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Wielandt Series and Defects of Subnormal Subgroups in Finite Soluble Groups.

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1. Introduction.

We are interested in this paper in the subnormal structure of finite soluble groups; we show that the Wielandt length of a finite soluble group is bounded in terms of its Fitting length and of the maximum defect of its subnormal subgroups.

The Wielandt subgroup $\omega(G)$ of a group G is the intersection of all normalizers of subnormal subgroups of G; clearly $\omega(G)$ is a characteristic subgroup of G. The Wielandt series:

$$\omega_0(G) \leq \ldots \leq \omega_n(G) \leq \omega_{n+1}(G) \leq \ldots,$$

is defined by (see [9; § 4.6]):

$$\omega_0(G) = 1 \quad \text{and} \quad \omega_{n+1}(G)/\omega_n(G) = \omega(G/\omega_n(G)).$$

H. Wielandt [13] proved that if G is a finite group then $\omega(G) \neq 1$, for in this case $\omega(G)$ contains all minimal normal subgroups of G. It there follows that for a finite group G there is a smallest positive integer n such that $\omega_n(G) = G$: such n is called the Wielandt length of G; we denote it by $wl(G)$. From now on, all groups considered will be finite.

We denote by $b(G)$ the maximum among the defects of subnormal subgroups of the group G, and by $\boldsymbol{B}_n$ the class of all groups G such that $b(G) \leq n$. H. Wielandt observed that $b(G) \leq wl(G)$ [13]; thus a bound on the Wielandt length implies a bound on the defects of the subnormal subgroups. When dealing with soluble groups, there is a bound on the derived length of a group in terms of its Wielandt length (the exact bound has been determined by R. Bryce and J. Cossey [2], see [4] for

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an extension to infinite groups). On the other hand, while soluble groups in $\boldsymbol{B}_1$ are metabelian [6] [10] and finite soluble groups in $\boldsymbol{B}_2$ have derived length at most 5 [5], no bound exists on the derived length of finite soluble groups in $\boldsymbol{B}_3$, as first shown by T. Hawkes [8] (see also R. Bryce [1] for examples of $\boldsymbol{B}_3$-groups with abelian Sylow subgroups and arbitrary derived length). Thus, a fortiori, no bound on the Wielandt length of a group exists depending only on the maximum defect of its subnormal subgroups. Also, no bound on $b(G)$ (and so on the Wielandt length $wl(G)$) exists depending only on the derived length of the group G: the wreath product of a cyclic group of order p (a prime number) by itself is metabelian and has subnormal subgroups of defect p.

The n-th term $\boldsymbol{F}_n(G)$ of the Fitting series of G is defined by $\boldsymbol{F}_0(G) = 1$ and $\boldsymbol{F}_{n+1}(G)/\boldsymbol{F}_n(G) = \mathrm{Fit}(G/\boldsymbol{F}_n(G))$. If G is soluble, the Fitting (or nilpotent) length of G is the smallest n such that $\boldsymbol{F}_n(G) = G$, and it is denoted by $l(G)$.

We shall show that there exists a function $\bar{f}\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$wl(G) \le \bar{f}(l(G), b(G))$$

for any finite soluble group G.

In fact this result is a corollary of a more general statement (Theorem 2): if $\boldsymbol{X}$ is a class of finite soluble groups, closed by homomorphic images and normal subgroups, such that all nilpotent groups belonging to it have nilpotency class at most c, then $wl(G) \le f(l(G), c)$ for every G in $\boldsymbol{X}$.

2. A strong Wielandt subgroup.

In order to find bounds on the Wielandt length of a finite group, we have found it useful to introduce a special subgroup of the Wielandt subgroup, which is in some sense easier to deal with. Then we define a corresponding ascending series, a bound on whose length implies a bound on the Wielandt length. In our definition and most of our arguments, we have chosen a «local» approach, as it is done in [2] for the Wielandt subgroup.

DEFINITIONS. Let G be a group, p a prime number. We put

$$\overline{\omega}^p(G) = \{x \in G;\ [H, x] \le O^p(H) \text{ for all } p'\text{-perfect subnormal subgroups } H \text{ of } G\}.$$

(We remind that, if π is a set of primes, a group H is said to be π-perfect if $O^{\pi}(H) = H$). And

$$\overline{\omega}(G) = \{x \in G;\ [H, x] \leq H^N \text{ for all } H \operatorname{sn} G\}.$$

(Here H^N is the nilpotent residual of H, i.e. the smallest normal subgroup K of H such that H/K is nilpotent.)

Then $\overline{\omega}^p(G)$ and $\overline{\omega}(G)$ are characteristic subgroups of G, and it is immediate that $\overline{\omega}^p(G)$ is contained in $\omega^p(G)$ (the intersection of the normalizers of all p'-perfect subnormal subgroups of G, as defined in [2]), and $\overline{\omega}(G) \leq \omega(G)$.

PROPOSITION 1. *Let G be a group, $\pi(G)$ the set of all primes dividing $|G|$, $\sigma(G)$ the socle of G. Then:*

i) $\overline{\omega}(G) = \bigcap_{p \in \pi(G)} \overline{\omega}^p(G)$.

ii) $\overline{\omega}(G) \geq \sigma(G)$.

PROOF. (i) If H is p'-perfect then $O^p(H) = H^N$. Thus $\overline{\omega}(G) \leq \overline{\omega}^p(G)$ for every $p \in \pi(G)$. Conversely, let $x \in \overline{\omega}^p(G)$ for every $p \in \pi(G)$, and $H \operatorname{sn} G$. For each p, $H_p = O^{p'}(H)$ is p'-perfect and subnormal in G; so $[H_p, x] \leq O^p(H_p) \leq H^N$. Since H is the normal product of the H_p's, $[H, x] \leq H^N$ and $x \in \overline{\omega}(G)$.

(ii) Let M be a minimal normal subgroup of G. We have to show that $M \leq \overline{\omega}(G)$. Let $H \operatorname{sn} G$; then MH is subnormal in G and so $M \leq \sigma(MH)$. We may therefore assume $MH = G$. Then $M \geq [M, H] \trianglelefteq G$, so $[M, H] = 1$ or $[M, H] = M$. Since the second case implies $H^N \geq M$, we have, in both cases, $[M, H] \leq H^N$, as wanted. ■

The next lemma states some trivial properties of $\overline{\omega}^p(G)$ and $\overline{\omega}(G)$. The proof is straightforward, and we omit it.

LEMMA 1. *Let N be a normal subgroup of G. Then:*

i) $\overline{\omega}^p(G)N/N \leq \overline{\omega}^p(G/N)$ and $\overline{\omega}(G)N/N \leq \overline{\omega}(G/N)$.

ii) $\overline{\omega}^p(G) \cap N \leq \overline{\omega}^p(N)$ and $\overline{\omega}(G) \cap N \leq \overline{\omega}(N)$.

The next proposition shows that the subgroups $\overline{\omega}^p(G)$ and $\overline{\omega}(G)$ enjoy a property which is not satisfied by the corresponding standard Wielandt subgroups. Since we do not need it in the proof of our main result, we again leave the easy proof to the reader.

PROPOSITION 2. *Let G and H be groups, p a prime. Then:*

i) $\overline{\omega}^p(G \times H) = \overline{\omega}^p(G) \times \overline{\omega}^p(H)$.

ii) $\overline{\omega}(G \times H) = \overline{\omega}(G) \times \overline{\omega}(H)$.

Let now n be a positive integer, we define $\overline{\omega}_n^p(G)$ and $\overline{\omega}_n(G)$ by putting: $\overline{\omega}_0^p(G) = \overline{\omega}_0(G) = 1$ and

$$\overline{\omega}_n^p(G)/\overline{\omega}_{n-1}^p(G) = \overline{\omega}^p(G/\overline{\omega}_{n-1}^p(G)), \quad \overline{\omega}_n(G)/\overline{\omega}_{n-1}(G) = \overline{\omega}(G/\overline{\omega}_{n-1}(G)),$$

for $n \geq 1$.

Observe that if G is nilpotent, then the series $\overline{\omega}_n(G)$. $n \in \mathbb{N}$, coincides with the upper central series of G.

LEMMA 2. *Let G be a p-soluble group, and $K = O_{p'}(G)$. Then $\overline{\omega}^p(G) \geq K$ and $\overline{\omega}^p(G)/K = \overline{\omega}^p(G/K) = \overline{\omega}(G/K)$.*

PROOF. We first show that, if $K = 1$, $\overline{\omega}^p(G) = \overline{\omega}(G)$. Let $T = \overline{\omega}^p(G)$; then T is a p-group, since $O_{p'}(T) = 1$ and all normal p-sections of T are central. If $H \operatorname{sn} G$, then T normalizes $O^p(H)$ because $O^p(H)$ is p-perfect subnormal and T is a normal p-subgroup. Also $[T, O^p(H)] \leq T \cap O^p(H) \leq O^{p'}(O^p(H))$. On the other hand, T normalizes $O^{p'}(H)$ by definition and $[T, O^{p'}(H)] \leq O^p(O^{p'}(H))$. In conclusion:

$$[T, H] \leq [T, O^p(H)][T, O^{p'}(H)] \leq H^N.$$

Thus $T \leq \overline{\omega}(G)$. Since the reverse inclusion is trivial, we get $T = \overline{\omega}(G)$.

Let now $K \neq 1$, and H a p'-perfect subnormal subgroup of G. Then H is normalized by K and so $[K, H] \leq K \cap H \leq O^p(H)$. Hence $K \leq \overline{\omega}^p(G)$. This, together with Lemma 1, gives:

$$\overline{\omega}^p(G)/K \leq \overline{\omega}^p(G/K) = \overline{\omega}(G/K),$$

by the case discussed above. To finish, we have to show that $\overline{\omega}(G/K) \subseteq \overline{\omega}^p(G)/K$. Let $gK \in \overline{\omega}(G/K)$, H a p'-perfect subnormal subgroup of G. Then $H = O^{p'}(HK)$ is normalized by g and

$$[H, g] \leq H \cap KH^N = O^p(H)K \cap H = O^p(H)(K \cap H) = O^p(H).$$

Hence $g \in \overline{\omega}^p(G)$, concluding the proof of the lemma. ■

LEMMA 3. *For every $i \in \mathbb{N}$: $\overline{\omega}_i^p(O^{p'}(G)) \leq \overline{\omega}_i^p(G)$.*

PROOF. This is because all p'-perfect subnormal subgroups of G lie in $O^{p'}(G)$. ■

By Proposition 1 (ii), for every finite group G there is a minimum positive integer n such that $\overline{\omega}_n(G) = G$. We call such n the strong Wielandt length of G, and denote it by $\overline{wl}(G)$. It is then clear that $\overline{wl}(G) \geq wl(G)$, and a bound on $\overline{wl}(G)$ implies a bound on $wl(G)$. However, we do not know the exact relation between $\omega_n(G)$ and $\overline{\omega}_n(G)$ (and thus between the two corresponding lengths). In particular we do not know whether there exists a positive integer m such that $\omega(G) \leq \overline{\omega}_m(G)$ for all groups G.

LEMMA 4. *Let P be a normal p-subgroup of the group G. Suppose that P has nilpotency class d and that P/P' is contained in $\overline{\omega}_r^p(G/P')$. Then $P \leq \overline{\omega}_{g(r,d)}^p(G)$, where $g(r, d) = (2^d - 1)r - 2^{d-1} + 1$.*

PROOF. By induction on d.

If $d = 1$, then $P' = 1$, so $P \leq \overline{\omega}_r^p(G)$ and, in fact, $g(r, 1) = (2-1)r - 1 + 1 = r$.

Let now $d > 1$. Write $W_0 = \gamma_d(P)$ (the last non trivial term of the lower central series of P). Then, by inductive hypothesis:

$$P/W_0 \leq \overline{\omega}_a^p(G/W_0), \qquad \text{where } a = g(r, d-1).$$

For $i = 1, \dots, a$ let $W_i/W_0 = \overline{\omega}_i^p(G/W_0) \cap P/W_0$. Put also $X_0 = 1$ and, for $n = 1, \dots, 2a-1$:

$$X_n = \langle [W_i, W_j];\ i, j \in \{1, \dots, a\},\ i + j = n + 1 \rangle.$$

The X_n is normal in G for each n. We show that X_n/X_{n-1} is contained in $\overline{\omega}^p(G/X_{n-1})$ for every $n = 1, \dots, 2a-1$.

Let H be a subnormal p'-perfect subgroup of G. Then $O^p(H)$ is normalized by P, because P is a normal p-subgroup and $O^p(H)$ is subnormal and p-perfect. Thus $O^p(H) \trianglelefteq PH$. Let $t, s \in \{1, \dots, a\}$ and $t + s = n + 1$ $(n = 1, \dots, 2a-1)$. Then:

$$[W_t, H, W_s] \leq [O^p(H)W_{t-1}, W_s] \leq O^p(H)[W_{t-1}, W_s] \leq O^p(X)X_{n-1}$$

(observe that, if $t = 1$, $[W_{t-1}, W_s] = [W_0, W_s] = 1$).

Similarly: $[W_s, H, W_t] \leq O^p(H)X_{n-1}$. Since $O^p(H)X_{n-1}$ is normal in PH, by the three subgroups lemma we get:

$$[W_t, W_s, H] \leq O^p(H)X_{n-1}.$$

This is true for all pairs (t, s) with $t + s = n + 1$ and so

$$[X_n, H] \leq O^p(H)X_{n-1}$$

showing that X_n/X_{n-1} is contained in $\overline{\omega}^p(G/X_{n-1})$.

We have therefore $X_n \leq \overline{\omega}_n^p(G)$ for every $n = 1, \dots, 2a-1$. In particular

$$P' = [W_a, W_a] = X_{2a-1} \leq \overline{\omega}_{2a-1}^p(G).$$

Since P/P' is contained in $\overline{\omega}_r^p(G/P')$ we conclude that

$$P \leq \overline{\omega}_{2a-1+r}^p(G)$$

and:

$$2a - 1 + r = 2(g(r, d-1)) - 1 + r = 2((2^{d-1} - 1)r - 2^{d-2} + 1) - 1 + r =$$

$$= (2^d - 1)r - 2^{d-1} + 1 = g(r, d). \quad \blacksquare$$

REMARKS. The same argument of the preceding proof can be used to show that if N is a nilpotent normal subgroup of G, and N/N' is contained in $\overline{\omega}_r(G/M')$ then $N \leq \overline{\omega}_k(G)$, where k depends only on r and on the nilpotency class of N. In this form, the result is in the same spirit of a well known nilpotency criterion of P. Hall [7], stating that if both N and G/N' are nilpotent then G is nilpotent of class which depends on the classes of N and of G/N'. We have not attempted to find the best bounds for k above or $g(r, d)$ of Lemma 4. Also, it is conceivable that a similar result holds for the standard Wielandt series, but we have not been able to find a proof in this case.

3. The main result.

We begin this section with three known results, that will be used in the proof of our main theorem.

LEMMA 5. *Let A be a normal p-subgroup of G, complemented by Q in G, and let S be a p-perfect subnormal subgroup of G. Then $S = (S \cap A)(S \cap Q)$.*

PROOF. Since S is subnormal in G:

$$O^p(AS) = O^p(A)\,O^p(S) = O^p(S) = S.$$

Then, in particular, $S \trianglelefteq AS$. Let $H = AS \cap Q$, then $AH = AS$ and we have that $H/(H \cap S) \simeq HS/S \leq AS/S$ is a p-group. But $H = H/(A \cap H) \simeq$ $7AH/A = AS/A \simeq S/(A \cap S)$ is p-perfect. Thus $H = H \cap S$, so $H \leq S$, whence $H = S \cap Q$ and $(A \cap S)H = AH \cap S = AS \cap S$, as wanted. $\blacksquare$

LEMMA 6. *Let H, K be subgroups of G, such that H, K are subnormal in $J = \langle H, K \rangle$, and let $\boldsymbol{X}$ be the class of soluble groups of p-length* *at most* n (*for a fixed* $n \in \mathbb{N}$). *Then*

$$J^{X} = \langle H^{X}, K^{X} \rangle .$$

PROOF. This follows from the fact that, for every set π of primes, $H, K \operatorname{sn} J = \langle H, K \rangle$ implies $O^{\pi}(J) = \langle O^{\pi}(H), O^{\pi}(K) \rangle$, and induction on n. ■

LEMMA 7 (Shult [12]; Carter and Hawkes [3; Theorem 5.15]). *Let* $\boldsymbol{F}$ *be a saturated formation and* G *a soluble group whose* $\boldsymbol{F}$*-residual* $G^{\boldsymbol{F}}$ *is abelian. Then* $G^{\boldsymbol{F}}$ *is complemented and any two complements are conjugate.*

THEOREM 1. *There exists a function* $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that if* G *is a soluble group of* p*-length* n, *and all subnormal* p*-sections of* G *have class at most* c, *then*

$$O_p(G) \leq \overline{\omega}_{h(c,\, n)}(G) .$$

PROOF. We proceed by induction on the p-length n of G.

Observe that if A is a normal p-subgroup of G, then $A \leq \overline{\omega}^{q}(G)$ for every prime $q \neq p$. Thus, to prove the Theorem, it is enough to show that $O_p(G)$ is contained in $\overline{\omega}^{p}_{h(c,\, n)}(G)$.

If $n = 1$, then by Lemmas 2 and 3:

$$Z_i(O_{p',\,p}(G)/O_{p'}(G)) \leq \overline{\omega}^{p}_{i}(G/O_{p'}(G)) = \overline{\omega}^{p}_{i}(G)/O_{p'}(G)$$

for $i = 1, 2, \ldots, c$. Hence $O_{p',\,p}(G) \leq \overline{\omega}^{p}_{c}(G)$. We put $h(c, 1) = c$.

Let now $n > 1$ and suppose that we have proved the assertion for $n - 1$ and thus found $h(c, n - 1)$. Let $\boldsymbol{X}$ be the class of all soluble groups of p-length $n - 1$; then $\boldsymbol{X}$ is a saturated formation, and it is closed by taking normal products. Given a group K we denote by $\overline{K}$ its $\boldsymbol{X}$-residual. We prove the following fact:

(∗) let G be a soluble group of p-length n. Assume that all subnormal p-sections of G have class at most c and that $\overline{G}$ is an abelian p-group. Then $\overline{G} \leq \overline{\omega}^{p}_{h(c,\, n-1)}(G)$.

Let G be a minimal counterexample. Write $A = \overline{G}$ and let $N = G_{\boldsymbol{X}}$ be the $\boldsymbol{X}$-radical of G; also put $d = h(c, n - 1)$.

1) $O_{p'}(G) = 1$ *and* G/N *is a* p*-group.*

Let $K = O_{p'}(G)$; then $AK/K = (\overline{G/K})$ is abelian and a p-group; if

$K \neq 1$ then $AK/K \leq \overline{\omega}_d^p(G/K)$ and so, by Lemma 2, $A \leq \overline{\omega}_d^p(G)$. Hence $K = 1$. Furthermore $A = \overline{O^{p'}(G)}$. Since, by Lemma 3, $\overline{\omega}_i^p(O^{p'}(G)) \leq \leq \overline{\omega}_i^p(G)$, our choice of G implies $G = O^{p'}(G)$. Now G/N has p-length 1, and $O_{p'}(G/N) = 1$, so G/N is a p-group.

2) *A is not decomposable as the direct product of two proper G-invariant subgroups.*

Suppose this false, and let $R \times S = A$ be a proper direct G-decomposition of A. Now $(\overline{G/R}) = A/R$ so, by our choice of G, $A/R \leq \leq \overline{\omega}_d^p(G/R)$. For $i = 1, \ldots, d$ let $R_i/R = \overline{\omega}_i^p(G/R) \cap A/R$ and $X_i = R_i \cap S$; thus $X_i R = R_i$. Now, by Lemma 7, A is complemented in G, say by Q_0. It there follows that R is complemented in G by $Q = SQ_0$. Let H be a p'-perfect subnormal subgroup of G. Then, for $i = 1, \ldots, d$ (and with $R_0 = R$):

$$[R_i/R_{i-1}, HR_{i-1}/R_{i-1}] \leq O^p(H) R_{i-1}/R_{i-1}.$$

In particular $[R_i, H] \leq O^p(H) R_{i-1}$. Now $O^p(H)$ is p-perfect, whence, by Lemma 5, $O^p(H) = (O^p(H) \cap Q)(O^p(H) \cap R)$. We have therefore:

$$[H_i, H] = [R_i \cap S, H] \leq [R_i, H] \cap S \leq O^p(H) R_{i-1} \cap S \leq$$

$$\leq (O^p(H) \cap Q) R_{i-1} \cap Q = (O^p(H) \cap Q)(R_{i-1} \cap Q) \leq O^p(H) X_{i-1},$$

showing that X_i/X_{i-1} is contained in $\overline{\omega}^p(G/X_{i-1})$ for all $i = 1, \ldots, d$. Thus $S = X_d \leq \overline{\omega}_d^p(G)$. Applying the same argument, we obtain also $R \leq \overline{\omega}_d^p(G)$ and so $A \leq \overline{\omega}_d^p(G)$, contradicting the choice of G. Thus 2) is proved.

Let now $\mathfrak{M} = \{H; N \leq H \leq G, \overline{H} < A \text{ and } \overline{T} = A \text{ for all } T > H\}$. Observe that, since $\overline{N} = 1$, $\mathfrak{M} \neq \emptyset$. Let $H \in \mathfrak{M}$. By Lemma 7, $\overline{H}$ is complemented in H by Q_H, say, and all such complements are conjugate. Let $C_H = A \cap Q_H$; then $1 \neq C_H \trianglelefteq H$ and $A = \overline{H} \times C_H$. Observe in particular that $\overline{H}$ and C_H are normal subgroups of N.

3) *For every* $H \in \mathfrak{M}$, $C_H \leq \overline{\omega}_d^p(G)$.

Since $H/\overline{H}$ has p-length $n-1$, $A/\overline{H}$ is contained in $\overline{\omega}_d^p(H/\overline{H})$ by our inductive hypothesis. Then $C_H \leq \overline{\omega}_d^p(H)$ follows by the same argument used in part 2). Thus $C_H \leq \overline{\omega}_d^p(G)$ if we show that $[C_H, S] \leq \leq O^p(S)$ for any p'-perfect subnormal subgroup S of G not contained in H. Let therefore S be such a subgroup, and put $S_1 = NS$. Then, in particular, $S_1 > N$ and $\langle S, H\rangle > H$. By Lemma 6:

$$\overline{S}_1 = \langle \overline{S}, \overline{N}\rangle = \overline{S}$$

and, since $H \in \mathcal{M}$: $\overline{S}\cdot\overline{H} = \langle \overline{S}, \overline{H}\rangle = A$.

Let $X/\overline{S} = O_{p'}(S_1/\overline{S})$. I claim that $\overline{S} = [A, X]$. In fact, if $V = [A, X]$, then $V \leq \overline{S}$ because $A/\overline{S}$ and $X/\overline{S}$ are normal subgroups of $S_1/\overline{S}$ of coprime order. On the other hand, $V \trianglelefteq S_1$, AX/V is nilpotent and, if $P/\overline{S} = O_{p',p}(S_1/\overline{S})$, P/AX is a p-group. It follows that P/V has p-length 1, and so S_1/V belongs to $\boldsymbol{X}$; this yields $\overline{S} \leq V$ and consequently $\overline{S} = V$.

Now, $X \leq N$, so X normalizes both $\overline{H}$ and C_H. Thus:

$$\overline{S} = [A, X] = [\overline{H}, X] \times [C_H, X] \quad \text{with } [C_H, X] \leq C_H.$$

Hence we have: $A = \overline{S}\cdot\overline{H} = \overline{H}[C_H, X]$, which gives $[C_H, X] = C_H$. Therefore $\overline{S} \geq C_H$. In particular $O^p(S) \geq C_H$, and so $[C_H, S] \leq O^p(S)$. This shows that $C_H \leq \overline{\omega}_a^p(G)$.

4) *Proof of* (*) *concluded.*

Let $\emptyset \neq \mathcal{N} \subseteq \mathcal{M}$. By induction or $r = |\mathcal{N}|$, we show that

$$A = \langle C_H;\ H \in \mathcal{N}\rangle \times \left(\bigcap_{H \in \mathcal{N}} \overline{H}\right).$$

If $r = 1$: $A = \overline{H} \times C_H$ (where $\mathcal{N} = \{H\}$) by definition of C_H. Let $r > 1$ and let $H \in \mathcal{N}$; $\mathcal{N}_0 = \mathcal{N} \setminus \{H\}$. By inductive hypothesis $A = C_0 \times H_0$, where $C_0 = \langle C_K;\ K \in \mathcal{N}_0\rangle$ and $H_0 = \bigcap_{H \in \mathcal{N}_0} \overline{K}$. Let $X/\overline{H} = O_{p'}(H/\overline{H})$. As we saw in point 3), $\overline{H} = [A, X]$. Also $[C_H, X] \leq C_H \cap \overline{H} = 1$, and this yields $C_H = C_A(X)$, as X acts by coprime action on the abelian group A. Now, C_0 and H_0 are normal in N, whence, in particular, they are normalized by X. Thus we have:

$$H_0 = [H_0, X] \times C_{H_0}(X) = (\overline{H} \cap H_0) \times (C_H \cap H_0)$$

and $C_H = (H_0 \cap C_H) \times (C_0 \cap C_H)$. Therefore:

$$A = (\overline{H} \cap H_0) \times \langle C_H \cap H_0, C_0\rangle = (\overline{H} \cap H_0) \times \langle C_H, C_0\rangle,$$

as wanted.

Let now $C = \langle C_H;\ H \in \mathcal{M}\rangle$. Clearly $C \trianglelefteq G$ and C is complemented by $T = \bigcap_{H \in \mathcal{M}} \overline{H}$ in A, by what we proved above. But T is also normal in G, so, by point 2), $T = 1$, because certainly $C \neq 1$. Hence $C = A$. By point 3), $C_H \leq \overline{\omega}_a^p(G)$ for all $H \in \mathcal{M}$; thus $A \leq \overline{\omega}_a^p(G)$, a contradiction which concludes the proof of claim (*).

We now finish the proof of the Theorem. We have $d = h(c, n-1)$ by inductive hypothesis. Let G be a soluble group of p-length n, all of whose subnormal p-sections have nilpotency class at most c.

By Lemma 2, we may assume $O_{p'}(G) = 1$. Let $P = O_p(G)$. Then G/P has p-length $n-1$. Hence $A = G^{\boldsymbol{X}} \leq P$, where $\boldsymbol{X}$ is the class of soluble groups of p-length $n-1$. By (*)

$$A/A' \leq \overline{\omega}_d^p(G/A').$$

By applying Lemma 4, se get $A \leq \overline{\omega}^p_{g(d,\,c)}(G)$. Since P/A is contained in $\overline{\omega}_d^p(G/A)$ by inductive hypothesis, we finally have

$$P \leq \overline{\omega}^p_{h(c,\,n)}(G)$$

where $h(c, n) = g(d, c) + d = g(h(c, n-1), c) + (c, n-1)$, which, by what we observed at the beginning of the proof, it is enough to conclude that P is contained in $\overline{\omega}_{h(c,\,n)}(G)$. ■

Now observe that the function h of Theorem 1 (as well as the function g of Lemma 4) does not depend on the prime p. Thus if we put:

$$f(c, n) = \left(\sum_{i=1}^{n} h(c, i)\right) + 1$$

we obtain the following

COROLLARY 1. *Let G be a soluble group.*

(i) *If G has p-length n, an all subnormal p-sections of G have class at most c:*

$$G = \overline{\omega}^p_{f(c,\,n)}(G).$$

(ii) *If G has Fitting length $l(G)$ and all nilpotent subnormal sections of G have class at most c:*

$$\overline{wl}(G) \leq f(c, l(G)).$$

PROOF. It follows easily from Lemma 2, Theorem 1, and induction on n. ■

We restate part of the corollary to make more transparent our application to the classes $\boldsymbol{B}_m$.

THEOREM 2. *Let $\boldsymbol{X}$ be a class of soluble groups, closed by homomorphic images and normal subgroups, and suppose that every nilpotent group in $\boldsymbol{X}$ has nilpotency class at most c. Then*:

$$\overline{wl}(G) \leq f(c, l(G))$$

for all G in $\boldsymbol{X}$.

This in particular applies to the classe $\boldsymbol{B}_m$ ($m \in \mathbb{N}$); they are closed by homomorphic images and normal subgroups and, by a theorem of J. Roseblade [11], nilpotent groups in $\boldsymbol{B}_m$ have nilpotency class bounded by a function of m. Thus Theorem 2 yields:

COROLLARY 2. *There exists a function $\bar{f}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for every soluble group G*:

$$wl(G) \leq \overline{wl}(G) \leq \bar{f}(l(G), b(G)).$$

An interesting particular case is the class $\boldsymbol{B}_2$. In fact if G is a soluble group in $\boldsymbol{B}_2$ (i.e. if $b(G) \leq 2$), then $l(G) \leq 4$ (see [5]). Hence we have

COROLLARY 3. *Soluble groups in $\boldsymbol{B}_2$ have bounded (strong) Wielandt length.*

By combined results of H. Heineken and S. K. Mahdavianary, nilpotent groups in $\boldsymbol{B}_2$ have class at most three. Thus $f(3, 4)$ of Theorem 2 provides such a bound. However, direct computation of it gives a rather unrealistic value; we leave it open the problem of determining the exact bound for the Wielandt length of a finite soluble $\boldsymbol{B}_2$-group (I do not know of any example of a finite soluble $\boldsymbol{B}_2$-group whose Wielandt length exceeds 3).

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