Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 87 (1992), p. 281-297

<http://www.numdam.org/item?id=RSMUP_1992__87__281_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 87 (1992)

An Application of Markov Operators in Differential and Integral Equations.

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1. Introduction.

One of the important techniques used in the theory of Markov operators is studying the properties of a semigroup $\{P^t\}_{t \geq 0}$ from the properties of a single operator P^{t_0} with some $t_0 > 0$.

The main result of the paper is Theorem 2.1 which concerns the asymptotic stability of a Markov operator. The main advantage of having Theorem 2.1 is that it allows to obtain many corollaries concerning the asymptotic behavior for semigroup from already proved results for iterates of a single operator.

Stochastic semigroup generated by Fokker-Planck equations are particularly convenient to study by Theorem 2.1. This is due to the fact that they are represented by the integral formula (3.6). In this case the asymptotic behavior of the solution depend on the existence of a stationary solution and its summability.

The organization of the paper is the following. In Section 2 we give a formulation of the main theorem concerning expanding Markov operators. Section 3 gives a simple criterion (Theorem 3.1) for the asymptotical stability of solutions of parabolic partial differential equation. In Section 4 we study the one dimensional example of the Fokker-Planck equation. In Section 5 we analyze a rule which a stochastic integral operator plays in the asymptotic behavior of a semigroup, generated by integro-differential equation. The last one contains an application of Theorem 2.1 to Markov integral operators with advanced argument.

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2. Existence of an invariant density for Markov operators.

Let (X, Σ, m) be a σ-finite measure space.

A linear operator P: $L^1(m) \mapsto L^1(m)$ is called a Markov operator if $P(D) \subset D$, where $D = \{f \in L^1(m)\colon f \geqslant 0, \|f\| = 1\}$ is the set of densities and $\| \|$ stands for the norm in $L^1(m)$. By a standard procedure using monotone seuqences of integrable functions, any linear positive operator on $L^1(m)$ extends (uniquely) beyond $L^1(m)$ to act on arbitrary non-negative (possible infinite) measurable functions.

Let a Markov operator be given. A density f is called stationary if $Pf = f$. If a Markov operator P has a positive invariant function f_* then we can define the Markov operator $\widetilde{P}$: $L^1(X, \Sigma, \mu) \mapsto L^1(X, \Sigma, \mu)$ by letting

$$\widetilde{P}h = \frac{P(f_* h)}{f_*} \tag{2.1}$$

where $d\mu = f_* \, dm$. Clearly $\widetilde{P}1 = 1$, so by the Riesz-Thorin convexity theorem $\widetilde{P}$ acts as a positive contraction on any $L^p(\mu)$, $1 \leqslant p \leqslant \infty$. We denote by $\widetilde{U}$ the $L^2(\mu)$-adjoined of $\widetilde{P}$ as well as its monotone extension to all the spaces $L^p(\mu)$. Now applying the well-known complex Hilbert space technique to $\widetilde{P}$ (see [Fog69], Chapter III]), we define

$$K = \{f \in L^2(\mu)\colon \|\widetilde{P}^n f\|_2 = \|\widetilde{U}^n f\|_2 = \|f\|_2, \ n = 1, 2, \ldots\}. \tag{2.2}$$

Then K is a closed sublattice of $L^2(\mu)$ and the operator $\widetilde{P}$ is unitary on K. For every $f \in K^{\perp}$, $\widetilde{P}^n f \mapsto 0$ and $\widetilde{U}^n f \mapsto 0$ weakly in $L^2(\mu)$. Now let

$$\Sigma_1(\widetilde{P}) = \{A \in \Sigma\colon 1_A \in K\}. \tag{2.3}$$

Then $\Sigma_1(P)$ is a subring of Σ on which $\widetilde{P}$ and $\widetilde{U}$ act as automorphisms. Moreover, K is the closed span in $L^2(\mu)$ of $\{1_A\colon A \in \Sigma_1\}$ and if $X_1 \in \Sigma$ is minimal in Σ such that $A \subset X_1 \pmod{\mu}$ for every $A \in \Sigma_1$ then X_1 is $\widetilde{P}$-invariant. The set X_1 will be referred to as the deterministic part of $\widetilde{P}$. Finally $X_2 = X \setminus X_1$.

A Markov operator P: $L^1(m) \mapsto L^1(m)$ is called *asymptotically stable* if there is a unique stationary $f_* \in D$ and if $\|P^n f - f_*\| \mapsto 0$ as $n \mapsto \infty$ for every $f \in D$. For every $f \in L^1(m)$ we define the support of f by setting

$$\operatorname{supp} f = \{x \in X\colon f(x) \neq 0\}.$$

This set is not defined in a completely unique manner, since f may be represented by functions that differ on a set of measure zero. This inaccuracy never leads to any difficulties in calculating measures and integrals. Of course $\operatorname{supp} f$ is defined up to a set of measure zero. Inequalities (equalities) between functions or sets are in the a.e. sense.

Now, we examine a limit behavior of Markov operators admitting a stationary density. We have the following

THEOREM 2.1. Let $P\colon L^1(m) \mapsto L^1(m)$ be a Markov operator. Suppose that P has a positive invariant density f_*. Assume also that for every $A, B \in \Sigma$ with positive finite measure the following condition is satisfied:

$$m(A - \operatorname{supp} P^n 1_B) \mapsto 0, \qquad n \mapsto \infty. \tag{2.4}$$

If in addition, $\Sigma_1(\widetilde{P})$ defined by (2.3) is atomic, then P is asymptotically stable.

REMARK. If a Markov operator is given by the integral formula

$$(Pf)(x) = \int_X k(x, y) f(y)\, m(dy), \qquad f \in L^1(m), \tag{2.5}$$

where k is stochastic kernel, i.e. $k\colon X \times X \mapsto R_+$ is jointly measurable and

$$\int_X k(x, y)\, m(dx) = 1, \qquad y \in X,$$

then $\widetilde{P}$ has the form

$$(\widetilde{P}h)(x) = \int_X \widetilde{k}(x, y)\, h(y)\, \mu(dy), \qquad h \in L^1(\mu), \tag{2.6}$$

$d\mu = f_*\, dm$, $\widetilde{k}(x, y) = k(x, y)/f_*(x)$ and f_* is positive and invariant for P. It is known [Fel65] that $\Sigma_1(\widetilde{P})$ defined for the integral operator (2.6) is atomic.

PROOF. *Uniqueness of invariant density.* Assume there are two stationary densities for P, namely, f_1 and f_2. Set $f = f_1 - f_2$, so we have $Pf = f$. Let further $f = f^+ - f^-$, where $f^+(x) = \max(0, f(x))$ and $f^-(x) = \max(0, -f(x))$. So that, if $f_1 \neq f_2$, then neither f^+ nor f^- are zero. Note that $Pf^+ = f^+$ and $Pf^- = f^-$. Further, there exist sets A, B with positive measure such that $\alpha 1_A \leq f^+$, and $\beta 1_B \leq f^-$. Thus, we have $\operatorname{supp} P^n 1_B \subset \operatorname{supp} f^-$, which contradicts the condition (2.4).

Asymptotical stability. Define, as above, an operator $\widetilde{P}$: $L^1(\mu) \mapsto L^1(\mu)$ by letting $\widetilde{P}h = P(f_* h)/f_*$, where $d\mu = f_* dm$. Note that $\widetilde{P}1 = P(f_*)/f_* = 1$ and $\mu(X) = 1$. Using (2.4) it is easy to prove that $\Sigma_1(\widetilde{P}) = \{\emptyset, X\}$. Thus the operator $\widetilde{P}$ is a Harris operator with trivial $\Sigma_1(\widetilde{P})$. Then, by [Fog69, Chapter VIII, Theorem E, p. 89], we have

$$(2.7) \qquad \|\widetilde{P}^n h - 1_X\|_{L^1(\mu)} \mapsto 0, \quad n \mapsto \infty, \; h \geq 0, \qquad \int_X h \, d\mu = 1.$$

Now, pick $f \in D$, then the function $h = f/f_*$ belongs to $L^1(\mu)$ and from (2.7) we get $\|P^n f - f_*\|_{L^1(m)} \mapsto 0$ as $n \mapsto \infty$. ■

REMARK. The condition (2.4) is immediately satisfied if a kernel $k(x, y)$ in (2.5) is positive.

3. Stochastic semigroups; asymptotic behavior of solutions of parabolic equations.

Let (X, Σ, m) be a σ-finite measure space. A family of Markov operators $\{P^t\}_{t \geq 0}$ is called a stochastic semigroup if $P^{t_1 + t_2} = P^{t_1} \circ P^{t_2}$ and $P^0 = 1$ for all $t_1, t_2 \geq 0$. A stochastic semigroup $\{P^t\}_{t \geq 0}$ is called asymptotically stable if there exists a unique $f_* \in D$ such that $P^t f_* = f_*$ for all $t \geq 0$ and $\lim_{t \mapsto \infty} \|P^t f - f_*\| = 0$ for all $f \in D$.

The following proposition 3.1 shows the relationship between the asymptotical stability of discrete semigroup $\{P^n\}_{n \in N}$ and the semigroup $\{P^t\}_{t \geq 0}$.

PROPOSITION 3.1. Let $\{P^t\}_{t \geq 0}$ be a semigroup of Markov operators. Assume there exist $t_0 > 0$ and a unique $f_* \in D$ such that $P^n f_* := P^{nt_0} f_* = f_*$ and $\|P^n f - f_*\| \mapsto 0$ for $f \in D$ if $n \mapsto \infty$. Then $P^t f_* = f_*$ for all $t \geq 0$ and $\lim_{t \mapsto \infty} \|P^t f - f_*\| = 0$ if $f \in D$.

PROOF. First we show that $P^t f_* = f_*$ for all $t \geq 0$. Fix $t' > 0$ and set $f_1 := P^{t'} f_*$. Therefore

$$\|P^{t'} f_* - f_*\| = \|P^{t'}(P^{nt_0}) f_* - f_*\| = \|P^{nt_0}(P^{t'} f_*) - f_*\| = \|P^n f_1 - f_*\|.$$

Since $\lim_{t \mapsto \infty} \|P^n f_1 - f_*\| = 0$, we must have $\|P^{t'} f_* - f_*\| = 0$, and hence $P^{t'} f_* = f_*$. At the end, to show asymptotical stability pick a function $f \in D$, so that

$$\|P^t f - f_*\| = \|P^t f - P^t f_*\|$$

is a nonincreasing function. Since for the sequence $t_n = nt_0$ we have

$$\lim_{t \mapsto \infty} \|P^{t_n} f - f_*\| = 0,$$

we have a nonincreasing function that converges to zero on a subsequence and, hence $\lim_{t \mapsto \infty} \|P^t f - f_*\| = 0$. ■

As the first applications of Theorem 2.1 we consider a partial differential equation of parabolic type

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{d} (b_i(x) u) =: Lu, \qquad t > 0, \ x \in R^d, \tag{3.1}$$

with the initial condition

$$u(0, x) = f(x), \qquad x \in R^d. \tag{3.2}$$

This equation appears while studying the stochastic differential equation

$$\frac{dx}{dt} = b(x) + \sigma(x) \xi \tag{3.3}$$

with initial condition

$$x(0) = x^0, \tag{3.4}$$

where $b(x)$: $R^d \mapsto R^d$, $\sigma(x)$ is a $(d \times d)$ matrix.

In (3.3), the «white noise» vector

$$\xi = \left(\frac{dw_1}{dt}, \ldots, \frac{dw_d}{dt}\right)$$

should be considered, from a mathematical point of view, as a pure symbol much like the letters «dt» in the notation for derivative. However, from an application standpoint, ξ denotes a very specific process consisting of «infinitely» many independent, or random impulses. We assume that the initial vector x^0 and the Wiener process $\{w(t)\}$ are independent. To examine the solution of equation (3.3), (3.4) we are required to introduce all the abstract concepts which are necessary to define the Itõ integral, and to give the solution of equation (3.3) and (3.4) in terms of a general formula, generated by the method of successive approximations, which contains infinitely many Itõ integrals. One can pass to a consideration of the density function of random process $x(t)$ which is a solution (3.3) and (3.4).

This density is define as the function $u(t,x)$ that satisfies

$$\text{Prob}\{x(t) \in B\} = \int_B u(t, z)\,dz. \tag{3.5}$$

The uniqueness of $u(t,x)$ up to set of measure zero is an immediate consequence of Randon-Nikodym theorem, but the existence requires some regularity conditions on coefficients $b(x)$ and $\sigma(x)$. One can show that $u(t,x)$ can be found without any knowledge concerning the solution $x(t)$. It turns out that $u(t,x)$ is given by the solution of a partial differential equation (3.1), known as Fokker-Planck equation, that is completely specified by the coefficients $b(x)$ and $\sigma(x)$ of equation (3.3). We must only insert to (3.1)

$$a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\,\sigma_{jk}(x)$$

and $b_i(x)$ be the same as in (3.3).

It is clear that $a_{ij} = a_{ji}$, and

$$\sum_{i,j=1}^{d} a_{ij} z_i z_j = \sum_{k=1}^{d} \left(\sum_{i=1}^{d} \sigma_{ik}(x)\, z_i \right)^2 \geqslant 0.$$

Moreover, if the initial condition (3.4), $x(0) = x^0$, which is a random variable, has a density f then $u(0, x) = f(x)$. Thus to understand the behavior of the density $u(t,x)$, we must study the initial-value (Cauchy) problem (3.1) and (3.2). Since our unknown function $u(t,x)$ is a density, so we are interested in nonnegative solutions of (3.1) and (3.2).

The uniqueness of non-negative solutions of parabolic equations has been considered in several papers, beginning with the work of [Wid44] on the equation of heat conduction. Various extensions of Widder's result to more general parabolic equations can be found in [Aro65], [Krz64] and [Ser54]. In all of these papers, however, the coefficients of the equation are assumed to be bounded. Here we shall also deal with equations whose coefficients grow at infinity in various ways. The most convenient results, from our point of view, concerning the uniqueness of solution may be found in [Fri64], [Aro65], [ArB66], [Arb67a], [Cha70a].

It is well know that a linear second order parabolic equation with bounded Hölder continuous coefficients possesses a fundamental solution.

One important consequence of the existence of a fundamental solution is that it gives an explicit formula for solutions of the Cauchy problem. Detailed descriptions of this theory, as well as further references can be found in [IKO62] and in the books of [Eid69] and [Fri64]. The fundamental solution for equations with unbounded coefficients was treated, among other, in [Eid69], [Bod66], [ArB67b], [Cha70b], [Bes75].

In order to state a relatively simple existence and uniqueness theorem we admit here the following classical conditions (for a uniqueness theorem see [Fri64], Chapter II, Theorem 10, for a theorem of the existence of the fundamental solution see [Eid69], p. 136-137):

(i) The coefficients $a_{ij}(x)$, $b_i(x)$ are C^3 functions for $x \in R^d$ and their derivatives of the third order are locally *Hölderian.*

(ii) The equation (3.1) is uniformly parabolic that is $a_{ij}(x) = a_{ji}(x)$ and

$$\sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \geqslant C|z|^2, \qquad x, z \in R^d$$

where C is a positive constant and $|z|$ denotes the Euclidean norm of the vector $z = (z_1, \dots, z_d)$.

(iii) The coefficients satisfy the growth conditions

$$|a_{ij}(x)| \leqslant M, \qquad \left| b_i(x) - \sum_{j=1}^{d} \frac{\partial a_{ij}(x)}{\partial x_j} \right| \leqslant M(1 + |x|),$$

$$i, j = 1, \dots, d, \quad x \in R^d$$

and

$$\left| \frac{1}{2} \cdot \sum_{i,j=1}^{d} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial b_i}{\partial x_i} \right| \leqslant M(1 + |x|^2),$$

for some constant $M > 0$.

For less restrictive conditions the reader is referred to [Cha70] and [Bes75].

A function $\omega = \omega(t, x)$ defined on $R \times R^d$ will be said to belong to class $\mathcal{E}_K^\beta$ if there exist constant β, $K > 0$ such that

$$|\omega(t, x)| \leqslant K e^{\alpha |x|^2}.$$

Under the assumptions (i)-(iii) for every continuous function $u(0, x) = f(x)$ from a class $\mathcal{E}_K^\beta$ there exists a unique classical solution

$u(t,x)$ of (3.1) and (3.2). The term classical means that for every $T>0$ the solution $u(t,x)$ also belongs to $\mathcal{E}_K^\beta$ for $t \in (0,T]$, $x \in R^d$. Moreover, $u(t,x)$ has continuous derivatives u_t, u_{x_i}, $u_{x_i x_j}$ and satisfies equation (3.1) for every $t>0$, $x \in R^d$; and $\lim_{t \mapsto 0} u(t,x) = f(x)$. The solution $u(t,x)$ is given by the formula

$$u(t,x) = \int_{R^d} \Gamma(t,x,y) f(y)\, dy, \qquad x \in R^d, \ t>0 \tag{3.6}$$

where Γ is the fundamental solution of (3.1). The function $\Gamma(t,x,y)$, defined for $t>0$, $x, y \in R^d$, is continuous, positive and differentiable with respect to t, is twice differentiable with respect to x, and satisfies (3.1) as a function of (t,x) for every fixed y.

Moreover is satisfies the inequality

$$|D_t D_x^s \Gamma(t,x,u)| \leqslant A \cdot t^{-(d+s+2)/2} \exp\left\{-\alpha \frac{|x-y|^2}{t}\right\}, \tag{3.7}$$

for $x, y \in R^d$, $t>0$, $|s| \leqslant 2$, with some positive constant A and α.

If f is not necessarily continuous but integrable, the formula (3.6) defines a generalized solution of (3.1) for $t>0$. In this case it satisfies the initial condition of the form

$$\lim_{t \mapsto 0} \|u(t,\cdot) - f\| = 0. \tag{3.8}$$

Using (3.7) it is easy to verify that $u(t,\cdot) \in L^1(R^d)$ for $t \geqslant 0$. Thus setting

$$P^t f(x) = u(t,x) = \int_{R^d} \Gamma(t,x,y)\, dy, \qquad t>0, \ P^0 f = f, \tag{3.9}$$

we define a family of operators $P^t \colon L^1 \mapsto L^1$ which describes the evolution in time of solution $u(t,x)$. Using the specific «divergent» form of equation (3.1) it is easy to verify that $\{P^t\}_{t \geqslant 0}$ is a stochastic semigroup.

By virtue of the considerations of § 2, the behavior of the solutions of the Cauchy problem (3.1), (3.2) can be stated as follows.

THEOREM 3.1. Assume that the conditions (i)-(iii) are fulfilled. Assume moreover that equation (3.1) has a positive stationary solution $u_*(x)$ which belongs to a class $\mathcal{E}_K^\beta$ and $\int_{R^d} u_*(x)\, dx = 1$. Then u_* is unique and for every $f \in D(R^d)$ the stochastic semigroup $\{P^t\}_{t \geq 0}$ defined by (3.6) is asymptotically stable with the limiting function u_*.

PROOF. First we are going to show that the kernel $\Gamma(t, x, y)$ in formula (3.9) is stochastic for each $t > 0$. We already know that Γ is positive and $\{P^t\}_{t \geq 0}$ is stochastic semigroup. Further for each $f \in L^1(R^d)$ we have

$$\int_{R^d} f(y)\, dy = \int_{R^d} P^t f(x)\, dx = \int_{R^d}\int_{R^d} \Gamma(t, x, y) f(y)\, dy\, dx$$

and consequently

$$\int_{R^d} \left[\int_{R^d} \Gamma(t, x, y)\, dx - 1 \right] f(y)\, dy = 0 .$$

Since $f \in L^1(R^d)$ is arbitrary this implies

$$\int_{R^d} \Gamma(t, x, y)\, dx = 1, \qquad \text{for } t > 0,\ y \in R^d .$$

Further, according to the definition of semigroup $\{P^t\}_{t \geq 0}$ the function

$$u(t, x) = P^t u_*(x)$$

is a solution of (3.1), (3.2) with $f = u_*$. Since $u_*(x)$ is stationary solution and the Cauchy problem is uniquely solvable we have

$$u_*(x) = P^t u_*(x), \qquad \text{for } t > 0 .$$

Thus, by (3.6) we have

$$u_*(x) = \int_{R^d} \Gamma(t, x, y) u_*(y)\, dy = \int_{R^d} \Gamma(1, x, y) u_*(y)\, dy .$$

Since $\Gamma(1, x, y)$ is strictly positive, then the assumptions of Theorem 2.1 are satisfied. Hence by Theorem 2.1

$$Pf(x) = \int_{R^d} \Gamma(1, x, y) f(y)\, dy$$

is asymptotically stable for any integrable function f. By Proposition 3.1, we obtain the conclusion of Theorem 3.1. ■

4. The one dimensional case.

We are going to show an application of Theorem 3.1 to the one dimensional case. Thus we consider a stochastic differential equation:

$$\frac{dx}{dt} = b(x) + \sigma(x)\,\xi, \qquad x \in R \tag{4.1}$$

when σ, b are scalar functions and ξ is one dimensional white noise. The corresponding Fokker-Planck equation has form:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x)\,u) - \frac{\partial}{\partial x}(b(x)\,u). \tag{4.2}$$

We assume that $a(x) = \sigma^2(x)$ and $b(x)$ satisfy the conditions (i)-(iii). Moreover

$$xb(x) \leqslant 0 \tag{4.3}$$

for sufficiently large x. In order to find a stationary solution of (4.2) we should solve the ordinary differential equation

$$\frac{1}{2}\frac{d^2}{dx^2}(\sigma^2(x)\,u) - \frac{d}{dx}(b(x)\,u) = 0$$

or

$$\frac{d\gamma}{dt} = \frac{2b(x)}{\sigma^2(x)}\gamma + c_1$$

where $\gamma = \sigma^2 u$ and c_1 is a constant. Therefore

$$\gamma(x) = e^{G(x)}\left[c_2 + c_1\int_0^x e^{-G(y)}\,dy\right]$$

where

$$G(x) = 2\int_0^x \frac{2b(y)}{\sigma^2(y)}\,dy.$$

The solution $\gamma(x)$ is positive iff

$$c_2 + c_1\int_0^x e^{-G(y)}\,dy > 0, \qquad \text{for } x \in R. \tag{4.4}$$

From condition (4.3) it follows that the integral

$$\int_0^x e^{-G(y)}\,dy$$

converges to $+\infty$ if $x \mapsto +\infty$ and to $-\infty$ if $x \mapsto -\infty$. This shows that inequality (4.4) is satiafied iff $c_1 = 0$. Thus the unique up to multiplicative constant positive stationary solution of equation (4.2) is given by

$$u_*(x) = \frac{c}{\sigma^2(x)} e^{G(x)}$$

with $c > 0$. Applying Theorem 3.1 to equation (4.2) we get the following.

COROLLARY. Assume that the coefficients $a = \sigma^2$ and b of equation (4.2) satisfy the conditions (i)-(iii). If $u_*(x)$ belongs to a class $\mathcal{E}_K^\beta$ for same $\beta, K > 0$ and $\int_R \sigma^{-2}(x)\, e^{G(x)}\,dx$ is finite, then the semigroup $\{P^t\}_{t \geqslant 0}$ generated by equation (4.2) is asymptotically stable.

5. Integro-differential equations.

In this section we consider a special case of the linear Boltzmann equation

$$\frac{\partial u(t, x)}{\partial t} + u(t, x) = Pu, \qquad \text{for } t \geqslant 0, \tag{5.1}$$

with the initial condition

$$u(0, x) = f(x), \tag{5.2}$$

where P is given Markov operator on $L^1(X, \Sigma, m)$.

We consider the solution $u(t, x)$ as a function from the positive real numbers, R_+ into $L^1(X, \Sigma, m)$.

By the Hille-Yosida theorem, the linear Boltzman equation (5.1) generates a continuous semigroup of Markov operators $\{P^t\}_{t \geqslant 0}$. For the initial condition $f \in L^1(m)$ the unique solution to equation (5.1) is given by formula

$$u(t, x) = P^t f(x) = e^{t(P-I)} = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} P^n f(x), \qquad \text{for } t \geqslant 0. \tag{5.3}$$

Where I is the identity operator on $L^1(m)$. For $t \geqslant 0$ the sum of the coefficients on the right-hand side of (5.3) is sequal to one. This is the reason that the operators $e^{t(P-I)}$ preserve integral.

In order to examine the behavior of solutions of equation (5.1) we may use Theorem 2.1.

PROPOSITION 5.1. Assume that the Markov operator P satisfies assumptions of Theorem 2.1. Then the semigroup $\{e^{t(P-I)}\}_{t\geq 0}$ is asymptotically stable.

PROOF. Assume that there exists a positive stationary density f_* for P. Then by Theorem 2.1 f_* is unique. By [LaM85, Corollary 8.7.2, p. 239] the semigroup $\{e^{t(P-I)}\}_{t\geq 0}$ is asymptotically stable. ∎

We consider now the special case of the linear Boltzman equation which is called *the linear Tjon-Wu equation*

$$\frac{\partial u(t,x)}{\partial t} + u(t,x) = \int_x^\infty \frac{dy}{y} \int_0^y e^{-(y-z)} u(t,z)\, dz, \qquad x > 0. \tag{5.4}$$

A Markov operator $P\colon L^1((0,\infty)) \mapsto L^1((0\infty))$ is defined by

$$Pf(x) = \int_x^\infty \frac{dy}{y} \int_0^y e^{-(y-z)} f(z)\, dz = \int_0^\infty k(x,y) f(y)\, dy, \qquad x > 0, \tag{5.5}$$

where

$$k(x,y) = \begin{cases} e^y \int_y^\infty \frac{e^{-z}}{z} dz, & 0 < x \leq y, \\ e^y \int_x^\infty \frac{e^{-z}}{z} dz, & 0 < y < x. \end{cases} \tag{5.6}$$

Since $f_* = e^{-x}$ is the stationary density for (5.5) and the kernel (5.6) is positive then the assumptions of Theorem 2.1 are satisfied. Therefore for an arbitrary initial condition $u(0,x) = f(x) \in D((0,\infty))$ the solution of (5.4) is asymptotically stable with the limiting density e^{-x}. By Theorem 2.1 the Markov operator (5.5) is asymptotically stable as well.

REMARK. A different situation occurs when we consider the Chandrasekhar-Münch equation (see Examples 7.9.2 and 11.10.2

in [LaM85]) describing the fluctuations in the brightness of the Milky Way

$$\frac{\partial u(t, x)}{\partial t} + \frac{\partial u(t, x)}{\partial x} + u(t, x) = Pu, \tag{5.7}$$

where $P\colon L^1[0, \infty) \mapsto L^1[0, \infty)$ is the integral Markov operator of the form

$$(Pf)(x) = \int_x^\infty \psi\left(\frac{x}{y}\right) f(y) \frac{dy}{y} = \int_0^\infty I_{[0, \infty)}(y) \psi\left(\frac{x}{y}\right) f(y) \frac{dy}{y}, \qquad x \geqslant 0 \tag{5.8}$$

and $\psi\colon [0,1] \mapsto R$ is an integragle function such that

$$\psi(z) \geqslant 0 \quad \text{and} \quad \int_0^1 \psi(z)\, dz = 1. \tag{5.9}$$

Here we will discuss the properties of (5.8) independently of this equation. We show that there is no invariant density for (5.8). Let $V\colon R_+ \mapsto R$ be a nonnegative, measurable and bounded function. We have

$$\int_0^\infty V(x)\, Pf(x)\, dx = \int_0^\infty V(x)\, dx \int_x^\infty \psi\left(\frac{x}{y}\right) f(y) \frac{dy}{y} = \int_0^\infty f(y)\, dy \int_0^y \psi\left(\frac{x}{y}\right) V(x) \frac{dx}{y}$$

or substituting $x/y = z$

$$\int_0^\infty V(x)\, Pf(x)\, dx = \int_0^\infty f(y)\, dy \int_0^1 \psi(z)\, V(zy)\, dz.$$

Assuming that for some density f_*: $Pf_* = f_*$ we get

$$\int_0^\infty V(x) f_*(x)\, dx = \int_0^\infty f_*(y)\, dy \int_0^1 \psi(z)\, V(zy)\, dz$$

or

$$\int_0^\infty f_*(y)\, dy \int_0^1 \psi(z)[V(y) - V(zy)]\, dz = 0. \tag{5.10}$$

Now choose $V\colon [0, \infty) \mapsto R$ to be positive, bounded and strictly increasing (e.g. $V(z) = z/(1 + z)$) then $V(y) - V(zy) > 0$ for $y > 0$, $0 \leqslant z <$

< 1 and the integral

$$I(y) = \int_0^1 \psi(z)[V(y) - V(zy)]\,dz,$$

is strictly positive for every $y > 0$. In particular the product $f_*(y)I(y)$ is a nonnegative and nonvenishing function. This shows that the equality (5.10) is impossible. Even more, one can show that $\int_c P^n f\,dx \mapsto 0$, $c > 0$ for every $f \in L^1[0, \infty)$. This seems to be quite interesting since it was proved in [LaM85] that the semigroup of Markov operators generated by (5.7)-(5.8) is asymptotically stable. Thus $\partial u/\partial x$ is a factor in (5.7) which determines the asymptotic stability of the semigroup generated by this equation.

6. Markov operator defined by Volterra type integral with advanced argument.

A more sophisticated example of a Markov operator satisfying the condition (2.4) is given by formula

$$Pf(x) = \int_\sigma^{\gamma(x)} k(x, y) f(y)\,dy \tag{6.1}$$

where $k(x, y)$ is a measurable kernel satisfying

$$k(x, y) > 0, \qquad \text{for } \sigma < y < \gamma(x),\ \sigma < x \tag{6.2}$$

and $k\colon [\sigma, \infty) \mapsto [\sigma, \infty)$ is a continuous, strictly increasing function such that

$$\gamma(x) > x, \qquad \text{for } \alpha < x. \tag{6.3}$$

It is easy to show that P is a Markov operator on $L^1[\sigma, \infty)$ when

$$\int_{\gamma^{-1}(y)}^{\infty} k(x, y)\,dx = 1, \qquad \text{for } y > \sigma.$$

We have the following

PROPOSITION 6.1. If k and γ satisfy conditions (6.2) and (6.3) then the Markov operator $P\colon L^1[\sigma, \infty) \mapsto L^1[\sigma, \infty)$, defined by (6.1), satisfies the condition (2.4).

PROOF. Let $g \in D$ be given and let

$$x_0 = \text{ess inf}\{x: g(x) > 0\}.$$

This means that x_0 is the largest possible real number satisfying

$$m(\operatorname{supp} g \cap [0, x_0]) = 0,$$

where m is the Lebesque measure. Further let $x_1 = \gamma^{-1}(x_0)$. From (6.1) we have $Pg(x) > 0$ for $\gamma(x) > x_0$ or $x > x_1$. Define $x_n = \gamma^{-n}(x_0)$. It is easy to prove by induction that $P^n g(x) > 0$ for $x > x_n$. Thus for arbitrary measurable set $\mathcal{A} \subset [\sigma, \infty)$ we have

$$m(A - \operatorname{supp} P^n g) \leqslant x_n - \sigma. \tag{6.4}$$

The sequence $\{x_n\}$ is bounded from below ($x_n \geqslant \sigma$) and it is decreasing since $x_n = \gamma^{-1}(x_{n-1}) \leqslant x_{n-1}$. Thus $\{x_n\}$ is convergent to a number $x_* \geqslant \sigma$.

Since $\gamma(x_n) = x_{n-1}$ we have $\gamma(x_*) = x_*$. From inequality (6.3) this is only possible if $x_* = \sigma$ which according to (6.4) shows that P satisfies (2.4). ■

To illustrate the utility of Theorem 2.1 for a the Markov operator of the form (6.1) let us consider the Markov operator $P: L^1[\sigma, \infty) \mapsto$ $\mapsto L^1[\sigma, \infty)$

$$Pf(x) = \int_\sigma^{x/\sigma} k(x, y) f(y)\, dy, \tag{6.5}$$

with

$$k(x, y) = \begin{cases} \dfrac{\alpha}{\sigma}\left(\dfrac{x}{\sigma}\right)^{-1-\alpha}, & \sigma \leqslant y \leqslant 1, \\ \dfrac{\alpha}{\sigma}\left(\dfrac{x}{\sigma}\right)^{-1-\alpha} y^\alpha, & 1 \leqslant y \leqslant \dfrac{x}{\sigma}, \\ 0, & \text{elsewhere in } [\sigma, \infty) \times [\sigma, \infty), \end{cases} \tag{6.6}$$

where $\alpha > 0$ and $1 > \sigma > 0$ are constants. This operator was introduced by J. J. Tyson and K. B. Hannsgen [TyH86] in the mathematical modelling of the cell cycle.

Following Tyson and Hannsgen we are looking for the invariant density of the form $f_*(x) = cx^{-1-\beta}$. From the equation $f_* = Pf_*$ we obtain

$$x^{-1-\beta} = \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha}\left[\int_\sigma^1 y^{-1-\beta}\, dy + \int_1^{x/\sigma} y^{\alpha-1-\beta}\, dy\right]$$

or

$$x^{-1-\beta} = \frac{(\alpha/\sigma)(x/\sigma)^{-1-\beta}}{\alpha-\beta} + \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha}\left[\frac{\sigma^{-\beta}-1}{\beta} - \frac{1}{\alpha-\beta}\right].$$

It is clear that the above condition is satisfied when β is a solution of the transcended equation

$$\sigma^{\beta} + \frac{\beta}{\alpha} = 1. \tag{6.7}$$

The left hand side of this equation is equal to 1 for $\beta = 0$ and tends to $+\infty$ as β tends to $+\infty$. Thus in order to have a positive solution of (6.7) it is sufficient to assume that

$$\frac{d}{d\beta}\left(\sigma^{\beta} + \frac{\beta}{\alpha}\right) < 0, \qquad \text{for } \beta = 0$$

which is equivalent to

$$\alpha \ln < -1. \tag{6.8}$$

Thus for α, σ satisfying (6.8) there exists $\beta > 0$, for which the function $f_*(x) = cx^{-1-\beta}$ is invariant with respect to P. It can be normalized on the interval $[\alpha, \infty)$, namely for $c = \beta\sigma^{-\beta}$

$$\int_{\sigma}^{\infty} f_*(x)\,dx = \beta\sigma^{-1}\int_{0}^{\infty} x^{-1-\beta}\,dx = 1.$$

Now we can apply Theorem 2.1. The function f_* is a positive invariant density, further according to Proposition 6.1 the operator (6.5) satisfies the condition (2.4). Therefore the sequence $\{P^n\}$ is asymptotically stable.

REFERENCES

[ArB66] D. G. Aronson - P. Besala, *Uniqueness of solutions of the Cauchy problem for parabolic equations*, J. Math. Anal. Appl., **13** (1966), pp. 516-526.

[ArB67a] D. G. Aronson - P. Besala, *Uniqueness of positive solutions of parabolic equations with unbounded coefficients*, Coll. Math., **18** (1967), pp. 125-135.

[ArB67b] D. G. Aronson - P. Besala, *Parabolic equations with unbounded coefficients*, J. Diff. Eqs., **3** (1967), pp. 1-14.

[Aro65] D. G. ARONSON, *Uniqueness of positive weak solutions of parabolic equations*, Ann. Polon. Math., **16** (1965), pp. 285-303.

[Bes75] P. BESALA, *On the existence of a fundamental solution for parabolic differential equation with unbounded coefficients*, Ann. Polon. Math., **29** (1975), pp. 403-409.

[Bod66] W. BODANKO, *Sur le probléme de Cauchy et les problémes de Fourier pour les équations paraboliques dans un domaine non borné*, Ann. Polon. Math., **18** (1966), pp. 79-74.

[Cha70a] J. CHABROWSKI, *Remarques sur l'unicité du probléme de Cauchy*, Colloq. Math., **21** (1970), pp. 133-139.

[Cha70b] J. CHABROWSKI, *Sur la construction de la solution fondamentale de l'équation parabolique aux coefficients non borné*, Colloq. Math., **21** (1970), pp. 141-148.

[Eid69] S. D. EIDELMAN, *Parabolic Systems*, North-Holland Publ. Company, Amsterdam and Wolters-Noordhoff Publ., Gröningen (1969).

[Fel65] J. FELDMAN, *Integral kernels and invariant measures for Markov transition functions*, Ann. Math. Statist., **36** (1965), pp. 517-523.

[Fog69] S. R. FOGUEL, *The Ergodic Theory of Markov Processes*, Van Nostrand Reinhold, New York (1969).

[Fri64] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs (1964).

[IKO62] A. M. IL'IN - A. S. KALASHNIKOV - O. A. OLEJNIK, *Second order linear equations of parabolic type*, Russian Math. Surveys (Uspiehi Mat. Nauk.), **17**, 3 (1962), pp. 1-143.

[Krz64] M. KRZYŻAŃSKI, *Sur les solutions non negative de l'équation lineaire normale parabolique*, Rev. Roum. Math. Pures Appl., **9** (1964), pp. 393-408.

[LaM85] A. LASOTA - M. C. MACKEY, *Probabilistic Properties of Deterministic Systems*, Cambridge Univ. Press (1985).

[Ser54] J. SERRIN, *A uniqueness theorem for the parabolic equation* $u_t = a(x)\,u_{xx} + b(x)\,u_x + c(x)\,u$, Bull. Amer. Math. Soc., **60** (1954), p. 344.

[TyH86] J. J. TYSON - K. B. HANNSGEN, *Cell growth and division: a deterministic/probabilistic model of the cell cycle*, J. Math. Biology, **23** (1986), pp. 231-246.

[Wid44] D. WIDDER, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc., **55** (1944), pp. 85-95.

Manoscritto pervenuto in redazione il 27 ottobre 1990 e, in forma revisionata, il 24 aprile 1991.