RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

FEDERICO MENEGAZZO

A property equivalent to commutativity for infinite groups

Rendiconti del Seminario Matematico della Università di Padova, tome 87 (1992), p. 299-301

<http://www.numdam.org/item?id=RSMUP_1992__87__299_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 87 (1992)

A Property Equivalent to Commutativity for Infinite Groups.

FEDERICO MENEGAZZO (*)

According to a well-known result of B. H. Neumann, the class of groups with centre of finite index can be characterized as the class of groups all whose infinite subsets contain a pair of permutable elements [2].

Similar, possibly more general classes have been introduced by J. C. Lennox, A. M. Hassanabadi and J. Wiegold in [1]. If G is a group and n is a positive integer, they define an n-set in G to be a subset of G of cardinality n. The class P_n^* is then defined by

$G \in P_1^*$ if and only if every infinite set of n-sets in G contains a pair X, Y of different members such that $XY = YX$.

In this terminology, P_n^* is the class of centre-by-finite groups. In [1] it is shown that infinite groups in P_n^* are abelian if $n = 2$ or $n = 3$. However, this result holds in general; in fact, we will prove the following

THEOREM. *Suppose G is an infinite non-abelian group. For every integer $n > 1$ there is an infinite set of pairwise non-permutable n-sets in G.*

1. – We will look first at a special case.

PROPOSITION. *If $n > 1$ and $G \in P_n^*$ has an element with finite centralizer, then G is finite.*

PROOF. Assume, by contradiction, that $G \in P_n^*$ is an infinite group, $a \in G$, and $C_G(a)$ is finite. For $g \in G$, we denote $S(a, g)$ the set $\{x \in$

(*) Indirizzo dell'A.: Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, I-35131 Padova.

$\in G \mid a^x = g\}$: $S(a, g)$ is either empty or a right coset of $C_G(a)$, and in any case it is finite. Let F be a fixed subset of G of cardinality $n-1$ and such that $a \in F$; the set $B = a^{-1}FF \cup \left(\bigcup_{b \in F} S(a, b) \right)$ is finite. We now define by induction a sequence of elements of G, beginning with an element $y_1 \notin F$: if $y_1, \ldots, y_{m-1}$ have already been defined, we choose y_m in the complement of the finite set $B \cup \left(\bigcup_{1 \leq k < m} a^{-1}Fy_k \right) \cup \cup \left(\bigcup_{1 \leq k < m} S(a, y_k) \right)$. Finally, for every integer m we set $X_m = F \cup \{y_m\}$. All these X_m'a are n-sets in G, and $X_s \neq X_t$ if $s \neq t$. Fix now $1 \leq s < t$; certainly $ay_t \in X_s X_t$. On the other hand, $ay_t \notin X_t X_s = FF \cup Fy_s \cup \cup y_t F \cup \{y_t y_s\}$, since

$ay_t \in FF$ implies $y_t \in a^{-1}FF$,
$ay_t \in Fy_s$ implies $y_t \in a^{-1}Fy_s$ with $s < t$,
$ay_t \in y_t F$ implies $y_t \in S(a, b)$ for some $b \in F$,
$ay_t = y_t y_s$ implies $y_t \in S(a, y_s)$ with $s < t$,

and all of the above possibilities contradict our choice of the y_m's.

2. – We come now to the proof of our theorem. Let G be any infinite non-abelian group, and n an integer, $n > 1$. By the Proposition, we may assume that G satisfies the following condition: for every infinite subgroup H of G and every $h \in H$, $C_H(h)$ is infinite. It is immediate to check that this condition implies that every element of G belongs to an infinite abelian subgroup of G. At this point, one can apply Theorem C of [1] and conclude. To get a self-contained proof, one can proceed as follows. Suppose $x, y \in G$ and $xy \neq yx$, and let A be an infinite abelian subgroup of G containing y; of course, $x \notin A$. Choose $a_1, \ldots, a_{n-1} \in A$ such that $a_1 = y$ and $a_1, a_1^x, \ldots, a_{n-1}$ are different elements of G. Now define a sequence of elements of G by induction: take $c_1 = 1$ and, if $c_1, \ldots c_{m-1}$ have already been defined, choose c_m in the complement in A of the finite set $\bigcup_{1 \leq k < m} a_1^{-1}\{a_1, \ldots, a_{n-1}\}c_k$.

For every m, set $Y_m = \{a_1, \ldots, a_{n-1}, c_m x\}$; then $|Y_m| = n$ for every m, and $Y_i \neq Y_j$ if $i \neq j$. Now for $i < j$ we have $a_1 c_j x \in Y_i Y_j$ but

$$a_1 c_j x \notin Y_j Y_i = \{a_1, \ldots, a_{n-1}\}^2 \cup c_j x\{a_1, \ldots, a_{n-1}\} \cup \cup \{a_1, \ldots, a_{n-1}\} c_i x \cup \{c_j x c_i x\};$$

in fact: $a_1 c_j x \in \{a_1, \ldots, a_{n-1}\}^2$ implies $x \in A$; $a_1 c_j x \in c_j x\{a_1, \ldots, a_{n-1}\}$ implies $a_1^x = a_r$ for some r, $1 \leq r \leq n-1$; $a_1 c_j x \in \{a_1, \ldots, a_{n-1}\} c_i x$ implies $c_j \in a_1^{-1}\{a_1, \ldots, a_{n-1}\} c_i$ with $i < j$; and finally $a_1 c_j x = c_j x c_i x$ implies $x \in A$; and all these contradict some earlier assumption.

REFERENCES

[1] J. C. Lennox - A. M. Hassanabadi - J. Wiegold, *Some commutativity criteria*, Rend. Sem. Mat. Univ. Padova, 84 (1990), pp. 135-141.

[2] B. H. Neumann, *A problem of Paul Erdös on groups*, J. Austral. Math. Soc. (Series A), 21 (1976), pp. 467-472.

Manoscritto pervenuto in redazione il 16 maggio 1991.