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On Some Questions Concerning Permutable Subgroups of Finitely Generated Groups and Projectivities of Perfect Groups.

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A subgroup H of a group G is called *permutable* (or *quasinormal*) in G if $HK = KH$ for every subgroup K of G.

Let G be a finitely generated group and H permutable subgroup of G with trivial core H_G. In [2] (see also [3], p. 217), the following questions are posed:

Is H finite? Is H contained in the hypercentre of G?

The aim of this note is to show the existence of a finitely generated p-group G containing a permutable subgroup H such that H/H_G is infinite and not contained in the hypercentre of G/H_G. Thus the answer to the questions stated above is negative. Moreover, the group G is perfect and there exists an autoprojectivity of G, namely an automorphism of its lattice of subgroups, not preserving normal subgroups. This observation allows us to solve a problem raised in a natural way when M. Suzuki [6, Theorem 15, p. 51] showed that in the case of finite perfect groups projectivities preserve normality, namely if this property holds without the hypothesis that the group is finite.

In our argument extended Tarski p-groups of exponent p^n, for $n \geqslant 3$, are considered [5]. We recall that a group G is an extended Tarski p-group if G/G^p is a Tarski group, $G^p \neq 1$ and for every subgroup X of G either $X \leqslant G^p$ or $X \geqslant G^p$. The existence of extended Tarski p-

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groups of any exponent for a sufficiently large prime p has been proved by Ashmanov and Ol'shanskii [1, Remark 4].

THEOREM. *There is a prime p and a perfect finitely generated p-group G containing a permutable subgroup H such that H/H_G is infinite and not contained in the hypercentre of G/H_G. Moreover, there is an autoprojectivity of G not preserving normal subgroups.*

PROOF. Let p be a prime such that there exists an extended Tarski p-group T of exponent p^3 and let $A = T^p$; then T/A is a Tarski group. Let M be the group algebra of T/A over the field with p elements and let N be the augmentation ideal of M. Since T/A is a finitely generated group, N is a finitely generated right ideal of M. Thus the semidirect product $G = N \rtimes T$, where the action of T over N is the one induced by the natural action of T/A over N by right multiplication, is a finitely generated group. Moreover, if $B = \langle e_0 \rangle$ is the unique subgroup of order p of T, then $\Omega_1(G) = NB$ is elementary abelian; also, since T/A is perfect, it is straightforward to show that $N = N^2$ as ideal of M, and so we have $[\Omega_1(G), G] = [N, G] = N$. Therefore, since T is perfect, G is also perfect. Let $H \gg N$ be a maximal subgroup of $\Omega_1(G)$ not containing B. If $\Omega_1(G)/H_G$ is finite, then for some integer t we have $H_G \geq$ $\geq [\Omega_1(G), tG] = N$, and so $H = N$, against the assumption. Thus H/H_G is infinite. Moreover, H/H_G intersects trivially the hypercentre AH_G/H_G of G/H_G.

We show that H is permutable in G, namely that $H\langle g \rangle = \langle g \rangle H$ for every $g \in G$. This is obvious if $g \in HA$. Otherwise g is of the form $g = xy$ with $x \in T \setminus A$ and $y \in N$; in this case, since x acts on N as an automorphism of order p, it follows that $\langle g^{p^2} \rangle = \langle x^{p^2} \rangle = B$. Thus $\langle g \rangle \Omega_1(G) =$ $= \langle g \rangle H$.

It remains to show that there exists an autoprojectivity of G not preserving normal subgroups.

Thus, let $\{e_1, e_2, \ldots\}$ be a basis for N and define the automorphism τ of $\Omega_1(G)$ by putting $e_0^\tau = e_0$, $e_1^\tau = e_0 e_1$, $e_i^\tau = e_i$ for $i \neq 0, 1$. Let σ be the identity of G/B. Clearly for every subgroup U of G either $U \leq \Omega_1(G)$ or $U \geq B$ and $U^\sigma = U^\tau$ whenever $B \leq U \leq \Omega_1(G)$ since τ induces the identity on the factor group $\Omega_1(G)/B$. Thus we can apply [4, Lemma 2.5]: the map ρ defined by putting $U^\rho = U^\tau$ if $U \leq \Omega_1(G)$ and $U^\rho = U^\sigma$ if U is not contained in $\Omega_1(G)$, is an autoprojectivity of G; and N is a normal subgroup of G such that its projective image $N^\rho = \langle e_0, e_1, e_2, \ldots \rangle$ is not normal in G.

We leave the following question open: is a finite core-free permutable subgroup of a group G contained in the hypercentre of G?

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