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A ZJ-Theorem for p^*, p-Injectors in Finite Groups.

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1. Introduction and notation.

All groups considered in this paper are finite. A group G is said to be p-stable, p prime, if whenever A is a p-subgroup of G and B is a p-subgroup of $N_G(A)$ such that $[A, B, B] = 1$ then $B \leq \leq O_p(N_G(A) \bmod. C_G(A))$. In [5] Glauberman obtained the following theorem:

Let p be an odd prime and let P be a Sylow p-subgroup of a group G. Assume that G is p-stable and that $C_G(O_p(G)) \leq O_p(G)$. Then $ZJ(P)$ is a characteristic subgroup of G, where $ZJ(P) = Z(J(P))$ and $J(P)$ is the Thompson's subgroup of P, that is, subgroup of P generated by the set $\mathfrak{A}(P)$ of all abelian subgroups of maximum order of P.

With the same conditions he also obtained a factorization of the group G.

In the same paper Glauberman introduces the characteristic subgroup $ZJ^*(P)$ and proves the following result:

Let p be an odd prime and let P be a Sylow p-subgroup of a group G. Suppose that $C_G(O_p(G)) \leq O_p(G)$ and that $SA(2, p)$ is not involved in G. Then $ZJ^*(P)$ is a characteristic subgroup of G and $C_G(ZJ^*(P)) \leq \leq ZJ^*(P)$.

Some related results were obtained by Ezquerro [4] and Pérez Ramos [9, 10].

Given a fixed prime p, we shall denote by $\mathfrak{S}_p$ the class of all p-

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groups, $\mathfrak{E}_{p^*}$ that all p^*-groups [2] and $\mathfrak{E}_{p^*}\mathfrak{S}_p$ that of the p^*,p-groups; the corresponding radicals in a group G are denoted, as usual, by $O_p(G)$, $O_{p^*}(G)$ and $O_{p^*,p}(G)$ respectively. Every group G possesses p^*,p-injectors which are the subgroups of the form $O_{p^*}(G)P$ where P describes the set of Sylow p-subgroups of G [8].

The aim of this paper is to establish the analogous to Glauberman's results with the subgroups $ZJ(K)$ and $ZJ^*(K)$ where K is a p^*,p-injector of a group G.

2. The factorization.

LEMMA 2.1. Let G be a group of $F^*(G)\leq H\leq G$. Then it follows:

$$\pi(ZJ(H))\subseteq\pi(F(G))=\pi(F(H)).$$

PROOF. Clearly $\pi(ZJ(H))\subseteq\pi(F(H))$ and $\pi(F(G))\subseteq\pi(F(H))$. On the other hand as $\pi(F(H))=\pi(Z(F(H)))$ and $Z(F(H))\leq C_G(F(G))\cap \cap C_G(E(G))=C_G(F^*(G))\leq F(G)$, then the result follows.

COROLLARY 2.2. Let G be a group and K a p^*,p-injector of G. If $p\in\pi(F(G))$ then $p\in\pi(ZJ(K))$. Moreover if $O_{p'}(F(G))\leq Z(K)$ then $\pi(F(G))=\pi(F(K))=\pi(ZJ(K))$ and in particular the following are equivalent: i) $F(G)\neq 1$, ii) $F(K)\neq 1$, iii) $ZJ(K)\neq 1$.

PROOF. We can assume that $K=O_{p^*}(G)P$ where P is a Sylow p-subgroup of G. If $O_p(G)\neq 1$ then we have:

$$1\neq Z(P)\cap O_p(G)\leq C_K(O_{p^*}(G))\cap C_K(P)=Z(K)\leq ZJ(K).$$

If moreover $O_{p'}(F(G))\leq Z(K)$ then $O_{p'}(F(G))\leq ZJ(K)$ and we can conclude that $\pi(F(G))\subseteq\pi(ZJ(K))$. Now Lemma 2.1 applies.

LEMMA 2.3. Let G be a group and K a p^*,p-injector of G. Assume that $O_{p'}(F(G))\leq Z(K)$. Let B be a nilpotent normal subgroup of G and let A be a nilpotent subgroup of K, then AB is a nilpotent subgroup of G.

PROOF. By a known result of Bialostocki [3] it is enough to prove that $AO_q(B)$ is nilpotent for every $q\in\pi(B)$.

Since K is a p^*,p-injector of G we have $B\leq F(G)\leq K$ and so $O_{p'}(B)\leq O_{p'}(F(G))\leq Z(K)$. Then $AO_q(B)$ is nilpotent for every prime q with $q\neq p$. Now assume that $K=O_{p^*}(G)P$ where P is a Sylow p-subgroup of G, then:

$$[O_p(B), A] \leq [O_p(B), O_{p^*}(G)P] = [O_p(B), P] \leq [P, P] = P'.$$

By induction on n we can prove that for every $n \geq 1$:

$$[O_p(B), A; n] \leq [O_p(B), P; n] \leq \Gamma_{n+1}(P)$$

and so, for some positive integer m is $[O_p(B), A; m] = 1$, thus A acts nilpotently on $O_p(B)$ and then $AO_p(B)$ is nilpotent.

PROPOSITION 2.4. Let G be a p-stable group. If K is a p^*, p-injector of G and A is an abelian normal subgroup of K then $A \trianglelefteq\trianglelefteq G$ and $A \leq F(G)$. In particular $ZJ(K) \leq F(G)$.

PROOF. Clearly $O_{p'}(A) \leq O_{p^*}(K) = O_{p^*}(G) \leq O_{p^*,p}(G) \leq K$ thus $O_{p'}(A) \trianglelefteq O_{p^*,p}(G)$. If $K = O_{p^*}(G)P$ then $O_p(A) \trianglelefteq P$. Set $Q = P \cap O_{p^*,p}(G)$ then $G = O_{p^*,p}(G) N_G(Q)$. Since $Q \leq N_G(O_p(A))$ and A is abelian then $[Q, O_p(A), O_p(A)] = 1$. As $O_p(A) \leq N_G(Q)$ and G is p-stable we have:

$$O_p(A)\, C_G(Q)/C_G(Q) \leq O_p(N_G(Q)/C_G(Q)) = M/C_G(Q).$$

On the other hand $C_G(Q) \leq O_{p^*,p}(G)$ [7], so $MO_{p^*,p}(G)/O_{p^*,p}(G)$ is a normal p-subgroup of $G/O_{p^*,p}(G)$, hence is trivial. So $O_p(A) \trianglelefteq O_{p^*,p}(G)$ thus $A \trianglelefteq\trianglelefteq G$ and $A \leq F(G)$.

THEOREM 2.5. Let G be a p-stable group, p odd and $F(G) \neq 1$. If K is a p^*, p-injector of G and $O_{p'}(F(G)) \leq Z(K)$ then $1 \neq ZJ(K) \trianglelefteq G$.

PROOF. (This proof is based, in part, on Glauberman's proof of his ZJ-theorem ([6], Th. 8,2.10)).

First note that as the p^*, p-injectors of G are coniugated, the statements $ZJ(K) \trianglelefteq G$ and $ZJ(K)$ char G are equivalent.

As a consequence of the above Proposition we know that $ZJ(K) \leq F(G)$ and by Corollary 2.2 $ZJ(K) \neq 1$. Now to obtain the theorem it is enough to prove that if B is a nilpotent normal subgroup of G then $B \cap ZJ(K)$ is normal in G.

Let G be a minimal counterexample and B a nilpotent normal subgroup of G of least order such that $B \cap ZJ(K)$ is not normal in G.

Set $Z = ZJ(K)$ and let B_1 be the normal closure of $B \cap Z$ in G, then $B \cap Z = B_1 \cap Z$ an by our minimal choice it follows $B = B_1$.

Now $B' < B$ then $B' \cap Z \trianglelefteq G$, hence for every g of G is $[(B \cap Z)^g, B] = [B \cap Z, B]^g \leq B' \cap Z$. Since B is the normal closure of $B \cap Z$

in G it follows that $B' \leq B' \cap Z$. In particular $B \cap Z$ centralizes B' and by an analogue argument we obtain that $[B, B, \mathrm{B}] = 1$.

Consider $A \in \mathfrak{A}(K)$. For Lemma 2.3 we know that AB is nilpotent. Thus there exists a positive integer n such that $[B, A; n] = 1$. Moreover as $O_{p'}(B) \leq Z(K)$ and p odd it follows that $[A, B]' = [A, O_p(B)]'$ has odd order.

By ([1], Cor. 2.8) there exists $A \in \mathfrak{A}(K)$ such that $B \leq N_G(A)$, therefore $[B, A, C] = 1$. In particular $[O_p(B), O_p(A), O_p(A)] = 1$. Since G is p-stable we have:

$$O_p(A)\, C/C \leq O_p(G/C) = T/C \trianglelefteq G/C$$

where $C = C_G(O_p(B))$. Now, since $O_{p'}(A)$ centralizes $O_p(B)$ it follows that $A \leq T$.

If $T = G$ then G/C is a p-group so $KC = G$. Moreover as $O_{p'}(B) \leq$ $\leq Z(K) \leq Z$ is $B \cap Z = O_{p'}(B)(O_p(B) \cap Z)$ a normal subgroup of KC, what is a contradiction. Thus $T < G$. Since $A \leq K \cap T$ it follows $\mathfrak{A}(K \cap T) \subseteq$ $\subseteq \mathfrak{A}(K)$, $J(K \cap T) \leq J(K)$ and $ZJ(K) \leq ZJ(K \cap T)$. By our minimal choice of $GZJ(K \cap T)$ char T and so it follows $ZJ(K \cap T)$ normal in G. Then $B \leq ZJ(K \cap T)$. In particular B is abelian. If $J(K \cap T) < J(K)$ then there exists $A_1 \in \mathfrak{A}(K)$ such that A_1 there is not a subgroup of T, then we must have $[B, A_1, A_1] \neq 1$. Set $\mathfrak{S} = \{A \in \mathfrak{A}(K) \mid [B, A, A] \neq 1\}$, we choose $A_1 \in \mathfrak{S}$ such that $|A_1 \cap B|$ is maximal. By ([1], Th. 2.5) there exists $A_2 \in \mathfrak{A}(K)$ such that $A_2 \leq N_G(A_1)$ and $B \cap A_1 < B \cap A_2$. Therefore $[B, A_2, A_2] = 1$ thus $A_2 \leq T$ and $ZJ(K \cap T) \leq A_2$. Hence:

$$[B, A_1, A_1] \leq [ZJ(K \cap T), A_1, A_1] \leq [A_2, A_1, A_1] = 1$$

what is a contradiction.

Consequently $J(K \cap T) = J(K)$ and $ZJ(K) = ZJ(K \cap T) \trianglelefteq G$. This is the last contradiction.

Corollary 2.6. Let G be a p-stable group, p odd, with $F(G) \neq 1$. If K is a p^*, p-injector of G and $O_{p'}(F(G)) \leq Z(K)$ then:

$$G = N_G(J(K))\, C_G(ZJ(K)) = N_G(J(K))\, C_G(Z(K)).$$

Proof. Let us write $Z = ZJ(K)$ and $C = C_G(Z)$. As $Z \trianglelefteq G$ is also $C \trianglelefteq G$ and therefore $G = CN_G(K \cap C)$. Now, as $J(K \cap C)$ char $K \cap C$, it follows $G = CN_G(J(K \cap C))$. Since $J(K) \leq K \cap C$, $J(K) = J(K \cap C)$. Moreover as $Z(K) \leq Z$ we have $C \leq C_G(Z(K))$ and we obtain:

$$G = N_G(J(K))\, C_G(ZJ(K)) = N_G(J(K))\, C_G(Z(K)).$$

COROLLARY 2.7 (Glauberman's ZJ-Theorem). Let G be a group with $O_p(G) \neq 1$, $O_{p'}(G) = 1$, which is p-constrained and p-stable, p odd. If P is a Sylow p-subgroup of G, then $ZJ(P) \trianglelefteq G$.

PROOF. Since G is p-constrained then $O_{p^*}(G) = O_{p'}(G) = 1$ ([2], Lemma 6.12) and so $K = O_{p^*}(G)P = P$ is a p^*,p-injector of G and Theorem 2.5 applies.

3. A self-centralizing characteristic subgroup.

DEFINITION 3.1 [4]. For any group K define two sequences of characteristic subgroups of K as follows. Set $ZJ^0(K) = 1$ and $K_0 = K$. Given $ZJ^i(K)$ and K_i $i \geq 0$ let $ZJ^{i+1}(K)$ and K_{i+1} the subgroups of K that contain $ZJ^i(K)$ and satisfy:

$$ZJ^{i+1}(K)/ZJ^i(K) = ZJ(K_i/ZJ^i(K)),$$

$$K_{i+1}/ZJ^i(K) = C_{K_i/ZJ^i(K)}(ZJ^{i+1}(K)/ZJ^i(K)).$$

Let n be the smallest integer such that $ZJ^n(K) = ZJ^{n+1}(K)$, then $ZJ^n(K) = ZJ^{n+r}(K)$ and $K_n = K_{n+r}$ for every $r \geq 0$. Set $ZJ^*(K) = ZJ^n(K)$ and $K_* = K_n$.

REMARKS. 1) Notice that if $C_G(ZJ^*(K)) \leq ZJ^*(K)$ then by ([4], Prop.II 3.7) $K_*/ZJ^*(K)$ is nilpotent, therefore $ZJ(K_*/ZJ^*(K) = 1$ implies $K_*/ZJ^*(K) = 1$, that is, $K_* = ZJ^*(K)$.

2) If K is a p^*,p-injector of G then $ZJ^i(K)$ is p-nilpotent for every $i \geq 0$. Later we will improve this statement.

LEMMA 3.2. Let K be a p^*,p-injector of G and $N \trianglelefteq G$ such that $C_K(N) \leq N \leq K$ then $C_G(N) = Z(N)$.

PROOF. Observe that $Z(N) = C_K(N) = C_G(N) \cap K$ is a p^*,p-injector of $C_G(N)$. On the other hand if $x \in C_G(N)$ then $\langle x, Z(N)\rangle$ is an abelian subgroup of G and $Z(N) \leq \langle x, Z(N)\rangle \leq C_G(N)$, therefore $\langle x, Z(N)\rangle = Z(N)$ and $x \in Z(N)$.

REMARK. Notice that if K is a p^*,p-injector of G then K is also p^*,p-injector of $N_G(K_*)$. Moreover by ([4], Prop.II 3.7) $C_K(K_*) \leq C_K(ZJ^*(K)) \leq K_*$. Thus, by the above Lemma, is $C_{N_G(K_*)}(K_*) = C_G(K_*) \leq K_*$.

THEOREM 3.3. Let p be an odd prime and K a p^*,p-injector of a group G. Assume that $SA(2,p)$ is not involved in G and that

$O_{p'}(F(G)) \leq Z(K)$. Then $ZJ^i(K)$ is a characteristic subgroup of G for every $i \geq 0$.

PROOF. Let G be a minimal counterexample. As $SA(2,p)$ is not involved in G, G is p-stable.Thus by Theorem 2.5 we have $ZJ(K)$ char G. If $ZJ(K) = 1$ then $ZJ^i(K) = 1$ contrary to the choice of G. So we can assume $1 \neq ZJ(K) \leq C_G(ZJ(K)) \trianglelefteq G$. Set $C = C_G(ZJ(K))$. Assume $C < G$ then for every $i \geq 0$ it follows $ZJ^i(K \cap C)$ char C. Since $J(K) \leq K \cap C$ we have $J(K) = J(K \cap C)$ and $ZJ(K) = ZJ(K \cap C)$. Moreover if $K_1 = = C_K(ZJ(K)) = C_{K \cap C}(ZJ(K \cap C))$ and by induction on i it follows $ZJ^i(K) = ZJ^i(K \cap C)$. Thus $ZJ^i(K)$ char $C \trianglelefteq G$ for every $i \geq 0$ and by the conjugacy of p^*, p-injectors of G, we obtain $ZJ^i(K)$ char G for every $i \geq 0$ contrary to our choice of G. Therefore $C = G$, and so $ZJ(K) = = Z(G)$. Since $ZJ(K) \neq 1$ we have $|G/Z(G)| < |G|$ and we can conclude that $ZJ^i(K/Z(G))$ char $G/Z(G)$ for every $i \geq 0$. Now, since $K_1 = K$, using ([4], Prop.II 3.6) it follows $ZJ^{i+1}(K)/Z(G) = ZJ^i(K/Z(G))$, thus for every $i \geq 0$, $ZJ^i(K)$ char G. That is the last contradiction.

PROPOSITION 3.4. Let K be a p^*, p-injector of a group G. Assume $O_{p'}(F(G)) \leq Z(K)$, then for every $i \geq 0$:

i) $ZJ^i(K)$ is nilpotent

ii) $F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$ is a p-group

iii) $O_{p^*}(G) \leq K_i$.

PROOF. Simultaneously we will prove i) and ii) by induction on i. First notice that $ZJ(K) \leq Z(C_K(ZJ(K))) = Z(K_1)$ thus $F(K_1/ZJ(K)) = = F(K_1)/ZJ(K)$ and $O_{p'}(F(K_1/ZJ(K))) = O_{p'}(F(K_1)/ZJ(K)) = 1$. Clearly $ZJ^1(K) = ZJ(K)$ is nilpotent.

Assume that $F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$ is a p-group, then $O_{p'}(ZJ^{i+1}(K)) \leq ZJ^i(K)$, therefore $O_{p'}(ZJ^{i+1}(K))$ is a nilpotent normal subgroup of K and we have $O_{p'}(ZJ^{i+1}(K)) \leq O_{p'}(F(K)) \leq Z(K)$. Now, since $ZJ^{i+1}(K)$ is p-nilpotent, we have that $ZJ^{i+1}(K)$ is nilpotent.

On the other hand we have:

$$F(K_{i+1}/ZJ^{i+1}(K)) \cong F(K_{i+1}/ZJ^i(K)/ZJ^{i+1}(K)/ZJ^i(K))$$

and as $ZJ^{i+1}(K)/ZJ^i(K) = Z(K_{i+1}/ZJ^i(K))$, it follows:

$$F(K_{i+1}/ZJ^i(K)/ZJ^{i+1}(K)/ZJ^i(K)) =$$

$$= F(K_{i+1}/ZJ^i(K))/ZJ^{i+1}(K)/ZJ^i(K).$$

But

$$F(K_{i+1}/ZJ^i(K)) = F(K_i/ZJ^i(K)) \cap K_{i+1}/ZJ^i(K) =$$

$$= F(K_i)/ZJ^i(K) \cap K_{i+1}/ZJ^i(K) = F(K_{i+1})/ZJ^i(K)$$

and we can conclude:

$$F(K_{i+1}/ZJ^{i+1}(K)) = F(K_{i+1})/ZJ^{i+1}(K).$$

Moreover as $O_{p'}(F(K_{i+1})) \leq O_{p'}(F(K)) \leq Z(K) \leq (ZJ^{i+1}(K))$ it follows that $F(K_{i+1}/ZJ^{i+1}(K))$ is a p-group.

iii) Clearly $O_{p^*}(G) = O_{p^*}(K) \leq C_K(O_p(ZJ^*(K)))$. Moreover as $ZJ^*(K)$ is nilpotent it follows: $O_{p'}(ZJ^*(K)) \leq O_{p'}(F(K)) \leq Z(K)$. Therefore $O_{p^*}(G) \leq C_K(ZJ^*(K)) \leq K_i$ for every $i \geq 0$.

PROPOSITION 3.5. Let G be a group with $O_{p'}(F(G)) \leq Z(K)$, where K is a p^*, p-injector of G, then the following are equivalent:

i) G is an $\mathfrak{K}$-constrained group

ii) $K_* = ZJ^*(K)$

iii) $C_G(ZJ^*(K)) \leq ZJ^*(K)$.

PROOF. i) $\Rightarrow$ ii) Set $K = O_{p^*}(G)P$ where P is a Sylow p-subgroup of G, then by ([2], Lemma 6.11) it follows $[P, O_{p^*}(G)] = 1$ and as $O_{p^*}(G) \leq \leq C_G(F(G)) \leq F(G)$ we have K is nilpotent, thus K_* is nilpotent, hence $K_* = ZJ^*(K)$.

ii) $\Rightarrow$ iii) By the remark of Lemma 3.2.

iii) $\Rightarrow$ i) We know that $[O_{p^*}(G), ZJ^*(K)] = 1$ therefore $E(G) \leq \leq O_{p^*}(G) \leq C_G(ZJ^*(K)) \leq ZJ^*(K)$, then $E(G) = 1$, i.e. G is an $\mathfrak{K}$-constrained group.

COROLLARY 3.6. Let p be an odd prime and K a p^*, p-injector of a $\mathfrak{K}$-constrained group G. Assume that $SA(2, p)$ is not involved in G and $O_{p'}(F(G)) \leq Z(K)$. Then $ZJ^*(K)$ is a characteristic subgroup of G and $C_G(ZJ^*(K)) \leq ZJ^*(K)$.

As a consequence of the above Corollary we can obtain the well-known result of Glauberman:

COROLLARY 3.7. Let p be an odd prime and P a Sylow p-subgroup of a group G. Suppose that $C_G(O_p(G)) \leq O_p(G)$ and that $SA(2, p)$ is not involved in G. Then $ZJ^*(P)$ is a characteristic subgroup of G and $C_G(ZJ^*(P)) \leq ZJ^*(P)$.

PROOF. Clearly $O_{p^*}(G) = O_{p'}(G) = 1$ so P is a p^*, p-injector of G. Now Corollary 3.6 applies.

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