

For any non-trivial complex Banach space  $G$ , if  $(g_0, l_0) \in (G \oplus_2 L)_0$ , then  $p(l_0) = 0$ .

PROOF. Choose  $g_1 \in G$ ,  $g_1^* \in G^*$  such that  $g_1^*(g_1) = \|g_1\| = \|g_1^*\| = 1$ .

Set  $F = G \oplus_2 L$  and  $E = \mathbb{C} \oplus_2 (\mathbb{C} \oplus_\infty \mathbb{C})$  and define

$$i_1: E \rightarrow F \quad p_1: F \rightarrow i_1(E)$$

by

$$i_1(z_1, z_2, z_3) = (z_1 \cdot g_1, i(z_2, z_3)), \quad p_1(g, l) = (g_1^*(g) \cdot g_1, p(l)).$$

$i_1$  and  $p_1$  fulfill the hypothesis of Lemma 2, and hence

$$(i_1^{-1} \circ p_1)(g_0, l_0) \in E_0.$$

It follows from Lemma 3 that

$$(g_1^*(g_0), (i^{-1} \circ p)(l_0)) \in \mathbb{C} \times \{0\} \times \{0\}$$

and hence  $(i^{-1} \circ p)(l_0) = 0$ , i.e.  $p(l_0) = 0$ . ■

**COROLLARY 1.** If in Lemma 4 we assume that, for any  $l \in L$ ,  $i$  and  $p$  can be chosen in such a way that  $l \in i(\mathbb{C} \oplus_\infty \mathbb{C})$ , then for any non-trivial complex Banach space  $G$  we have

$$(G \oplus_2 L)_0 \subseteq G \times \{0\}.$$

**PROOF.** If  $(g_0, l_0) \in (G \oplus_2 L)_0$  and  $l_0 \in i(\mathbb{C} \oplus_\infty \mathbb{C})$  then  $l_0 = p(l_0) = 0$ . ■

**LEMMA 5.** In order to prove that  $(G \oplus_2 L)_0 \subseteq G_0 \times \{0\}$  it suffices to show that  $(G \oplus_2 L)_0 \subseteq G \times \{0\}$ .

**PROOF.** Assume  $(G \oplus_2 L)_0 \subseteq G \times \{0\}$  and let  $(g_0, 0) \in (G \oplus_2 L)_0$ ; since the projection

$$p: G \oplus_2 L \rightarrow G \quad (g, l) \mapsto g$$

has norm 1 then Lemma 2 implies that

$$g_0 = p(g_0, 0) \in G_0$$

and then  $(g_0, 0) \in G_0 \times \{0\}$  so that  $(G \oplus_2 L)_0 \subseteq G_0 \times \{0\}$ . ■

### 3. Direct sums.

Let  $H$  and  $K$  be complex Hilbert spaces. If either  $H$  or  $K$  is 1-dimensional  $\mathcal{L}(H, K)$  is Hilbertian, and hence, by Theorem 4 of [Pe2], when a direct sum with 2-norm is performed, homogeneity along  $\mathcal{L}(H, K)$  is

preserved, *i.e.* for any complex Banach space  $G$

$$(G \oplus_2 \mathcal{L}(H, K))_0 \supseteq \{0\} \times \mathcal{L}(H, K);$$

if this is not the case, the situation is radically modified.

**THEOREM 2.** If  $H$  and  $K$  are at least 2-dimensional then for any non-trivial complex Banach space  $G$  we have

$$(G \oplus_2 \mathcal{L}(H, K))_0 \subseteq G_0 \times \{0\}.$$

**PROOF.** Let  $F = G \oplus_2 \mathcal{L}(H, K)$ , and suppose  $(g_0, A) \in F_0$ ; according to Lemma 5 it is enough to show that  $A = 0$ . In order to check this, it suffices to prove that for every arbitrary pair of unit vectors  $\phi_1 \in H$  and  $\psi_1 \in K$ , we have  $(A\phi_1 | \psi_1) = 0$ .

Fix such  $\phi_1$  and  $\psi_1$  and find  $\phi_2 \in H$  and  $\psi_2 \in K$  in such a way that  $\{\phi_1, \phi_2\}$  and  $\{\psi_1, \psi_2\}$  are orthonormal pairs.

Now, we define the mappings:

$$i: \mathbb{C} \oplus_\infty \mathbb{C} \rightarrow \mathcal{L}(H, K)$$

$$(x, y) \mapsto x \cdot \psi_1 \otimes \overline{\phi_1} + y \cdot \psi_2 \otimes \overline{\phi_2},$$

$$p: \mathcal{L}(H, K) \rightarrow i(\mathbb{C} \oplus_\infty \mathbb{C})$$

$$B \mapsto (B\phi_1 | \psi_1) \cdot \psi_1 \otimes \overline{\phi_1} + (B\phi_2 | \psi_2) \cdot \psi_2 \otimes \overline{\phi_2}.$$

It is easily checked that  $i$  and  $p$  fulfill the hypothesis of Lemma 4, and hence  $p(A) = 0$ ; then  $(A\phi_1 | \psi_1) = 0$  and the theorem is proved. ■

**COROLLARY 2.** If  $H$  and  $K$  are at least 2-dimensional then for any non-trivial complex Banach space  $G$  we have

$$(G \oplus_2 \mathcal{L}_0(H, K))_0 \subseteq G_0 \times \{0\}.$$

**PROOF.** Since the only property used for  $\mathcal{L}(H, K)$  in the above proof is the fact that it contains finite-rank operators, the very same method works for  $\mathcal{L}_0(H, K)$ . ■

Let  $H$  be a complex Hilbert space. If  $H$  has dimension 1,  $\mathcal{L}^{(s)}(H)$  has dimension 1 too, and hence it is a Hilbert space. It is easily verified that if  $\dim(H) \geq 2$  then  $\mathcal{L}^{(s)}(H)$  is not a Hilbert space. Our aim is to prove an analogue of Theorem 2 with  $\mathcal{L}^{(s)}(H)$  replacing  $\mathcal{L}(H, K)$ .

Since for  $\dim(H) = 2$   $\mathcal{L}^{(s)}(H)$  is isometrically isomorphic to a three-dimensional type IV Cartan factor (see [Ca-Ve]) we shall not consider

this case now. Anyway it will follow from Theorem 5 that the next theorem holds for  $\dim(H) = 2$  too.

**THEOREM 3.** If  $H$  is at least 3-dimensional then for any non-trivial complex Banach space  $G$  we have

$$(G \oplus_2 \mathcal{L}^{(s)}(H))_0 \subseteq G_0 \times \{0\}.$$

**PROOF.** As in Theorem 2, we set  $F = G \oplus_2 \mathcal{L}^{(s)}(H)$ , we suppose  $(g_0, A) \in F_0$  and we use Lemma 4 to prove that  $A = 0$ .

Let  $\{\phi_x\}$  be the basis with respect to which symmetry is considered. If, by absurd,  $A \neq 0$ , we can find two indices  $\alpha_1, \alpha_2$  such that  $(A\phi_{\alpha_1} | \phi_{\alpha_2}) \neq 0$ . We distinguish the cases  $\alpha_1 = \alpha_2$  and  $\alpha_1 \neq \alpha_2$ .

For  $\alpha_1 = \alpha_2$  we choose  $\alpha_3$  different from  $\alpha_1$  and we set

$$\begin{aligned} i: \mathbb{C} \oplus_\infty \mathbb{C} &\rightarrow \mathcal{L}^{(s)}(H) \\ (x, y) &\mapsto x \cdot \phi_{\alpha_1} \otimes \overline{\phi_{\alpha_1}} + y \cdot \phi_{\alpha_3} \otimes \overline{\phi_{\alpha_3}}, \\ p: \mathcal{L}^{(s)}(H) &\rightarrow i(\mathbb{C} \oplus_\infty \mathbb{C}) \\ B &\mapsto (B\phi_{\alpha_1} | \phi_{\alpha_1}) \overline{\phi_{\alpha_1}} \otimes \overline{\phi_{\alpha_1}} + (B\phi_{\alpha_3} | \phi_{\alpha_3}) \phi_{\alpha_3} \otimes \overline{\phi_{\alpha_3}}. \end{aligned}$$

Lemma 4 applies; we obtain  $p(A) = 0 \Rightarrow (A\phi_{\alpha_1} | \phi_{\alpha_1}) = 0$ .

For  $\alpha_1 \neq \alpha_2$  we choose  $\alpha_3$  different from both of them (we recall that  $H$  is at least 3-dimensional) and we set

$$\begin{aligned} i: \mathbb{C} \oplus_\infty \mathbb{C} &\rightarrow \mathcal{L}^{(s)}(H) \\ (x, y) &\mapsto x \cdot (\phi_{\alpha_1} \otimes \overline{\phi_{\alpha_2}} + \phi_{\alpha_2} \otimes \overline{\phi_{\alpha_1}}) + y \cdot \phi_{\alpha_3} \otimes \overline{\phi_{\alpha_3}}, \\ p: \mathcal{L}^{(s)}(H) &\rightarrow i(\mathbb{C} \oplus_\infty \mathbb{C}) \\ B &\mapsto (B\phi_{\alpha_1} | \phi_{\alpha_2})(\phi_{\alpha_1} \otimes \overline{\phi_{\alpha_2}} + \phi_{\alpha_2} \otimes \overline{\phi_{\alpha_1}}) + (B\phi_{\alpha_3} | \phi_{\alpha_3}) \phi_{\alpha_3} \otimes \overline{\phi_{\alpha_3}} \end{aligned}$$

and as above we obtain  $(B\phi_{\alpha_1} | \phi_{\alpha_2}) = 0$ .

In both cases we got a contradiction and hence the theorem is proved. ■

Let  $H$  be a complex Hilbert space. If  $\dim(H) = 1$  then  $\dim(\mathcal{L}^{(a)}(H)) = 0$ , and if  $\dim(H) = 2$  then  $\dim(\mathcal{L}^{(a)}(H)) = 1$ ; in both cases  $\mathcal{L}^{(a)}(H)$  is a Hilbert space; since for  $\dim(H) = 2$   $\mathcal{L}^{(a)}(H) \cong \mathcal{L}(\mathbb{C}, \mathbb{C}^3) \cong \mathbb{C}^3$  (see [Ca-Ve]),  $\mathcal{L}^{(a)}(H)$  is a Hilbert space in this case too. It follows that an analogue of Theorems 2 and 3 can hold for  $\mathcal{L}^{(a)}(H)$  only if  $\dim(H)$  is bigger than 3.



**THEOREM 4.** If  $H$  is at least 4-dimensional then for any non-trivial complex Banach space  $G$  we have

$$(G \oplus_2 \mathcal{L}^{(a)}(H))_0 \subseteq G_0 \times \{0\}.$$

**PROOF.** Let  $\{\phi_x\}$  be the orthonormal basis of  $H$  with respect to which skew-symmetry in  $\mathcal{L}(H)$  is defined, set  $F = G \oplus_2 \mathcal{L}^{(a)}(H)$  and suppose  $(g_0, A) \in F_0$ . As we remarked above, it is enough to show that  $A = 0$ .

In order to prove this it suffices to check that for an arbitrary fixed pair of indices  $\alpha_1 \neq \alpha_2$ , it happens that  $(B\phi_{\alpha_1} | \phi_{\alpha_2}) = 0$ .

Since  $H$  is at least 4-dimensional, we can find  $\alpha_3$  and  $\alpha_4$  in such a way that  $\alpha_1, \dots, \alpha_4$  are different from each other. For the sake of simplicity, we set  $\phi_i = \phi_{\alpha_i}$ .

As above, we define two linear mappings

$$i: \mathbb{C} \oplus_\infty \mathbb{C} \rightarrow \mathcal{L}^{(a)}(H)$$

$$(x, y) \mapsto x(\phi_2 \otimes \bar{\phi}_1 - \phi_1 \otimes \bar{\phi}_2) + y(\phi_4 \otimes \bar{\phi}_3 - \phi_3 \otimes \bar{\phi}_4),$$

$$p: \mathcal{L}^{(a)}(H) \rightarrow i(\mathbb{C} \oplus_\infty \mathbb{C})$$

$$B \mapsto (B\phi_1 | \phi_2)(\phi_2 \otimes \bar{\phi}_1 - \phi_1 \otimes \bar{\phi}_2) + (B\phi_3 | \phi_4)(\phi_4 \otimes \bar{\phi}_3 - \phi_3 \otimes \bar{\phi}_4).$$

It is readily verified that the hypothesis of Lemma 4 are fulfilled. It follows that  $p(A) = 0$ , hence  $(B\phi_{\alpha_1} | \phi_{\alpha_2}) = 0$  and the proof is complete. ■

**THEOREM 5.** Let  $\mathcal{U}$  be a Cartan factor of type IV, and assume that  $\mathcal{U}$  is at least 2-dimensional. Then for any non-trivial complex Banach space  $G$  we have

$$(G \oplus_2 \mathcal{U})_0 \subseteq G_0 \times \{0\}.$$

**PROOF.** Since if  $\dim(\mathcal{U}) = 2$  then  $\mathcal{U} \cong \mathbb{C} \oplus_\infty \mathbb{C}$  (see [Ca-Ve]), the theorem is certainly true in this case (once again Lemma 4 is used).

Using Corollary 1, the general case will be deduced from the following fact: given  $x \in \mathcal{U} \setminus \{0\}$  there exists a subspace  $\mathcal{V}$  of  $\mathcal{U}$  with the property that:

- a)  $x \in \mathcal{V}$ ;
- b)  $\mathcal{V}$  is 2-dimensional;

c)  $\mathfrak{V}$  is a type IV Cartan factor with respect to the induced norm;

d) there exists a surjective linear projection  $p: \mathfrak{U} \rightarrow \mathfrak{V}$  with  $\|p\| \leq 1$ .

To see this, we represent  $\mathfrak{U}$  as a Hilbert space  $K$  with conjugation  $\tau$ , as we mentioned in Section 1. We define a 2-subspace  $M \subseteq K$  containing  $x$  in the following way: if  $\tau x \notin \mathbb{C}x$  then  $M$  is generated by  $x$  and  $\tau x$ ; otherwise we remark that  $x^\perp$  is  $\tau$ -invariant, so that we can find  $x' \in x^\perp \setminus \{0\}$  such that  $\tau x' = x'$ , and then  $M$  is generated by  $x$  and  $x'$ . Since  $M$  is  $\tau$ -invariant, it is a type IV Cartan factor with respect to the induced norm. We are left to check property d).

We define  $p: K \rightarrow M$  as the orthogonal projection; our aim is to prove that for  $y \in M$  and  $z \in M^\perp$  the continuous real function

$$f(t) = \|y + tz\|_{\mathfrak{U}}^2$$

has minimum in 0. We can assume that  $y$  and  $z$  are both non-zero; since  $M$  and  $M^\perp$  are  $\tau$ -invariant,  $(y|z) = (\tau y|z) = (y|\tau z) = 0$ , and then  $f$  can be re-written as

$$f(t) = |y|^2 + |z|^2 t^2 + ((|y|^2 + |z|^2 t^2)^2 - |(y|\tau y) + (z|\tau z)t^2|^2)^{1/2}$$

( $|w|$  denotes the norm of  $w$  in  $K$ ). If for some point  $t_0 \neq 0$  the argument of the square root vanishes in  $t_0$ , it is easily verified that  $f$  is expressed by

$$f(t) = |y|^2 + |z|^2 t^2$$

and hence it does have minimum in 0. Conversely, suppose the argument of the square root does not vanish in  $\mathbb{R} \setminus \{0\}$ ; then  $f$  is differentiable in  $\mathbb{R} \setminus \{0\}$ ; by direct calculation we obtain that  $f'(t) = t \cdot q(t)$ , where  $q$  is a strictly positive continuous function. It follows that  $f$  has minimum in 0 in this case too, and hence the proof is complete. ■

Theorems 2, 3, 4 and 5 of the present paper (together with the accompanying remarks) are summarized by the following.

**THEOREM 6.** Let  $G$  be a non-trivial complex Banach space and let  $F$  be a Cartan factor of type I, II, III or IV. The following mutually exclusive possibilities are given:

a)  $F$  is a Hilbert space, and

$$(G \oplus_2 F)_0 \supseteq \{0\} \times F;$$

b)  $F$  is not a Hilbert space, and

$$(G \oplus_2 F)_0 \subseteq G_0 \times \{0\}.$$

The above result provides the machinery for the proof of an analogue of Theorem 8 in [Pe2], with the spaces  $L^p(\Omega, \mu)$  replaced by Cartan factors.

We shall denote by  $\mathcal{C}$  the category of all non-zero Cartan factors of type I, II, III or IV (the morphisms being the linear isometries) and by  $\mathcal{S}$  the category whose objects are the Banach spaces obtained from the objects of  $\mathcal{C}$  by a finite number of operations of direct sum of the type  $\oplus_r$  (with  $r \in [1, \infty]$ ), and whose morphisms are the linear isometries again. (If the collection  $\mathcal{B}$  of all Banach spaces were a set, and not only a category, we would have defined  $\mathcal{S}$  as the closure of  $\mathcal{C}$  in  $\mathcal{B}$  with respect to the operations  $\oplus_r$ .)

If  $E$  is an object of  $\mathcal{S}$ ,  $E$  is linearly and topologically isomorphic to a product  $F_1 \times \dots \times F_k$  where  $F_1, \dots, F_k$  are objects of  $\mathcal{C}$ ; for  $i \in \{1, \dots, k\}$  we can think of  $E$  as the space built up starting from  $F_i$  and adding to it other objects  $G_1, \dots, G_h$  of  $\mathcal{S}$ ; that is, we can represent  $E$  by

$$(\dots((F_i \oplus_{r_1} G_1) \oplus_{r_2} G_2) \oplus_{r_3} \dots) \oplus_{r_h} G_h$$

(or by a similar formula where the sums are not all performed at the right side). In such a case we will say that  $\langle \oplus_{r_1}, \dots, \oplus_{r_h} \rangle$  are, in the order, the direct sums which appear in  $E$  after  $F_i$ .

**THEOREM 7.** Let  $E$  be an object of  $\mathcal{S}$ , topologically and linearly isomorphic to a product of objects of  $\mathcal{C}$ ,  $F_1 \times \dots \times F_k$ . Then

$$E_0 = R_1 \times \dots \times R_k,$$

where  $R_i \subseteq F_i$  is either  $\{0\}$  or the whole  $F_i$ .

Precisely,  $R_i$  is equal to  $F_i$  if, and only if, one of the following conditions is fulfilled:

- (a)  $F_i$  is a Hilbert space and after  $F_i$  there are first some  $\oplus_2$  (possibly none) and then some  $\oplus_\infty$  (possibly none);
- (b) after  $F_i$  there are only  $\oplus_\infty$  (possibly none).

**PROOF.** We confine ourselves to a sketch since the argument imitates closely the one presented for Theorem 8 in [Pe2].

Let us consider by simplicity the first coordinate  $F_1$  and prove that if (a) or (b) are satisfied homogeneity along  $F_1$  is preserved, while if neither (a) nor (b) are satisfied homogeneity along  $F_1$  is lost.

In case (a) the conclusion follows from Theorems 3 and 4 of [Pe2], while case (b) is immediately settled.

If neither (a) nor (b) are fulfilled, one of the following cases occurs:

- (c) after  $F_1$  there is some  $\oplus_p$  with  $p \neq 2, \infty$ ;
- (d) after  $F_1$  there is a  $\oplus_\infty$  followed by a  $\oplus_2$ ;
- (e)  $F_1$  is not a Hilbert space and after  $F_1$  there is a  $\oplus_2$ .

In all these cases homogeneity along  $F_1$  is lost: case (c) follows from Theorem 2 of [Pe2], case (d) from Theorem 6 of [Pe2] and case (e) from Theorem 6 above. ■

#### 4. Duality theory for Cartan factors.

Theorem 1 establishes a duality theory for Cartan factors of type I; we will prove that a completely analogous result holds for Cartan factors of type II and III. Afterwards we will consider the case of type IV Cartan factors.

Let  $H$  be a non-trivial Hilbert space, let  $\tau$  be a conjugation on  $H$  and let  $A \mapsto {}^tA$  be the transposition associated to  $\tau$ .

We set  $\mathcal{L}_i^{(s)}(H) = \mathcal{L}^{(s)}(H) \cap \mathcal{L}_i(H)$  and  $\mathcal{L}_i^{(a)}(H) = \mathcal{L}^{(a)}(H) \cap \mathcal{L}_i(H)$  for  $i = 0, 1$ . We want to prove the following.

**THEOREM 8.**  $\mathcal{L}_i^{(s)}(H)$  and  $\mathcal{L}_i^{(a)}(H)$  (for  $i = 0, 1$ ) are complex Banach spaces with respect to the induced norms, and the following isometrical isomorphisms hold:

- (1)  $\mathcal{L}_0^{(s)}(H)^* \cong \mathcal{L}_1^{(s)}(H),$
- (2)  $\mathcal{L}_1^{(s)}(H)^* \cong \mathcal{L}_0^{(s)}(H),$
- (3)  $\mathcal{L}_0^{(a)}(H)^* \cong \mathcal{L}_1^{(a)}(H),$
- (4)  $\mathcal{L}_1^{(a)}(H)^* \cong \mathcal{L}_0^{(a)}(H),$

the value of  $A$  on  $B$  being given in any case by  $\text{tr}(AB)$ .

For the proof of this theorem we need a few technical preliminaries.

We recall that for  $A \in \mathcal{L}(H)$ ,  $[A] \equiv (A^*A)^{1/2}$ .

**LEMMA 6.**  $\forall A \in \mathcal{L}(H)$ ,  $[{}^tA] = \tau[A^*]\tau$ .

**PROOF.** Since  $(\tau x | \tau y) = (y | x)$  we have

$$\begin{aligned} (({}^tA)^* x | y) &= ((\tau A^* \tau)^* x | y) = (x | \tau A^* \tau y) = (A^* \tau y | \tau x) = (\tau y | A \tau x) = \\ &= (\tau A \tau x | y) \Rightarrow ({}^tA)^* = \tau A \tau \Rightarrow [{}^tA]^2 = \tau A \tau \tau A^* \tau = (\tau[A^*] \tau)^2 \end{aligned}$$

and conclusion follows from the fact that  $\tau[A^*]\tau \geq 0$ . ■

LEMMA 7. (a)  $A \in \mathcal{L}_1(H) \Rightarrow {}^tA \in \mathcal{L}_1(H)$ ,  $\|{}^tA\|_1 = \|A\|_1$ ,  $\text{tr}({}^tA) = \text{tr}(A)$ ;

(b)  $A \in \mathcal{L}_0(H) \Rightarrow {}^tA \in \mathcal{L}_0(H)$ ,  $\|{}^tA\| = \|A\|$ .

PROOF. (a) Let  $\{\phi_\alpha\}$  be an orthonormal basis of  $H$ ; then if  $\psi_\alpha = \tau\phi_\alpha$ ,  $\{\psi_\alpha\}$  is an orthonormal basis too, and hence

$$\begin{aligned} \|{}^tA\|_1 &= \text{tr}({}^t[A]) = \sum_\alpha (\tau[A^*] \tau\phi_\alpha | \phi_\alpha) = \sum_\alpha (\psi_\alpha | [A^*] \psi_\alpha) = \\ &= \sum_\alpha ([A^*] \psi_\alpha | \psi_\alpha) = \text{tr}([A^*]) = \|A^*\|_1 = \|A\|_1. \end{aligned}$$

The first two assertions are proved; as for the third one

$$\begin{aligned} \text{tr}({}^tA) &= \sum_\alpha ({}^tA\phi_\alpha | \phi_\alpha) = \sum_\alpha (\tau A^* \tau\phi_\alpha | \phi_\alpha) = \\ &= \sum_\alpha (\tau\phi_\alpha | A^* \tau\phi_\alpha) = \sum_\alpha (A\psi_\alpha | \psi_\alpha) = \text{tr}(A). \end{aligned}$$

(b) is obvious. ■

Now, for  $A \in \mathcal{L}(H)$  we set  $A^s = 1/2 \cdot (A + {}^tA)$ ,  $A^a = 1/2 \cdot (A - {}^tA)$ . It follows from Lemma 7 that if  $A \in \mathcal{L}_i(H)$  then  $A^s, A^a \in \mathcal{L}_i(H)$  (for  $i = 0, 1$ ).

LEMMA 8. (a) Given  $A \in \mathcal{L}_1(H)$  we have

$$\text{tr}(AB) = 0 \quad \forall B \in \mathcal{L}_0^{(s)}(H) \Leftrightarrow {}^tA = -A.$$

(b) Given  $A \in \mathcal{L}_1(H)$  we have  $\text{tr}(AB) = 0 \quad \forall B \in \mathcal{L}_0^{(a)}(H) \Leftrightarrow {}^tA = A$ .

(c) Given  $A \in \mathcal{L}(H)$  we have  $\text{tr}(AB) = 0 \quad \forall B \in \mathcal{L}_1^{(s)}(H) \Leftrightarrow {}^tA = -A$ .

(d) Given  $A \in \mathcal{L}(H)$  we have  $\text{tr}(AB) = 0 \quad \forall B \in \mathcal{L}_1^{(a)}(H) \Leftrightarrow {}^tA = A$ .

PROOF. (a)  $\Leftarrow$ .

$$\text{tr}(AB) = -\text{tr}({}^tA {}^tB) = -\text{tr}({}^t(BA)) = -\text{tr}(BA) = -\text{tr}(AB).$$

$\Rightarrow$ . Suppose  $A^s \neq 0$ ; since  $A^s \in \mathcal{L}_1(H)$  and  $\mathcal{L}_1(H) \cong \mathcal{L}_0(H)^*$  we can find  $B \in \mathcal{L}_0(H)$  with  $\text{tr}(A^s B) \neq 0$ ; but by the hypothesis and by the first implication

$$\text{tr}(A^s B) = \text{tr}(A^s B^s) = \text{tr}(AB^s) = 0$$

and this is absurd.

The proof of (b), (c), and (d) is completely analogous. ■

LEMMA 9. (a)  $A \in \mathcal{L}^{(s)}(H), B \in \mathcal{L}^{(a)}(H) \Rightarrow \|A + B\| \geq \max \{\|A\|, \|B\|\}$ .

(b)  $A \in \mathcal{L}_1^{(s)}(H), B \in \mathcal{L}_1^{(a)}(H) \Rightarrow \|A + B\|_1 \geq \max \{\|A\|_1, \|B\|_1\}$ .

PROOF. (a) Since the transposition is an isometry

$$\begin{aligned} \|A + B\| &= \|^t A - {}^t B\| = \|^t(A - B)\| = \|A - B\| \Rightarrow \\ \Rightarrow \|A\| &= \frac{1}{2} \|A + B + A - B\| \leq \frac{1}{2} (\|A + B\| + \|A - B\|) = \|A + B\|. \end{aligned}$$

The same holds for  $\|B\|$ .

(b) As the transposition is an isometry for  $\|\cdot\|_1$  too, the proof works as above. ■

PROOF OF THEOREM 8. Lemma 7 implies that  $\mathcal{L}_i^{(s)}(H)$  and  $\mathcal{L}_i^{(a)}(H)$  are closed subspaces of  $\mathcal{L}_i(H)$  for  $i = 0, 1$ , and hence the first assertion is obvious.

(1) By Theorem 1, Lemma 8 and the Hahn-Banach theorem there exists a one-to-one mapping  $\alpha$  from  $\mathcal{L}_0^{(s)}(H)^*$  onto  $\mathcal{L}_1^{(s)}(H)$  such that

$$\phi(B) = \text{tr}(\alpha(\phi)B) \quad \forall \phi \in \mathcal{L}_0^{(s)}(H)^*, B \in \mathcal{L}_0^{(s)}(H).$$

$\alpha$  is obviously a linear isomorphism. We are left to prove that  $\alpha$  is an isometry, i.e.

$$\|A\|_1 = \sup_{B \in \mathcal{L}_0^{(s)}(H)} \frac{|\text{tr}(AB)|}{\|B\|} \quad \forall A \in \mathcal{L}_1^{(s)}(H).$$

By Theorem 1, since  $\mathcal{L}_0^{(s)}(H) \subseteq \mathcal{L}_0(H)$ , inequality  $\geq$  is obvious.

As for the converse, using part (a) of Lemma 9,

$$\begin{aligned} \|A\|_1 &= \sup_{B \in \mathcal{L}_0^{(s)}(H)} \frac{|\text{tr}(A(B^s + B^a))|}{\|B^s + B^a\|} \leq \sup_{B \in \mathcal{L}_0^{(s)}(H)} \frac{|\text{tr}(AB^s)|}{\|B^s\|} = \\ &= \sup_{B \in \mathcal{L}_0^{(s)}(H)} \frac{|\text{tr}(AB)|}{\|B\|} \end{aligned}$$

(2) As above, it suffices to prove that

$$\|A\| \leq \sup_{B \in \mathcal{L}_1^{(s)}(H)} \frac{|\text{tr}(AB)|}{\|B\|_1} \quad \forall A \in \mathcal{L}^{(s)}(H).$$

The proof works as above, using part (b) of Lemma 9.

(3) and (4) are completely analogous. ■

Now we turn to the case of type IV Cartan factors.

As we mentioned in Section 1, if  $\mathcal{U}$  is a type IV Cartan factor then  $\mathcal{U}$  is linearly and topologically isomorphic to a Hilbert space  $K$ ; for a suitable conjugation  $\tau$  on  $K$  the  $\mathcal{U}$ -norm is given by

$$\|x\|_{\mathcal{U}} = ((x|x) + ((x|x)^2 - |(x|\tau x)|^2)^{1/2})^{1/2} \quad (x \in K).$$

Since  $K$  is a reflexive space  $\mathcal{U}$  is reflexive too; moreover the dual space  $\mathcal{U}^*$  of  $\mathcal{U}$  is linearly and topologically isomorphic to the dual space  $\overline{K}$  of  $K$ ; we recall that  $\overline{K}$  is the Hilbert space which coincides with  $K$  as a real Hilbert space, in which multiplication by complex number and inner product are given by

$$\lambda \cdot_{\overline{K}} x = \overline{\lambda} \cdot_K x \quad (\lambda \in \mathbb{C}, x \in K),$$

$$(x|y)_{\overline{K}} = (y|x)_K \quad (x, y \in K).$$

We remark that  $\tau$  is a conjugation on  $\overline{K}$  too.

The  $\mathcal{U}^*$ -norm on  $\overline{K}$  is given by

$$\|y\|_{\mathcal{U}^*} = \sup_{x \in K \setminus \{0\}} \frac{|(x|y)|}{\|x\|_{\mathcal{U}}} \quad (y \in \overline{K}).$$

In order to determine completely the dual space of  $\mathcal{U}$  we only have to compute explicitly this norm.

**THEOREM 9.**  $\|y\|_{\mathcal{U}^*}^2 = 1/2 \cdot ((y|y) + |(y|\tau y)|).$

**PROOF.** The theorem is certainly true if  $\dim \mathcal{U} = 1$ .

Now, assume  $\dim \mathcal{U} = 2$  and let  $\{\phi_1, \phi_2\}$  be an orthonormal basis of  $K$  such that  $\tau\phi_i = \phi_i$  for  $i = 1, 2$ . An isometrical isomorphism

$$j: \mathcal{U} \rightarrow \mathbb{C} \oplus_{\infty} \mathbb{C}$$

is explicitly given by

$$j(z_1\phi_1 + z_2\phi_2) = (z_1 + iz_2, z_1 - iz_2).$$

Since  $(\mathbb{C} \oplus_{\infty} \mathbb{C})^* = \mathbb{C} \oplus_1 \mathbb{C}$ ,

$$j^*: \mathbb{C} \oplus_1 \mathbb{C} \rightarrow \mathcal{U}^*$$

is an isometrical isomorphism. By direct computation we deduce from this that the  $\mathcal{U}^*$ -norm on  $\overline{K}$  has the required expression.

Now, assume that  $\dim \mathcal{U} \geq 3$ ; for  $y \in \overline{K} \setminus \{0\}$  we define  $M$  as the subspace of  $K$  generated by  $y$  and  $\tau y$  and we consider the orthogonal projection  $p$  of  $K$  onto  $M$ . During the proof of Theorem 5 it was checked

that  $\|p\| = 1$ , *i.e.*

$$\|p(x)\|_u \leq \|x\|_u \quad \forall x \in K.$$

It follows that

$$\sup_{x \in K \setminus \{0\}} \frac{|(x|y)|}{\|x\|_u} = \sup_{x \in M \setminus \{0\}} \frac{|(x|y)|}{\|x\|_u}.$$

Everything reduces to the 2-dimensional (or, possibly, 1-dimensional) case, and hence the theorem holds in the general case too. ■

#### REFERENCES

- [Ca-Ve] E. CALABI - E. VESENTINI, *On compact, locally symmetric Kähler manifolds*, Ann. Math., **71** (1960), pp. 472-507.
- [Ha] L. A. HARRIS, *Bounded symmetric homogeneous domains in infinite dimensional spaces*, Lectures Notes in Mathematics, **364**, Springer, Berlin-Heidelberg-New York (1973), pp. 13-40.
- [Pe1] C. PETRONIO, *Holomorphic automorphism groups in certain compact operator spaces*, Rend. Accad. Naz. Lincei, Serie IX, Vol. I, pp. 125-130.
- [Pe2] C. PETRONIO, *Holomorphic automorphisms of the unit ball of a direct sum*, Rend. Accad. Naz. Lincei, Serie IX, Vol. I, pp. 203-213.
- [Sc1] R. SCHATTEN, *A Theory of Cross-Spaces*, Princeton University Press (1950).
- [Sc2] R. SCHATTEN, *Norm ideals of completely continuous operators*, Ergebn. der Math., **27**, Springer-Verlag (1960).
- [St] L. L. STACHÒ, *A short proof of the fact that biholomorphic automorphisms of the unit ball in certain  $L^p$  spaces are linear*, Acta Sci. Math., **41** (1979), pp. 381-383.

Manoscritto pervenuto in redazione il 12 novembre 1990 e, in versione corretta, il 13 febbraio 1991.