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# Finite Groups with Few Conjugacy Classes of Subgroups.

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## Introduction.

Let $(X, \leqslant)$ be a finite partially ordered set (poset). A subset $A$ of $X$ is called an *antichain* if no two distinct elements of $A$ are comparable with respect to the order of $X$. Thus if $x, y \in A$ and $x \leqslant y$ then $x = y$. Such a subset intersects every chain of $X$ in one element at most. The Dilworth number $D(X)$ is a nonnegative integer defined to be the maximum taken over all cardinalities of antichains in $X$. As we will consider finite posets, $D(X)$ is a positive integer.

In this note we consider the poset $C(G)$ of all conjugacy classes of subgroups of a finite group $G$. If $H$ is a subgroup of $G$, the set $\{H^g \mid g \in \in H\}$ will be denoted by $[H]$. Thus $C(G) = \{[H] \mid H \leqslant G\}$. The ordering in $C(G)$ is defined by setting $[H] \leqslant [K]$ if $H$ is contained in some conjugate of $K$. It is clear that $C(G)$ is a poset; however, in general, $C(G)$ need not be a lattice.

DEFINITION. The Dilworth number $D(C(G))$ of the poset $C(G)$ will be denoted by $w_{C(G)}(G)$. If it is clear which poset is meant, we shall write $w_C(G)$.

It is easy to see that $w_C(G) = 1$ implies that $G$ is a cyclic $p$-group for some prime $p$, and, in general, groups $G$ with $w_C(G) \leqslant n$ for a fixed $n$, seem to have a rather restricted structure. For example, in [1] it was shown that although all 2-groups $G$ of maximal class satisfy $w_C(G) = 3$, for every $n \geqslant 4$ there are only finitely many $p$-groups $G$ with $w_C(G) =$

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$= n$. This may be compared with a similar result of Landau (see [6]) on groups with a given number of conjugacy classes of elements.

Here, we are interested in arbitrary finite groups $G$. Clearly, the set of conjugacy classes of Sylow subgroups for all primes forms an antichain in $C(G)$, and hence we have $w_C(G) \geqslant |\Pi(G)|$, where $\Pi(G)$ denotes the set of all primes dividing the order of $G$. In this paper we give a characterization of groups for which the aforementioned inequality becomes an equality.

During the proofs, we frequently need to consider the poset of the conjugacy classes of subgroups of $G$ contained in a fixed subgroup $H$ of $G$. The corresponding Dilworth number will be denoted by $w_{C(G)}(H)$. Thus $w_{C(G)}(G = w_C(G)$. The case when $H$ is a Sylow $p$-subgroup is crucial and we determine the possibilities when $w_{C(G)}(H) = 1$ (see Theorem 3). This may be of interest in the theory of defect groups in modular representation theory.

**Results.**

First we give some simple upper bounds for $w_C(G)$. Let $\mathcal{A}$ be any antichain in $C(G)$, let $\nu(\mathcal{A})$ be the maximal number of subgroups of $G$ such that no one of them is isomorphic to a subgroup of any other and whose conjugacy classes are belonging to $\mathcal{A}$. Let then $\nu(G) = \max\{\nu(\mathcal{A}) \mid \mathcal{A}$ antichain of $C(G)\}$. The following is obvious:

LEMMA 1. Let $G$ be a finite group. We have:

*a*) $w_C(G) \geqslant |\Pi(G)|$.

*b*) If $H \lhd G$ then $w_C(G) \geqslant w_{C(G/H)}(G/H)$.

*c*) If $H$ is a nontrivial complemented normal subgroup of $G$, then $w_C(G) \geqslant w_{C(G/H)}(G/H) + w_{C(G)}(H)$.

*d*) For every subgroup $M$ and for every quotient $G/H$ of $G$ we have $w_C(G) \geqslant \nu(M)$ and $w_C(G) \geqslant \nu(G/H)$.

COROLLARY 2. Let $G$ be a finite soluble group and let $f(G)$ be the Fitting length of $G$. Then $f(G) \leqslant w_C(G)$.

PROOF. By induction on $f(G)$. The result is true if $G$ is nilpotent. Since $F(G)/\Phi(G)$ is complemented in $G/\Phi(G)$, we have:

$$w_C(G) \geqslant w_{C(G/\Phi(G))}(G/\Phi(G)) \geqslant$$

$$\geqslant w_{C(G/F(G))}(G/F(G)) + w_{C(G/\Phi(G))}(F(G)/\Phi(G)) \geqslant f(G/F(G)) + 1 = f(G).$$

Our aim is now the classification of groups $G$ such that $w_C(G) = |\Pi(G)|$. Before answering this question, we examine the case of a group $G$ containing a Sylow subgroup $P$ such that $w_{C(G)}(P) = 1$, by classifying these subgroups $P$ and showing their existence by examples.

THEOREM 3. Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$ such that $w_{C(G)}(P) = 1$. Then one of the following holds:

1) $P$ is cyclic.

2) $P$ is a quaternion group of order 8.

3) $P$ is elementary abelian of order $p^2$, $p^3$, $2^5$.

4) $G$ is not $p$-soluble and $P$ is extraspecial of order $p^3$ and exponent $p$.

PROOF. Certainly the subgroups of $P$ of the same order must be isomorphic. If $\exp P \geqslant p^2$ then the subgroups of order $p^2$ are cyclic, therefore by [5], p. 311, we have that $P$ is cyclic or generalized quaternion. But in the last case, if $|P| > 8$ then $P$ would have two nonisomorphic subgroups of order 8, the first one being cyclic and the other one isomorphic to $Q_8$, so $|P| = 8$.

If $\exp P = p$ and if $P$ is elementary abelian of order $p^n$, then two subgroups of $P$ are conjugate in $G$ if and only if they are conjugate in $N_G(P)$ ([5], pag. 418, Lemma 2.5 (a theorem of Burnside)), so we may assume that $P$ is normal in $G$. Then for every $k$, our group $G$ acts transitively on the subgroups of order $p^k$ of $P$. Therefore if $n > 3$, a theorem of Cameron & Kantor (see [2]) yields that, if $p$ is odd, $G/P$ contains $SL(n,p)$ and, if $p = 2$, then $|P| = 2^4$ and $G/P$ is isomorphic to $A_7$, or $|P| = 2^5$ and $G/P$ is isomorphic to $\Gamma L(1, 2^5)$. But $p$ divides $|SL(n,p)|$ and 2 divides $|A_7|$, so that in these two cases, $P$ is not a Sylow $p$-subgroup of $G$. Thus $|P| = p^2$, $p^3$ or $2^5$.

Finally, if $P$ is nonabelian of exponent $p$, then $P$ contains a subgroup, which is extraspecial of order $p^3$ (to see this, let $x \in Z_2(P) \setminus Z(P)$ and $y \in P$ such that $[x,y] \neq 1$. Then $\langle x,y \rangle$ has the property, since $[[x,y],x] = [[x,y],y] = x^p = y^p = [x,y]^p = 1$). It follows that the maximal order of an abelian normal subgroup of $P$ is $p^2$. Let $A$ be one of them. Then $|\mathrm{Aut}(A)| = p(p-1)^2(p+1)$ and $A = C_P(A)$ and hence it follows that $P/A$ has order $p$ at most, therefore $|P| = p^3$. But in this case there cannot exist a normal subgroup $H$ in $G$ whose Sylow $p$-subgroup is nontrivial and has order $< p^3$, because otherwise $P$ would contain at least two conjugacy classes of subgroups of order $p$. Thus $G$ is not $p$-soluble.

EXAMPLES. In Case 2) we have for example $G = SL(2,3)$.

In Case 3), when $|P| = p^k$, $k = 2, 3$, we can choose the semi-direct product of the additive group by the multiplicative group of the field $GF(p^k)$; if $|P| = 2^5$ we can take the relative holomorph of $P$ by $\Gamma L(1, 2^5)$.

Finally, in Case 4), for $p = 5$, there is the Thompson simple group $G$, whose Sylow 5-subgroups are of the required type (see [7]). Moreover, we show that all subgroups of order 5 (respectively 25) are conjugate. Using the notation of [7], let $z$ be an involution of $G$, $H = C_G(z)$, $b \in H$, $|b| = 5$, $B = \langle b \rangle$. Then by [7], (5.2), $P = O_5(C_G(b))$ is a Sylow 5-subgroup of $G$, $B = Z(P)$, and $P$ is normal in $N_G(B)$. Moreover, $G$ contains one class of elements of order 5 and an element $b_1 \in P \setminus B$ has 120 conjugates in $P$ under the action of $N_G(B)$. Thus all subgroups of order 5 and 25 of $P$ are conjugate in $G$ and we have $w_{C(G)}(P) = 1$.

Returning to our classification, we observe that if $w_C(G) = |\Pi(G)|$, $p \in \Pi(G)$ and $P$ is a Sylow $p$-subgroup of $G$ then $w_{C(G)}(P) = 1$, so $P$ is as in Theorem 3. Moreover, we have:

LEMMA 4. If $G$ is a finite group such that $w_C(G) = |\Pi(G)|$ then:

*a*) if $M \leqslant G$ then $\nu(M) \leqslant |\Pi(M)|$.

*b*) If $H$ is a proper normal soluble subgroup of $G$, then $w_{C(G/H)}(G/H) = |\Pi(G/H)|$.

*c*) If $1 \neq H \lhd G$ and $p$ is a prime divisor of both $|H|$ and $[G:H]$, then the Sylow $p$-subgroups of $G$ are cyclic or quaternion groups.

PROOF. *a*) If $\nu(M) > |\Pi(M)|$, then

$$w_C(G) \geqslant \nu(M) + |\Pi(G) \setminus \Pi(M)| > |\Pi(G)|.$$

*b*) Let $H \lhd G$. First assume that $H$ has a complement $M$ in $G$: if $\mathcal{A}$ is any antichain of $G/H$ with $m = w_{C(G/H)}(G/H)$ elements and $K_1, \ldots, K_m$ are subgroups of $M$ such that $[HK_i/H] \in \mathcal{A}$, then these subgroups are not conjugate in $G$. Let $\Pi'$ be the set of all prime divisors of $|G|$ that are not divisors of $|G/H|$ and let $n = |\Pi'|$. For every $p_i \in \Pi'$ we consider a Sylow $p_i$-subgroup $S_i$ of $G$. Then the set $\{[K_1], \ldots, [K_m], [S_1], \ldots, [S_n]\}$ is an antichain of $C(G)$, therefore in this case the assertion follows from:

$$w_C(G) \geqslant m + n = w_{C(G/H)}(G/H) + n \geqslant |\Pi(G/H)| + n = |\Pi(G)| = w_C(G).$$

Now assume that $H$ is contained in $\Phi(G)$. Then:

$$|\Pi(G)| = |\Pi(G/H)| \leqslant w_{C(G/H)}(G/H) \leqslant w_C(G) = |\Pi(G)|,$$

and the assertion follows also in this case. Finally, let $H$ be soluble and let $P$ be a minimal normal subgroup of $G$ contained in $H$. Then $P \leqslant \Phi(G)$ or $P$ has a complement in $G$, therefore in every case $w_{C(G/P)} = |\Pi(G/P)|$. So the assertion follows by induction.

*c)* Let $P$ be a Sylow $p$-subgroup of $G$. From $w_{C(G)}(P) = 1$ it follows that $P \cap H$ is the only subgroup of $P$ of its order. Thus $P$ is cyclic or a quaternion group.

Now we distinguish two subcases: $G$ soluble and $G$ non-soluble. Let $O(G)$ and $O'(G)$ be respectively the maximal normal subgroup of odd order and the minimal normal subgroup with odd quotient. For the non-soluble case, we need the following:

LEMMA 5. Let $G$ be a finite group such that $w_C(G) = |\Pi(G)|$. Then:

*a)* If $G/O(G)$ is isomorphic to $PSL(2,5)$ or to $SL(2,5)$ then $O(G) = 1$.

*b)* If $G$ contains a normal subgroup $K$ isomorphic to $PSL(2,5)$ then $K = G$.

PROOF. *a)* Let $G/O(G) \cong PSL(2,5)$. As $O(G)$ is soluble, it has a subgroup $H$ normal in $G$ whose index in $O(G)$ is a power of an odd prime $p$. First let $p = 3$ or $p = 5$. The Sylow $p$-subgroups of $G/H$ are cyclic, by Lemma 4*c*) so $C_{G/H}(O(G)/H)$ contains the Sylow $p$-subgroups of $G/H$, therefore it is equal to $G/H$. Thus $O(G)/H$ is in the center of $G/H$, but then for $p = 3$ and $p = 5$ respectively, $G/H$ contains subgroups of order 4, 9, 5 and 6 and of order 4, 3, 25, 10, against Lemma 4. If $p > 5$ then $p$ does not divide $|PSL(2,5)|$. So $O(G)/H$ is the Sylow $p$-subgroup of $G/H$, and it must have order $p$, $p^2$ or $p^3$. Trivially we do not have $PSL(2,5)H/H \lhd G/H$, because otherwise $G$ would contain subgroups of order $2p$, $3p$, $5p$, 4, 10, 6; a contradiction. It is well known that, in characteristic $p > 5$, $PSL(2,5)$ has no faithful representations of degree $< 3$, therefore $|O(G)/H| = p^3$. Then there are $p^2 + p + 1$ subgroups of order $p$, so if $x \in G/H$, $x^2 = 1$, then $x$ must fix at least one of them. Thus in $G/H$ there is also a subgroup of order $2p$, so $w_{C(G/H)}(G/H) \geqslant 5$, while $|\Pi(G/H)| = 4$, against Lemma 4. Thus $O(G) = 1$. We can use the same argument if $G/O(G)$ is isomorphic to $SL(2,5)$.

*b)* Recall that $\mathrm{Aut}(PSL(2,5)) \cong \mathrm{Aut}(A_5) \cong S_5$. Let $p$ be a prime divisor of $|G/K|$. If $p = 2$ then $G$ would contain the dihedral group of order 8, against Lemma 2, or there would be an involution out of $K$, so in every case $w_C(G) \geq |\Pi(G)| + 1$. If $p$ were odd, let $M \leq G$ such that $|M/K| = p$. If $p = 3$ then $M$ would contain subgroups of order 4, 6, 10, 9, so in every case $\nu(M) > 3 = |\Pi(M)|$, against Lemma 4$a$). The same argument works if $p = 5$. Finally, if $p > 5$, we would have $M = PSL(2, 5) \times P$, with $|P| = p$ and $\nu(M) = 6 > 4 = |\Pi(M)|$, against Lemma 4$a$) again. Thus $K = G$.

THEOREM 6. Let $G$ be a non-soluble group. Then $w_C(G) = |\Pi(G)|$ if and only if $G$ is isomorphic to $PSL(2,5)$ or to $SL(2,5)$.

PROOF. If $G$ is isomorphic to $PSL(2,5)$ or $SL(2,5)$, the direct examination of the poset of conjugacy classes of subgroups shows that $w_C(G) = 3 = |\Pi(G)|$. We prove that these two groups are the only non-soluble groups with this propery. So let $G$ be a non-soluble group such that $w_C(G) = |\Pi(G)|$. Then $|G|$ is even, so let $P$ be a Sylow 2-subgroup of $G$. We know that $w_{C(G)}(P) = 1$. As $P$ is not cyclic, we have the following two cases:

I) $P$ is elementary abelian. Here Walter's theorem (see [8]) tell us that the group $R = O'(G/O(G))$ is isomorphic to the direct product $A \times S_1 \times \ldots \times S_t$, where $A$ is abelian and the $S_i$ are isomorphic to the first Janko group $J_1$, or to a group $PSL(2, 2^k)$, or to $PSL(2,q)$, $q$ an odd prime, or to a Ree group. But the Sylow 3-subgroups of the Ree groups are nilpotent of class three (see [9]), against Theorem 3. Moreover, there are 7 isomorphism classes of maximal subgroups in $J_1$ ([3], p. 36), so $w_C(J_1) \geq 7 > 6 = |\Pi(J_1)|$ eliminates $J_1$. Since the involutions of $G$ must be all conjugate, $A \lhd G/O(G)$ implies $A = 1$. The direct product $K = S_1 \times \ldots \times S_t$ contains $t$ classes of involutions that are not conjugate because their centralizers form a chain: let $x_i \in S_i$, $o(x_i) = 2$ and let $e_i = x_1 x_2 \ldots x_i$; if $C_i = C_{S_i}(x_i)$, we have $C_K(e_i) = C_1 \times C_2 \times \ldots \times C_i \times S_{i+1} \times \ldots \times S_t$. Thus if $t > 1$ the subgroups $\langle e_1 \rangle, \ldots, \langle e_t \rangle$ cannot be conjugate in $G/O(G)$, so $t = 1$ and therefore $R$ is isomorphic to $PSL(2, 2^k)$ or to $PSL(2,q)$, where $q$ is an odd prime. In the first case, we have $|P| = 2^k$, then $|N_G(P)/C_G(P)| = 2^k - 1$ (see [5], p. 191). Now, $P$ is elementary abelian of order $2^k$, so it has $((2^k-1)(2^k-2))/6$ subgroups of order $2^2$, that must be conjugate in $G$, so in $N_G(P)$. This implies $((2^k-1)(2^k-2))/6 \leq 2^k - 1$, i.e. $k \leq 3$. Then $R$ is isomorphic to $PSL(2,4)$ ($\cong PSL(2,5)$) or to $PSL(2,8)$. The last group however contains subgroups of order 9, 4, 6, 7, so $\nu(PSL(2,8)) > |\Pi(PSL(2,8))|$, against Lemma 4. Now let $R$ be isomorphic to $PSL(2,q)$. Then it contains cyclic subgroups of order $(q+1)/2$ and $(q-1)/2$, dihedral subgroups of order $2r$ for every odd prime $r \in \Pi(R)$ and the Sylow 2-subgroups, which contain elementary abelian subgroups of order 4. Thus $\nu(R) > |\Pi(R)|$ unless one of the numbers $(q+1)/2$ and $(q-1)/2$ is odd and the other one is 2. This is possible only for $q = 3$ or $q = 5$, but $PSL(2,3)$ is soluble, so $R \cong PSL(2,5)$. By Lemma 5 we have $G \cong PSL(2,5)$.

II) $P \cong Q_8$. In this case, Glauberman's $Z^*$-theorem (see [4]) implies that the centre of the group $G/O(G)$ has even order. Let $T \leqslant G$ be such that $Z(G/O(G)) = T/O(G)$. Then $w_{C(G/T)}(G/T) = |\Pi(G/T)|$, by Lemma 4. As the Sylow 2-subgroup of $G/T$ is elementary abelian, by I) we have $O'((G/T)/O(G/T)) \cong PSL(2,5)$. By Lemma 5, we have $G/T \cong$ $\cong PSL(2,5)$, and then $G/O(G) \cong SL(2,5)$. Again by Lemma 5 we must have $O(G) = 1$. This concludes the proof.

Now we examine the case when $G$ is soluble. Certainly, we have $m = w_C(G) = |\Pi(G)| \leqslant 3$, because the conjugacy classes of Hall subgroups whose order is a multiple of two distinct prime powers form an antichain of $\binom{m}{2}$ elements.

The case $m = 1$ is trivially that of cyclic $p$-groups. We examine now the case $m = 2$. Let $G$ be a soluble group such that $w_C(G) = 2$. Then $G$ is not a $p$-group, since a non-cyclic $p$-group has three maximal normal subgroups at least. Therefore $|G| = p^a q^b$ and $G$ is soluble, $f(G) \leqslant 2$ by Lemma 2 and its Sylow subgroups must be as in Theorem 3, (except for the case 4). Moreover, we have:

LEMMA 7. Let $w_C(G) = 2$. Then:

*a)* If $G$ is nilpotent then it is cyclic of order $p^a q$.

*b)* $G/F(G)$ is cyclic.

PROOF. *a)* In this case the maximal subgroups of a Sylow subgroup $P$ of $G$ are normal in $G$, therefore $P$ must contain just one maximal subgroup and hence it is cyclic. Thus $G$ is cyclic too. If $|G| = p^a q^b$ with $a, b > 1$ then $G$ would have subgroups of order $p^2$, $q^2$ and $pq$, therefore $w_C(G) \geqslant 3$. Now *b)* follows from *a)* and Corollary 2.

The Fitting subgroup of $G$ can be a $p$-group or not. We first examine the case when $F(G)$ is not a $p$-group. We have:

PROPOSITION 8. Let $w_C(G) = 2$, and assume that $G$ is non-nilpotent and $F(G)$ is not a $p$-group. Then $|G| = p^n q$ where $q \equiv 1 \pmod p$ and the Sylow $p$-subgroups are cyclic. Moreover $|G'| = q$ and $G$ has the following presentation:

$$G = \langle a, b \mid a^q = b^{p^n} = 1,\ b^{-1}ab = a^r \rangle, \quad 1 < r < q, \quad r^{p^k} \equiv 1 \pmod q$$

for some $k$.

If $h$ is the minimal integer such that $r^{p^h} \equiv 1 \pmod q$ then $Z(G) = \Phi(G)$ is a $p$-subgroup of order $p^{n-h}$. Also, $F(G) = \Phi(G) \times G'$.

PROOF. Since $F(G)$ is not a $p$-group, we have $F(G) = P \times Q$, where $P$ and $Q$ are normal in $G$. One of them must have prime order, because otherwise $G$ would have subgroups of order $p^2$, $q^2$ $pq$ and then $w_C(G) \geqslant 3$. Let us assume that $|Q| = q$. The same argument shows that $G$ must have order $p^n q$, $Q$ is the Sylow $q$-subgroup and is normal in $G$. Thus $Q$ is not contained in $\Phi(G)$. Therefore $\Phi(G) \leqslant P$. Let $S$ be a Sylow $p$-subgroup of $G$. Since $G$ is non-nilpotent then $P \neq S$; being $P \lhd G$, $P$ is unique in its order in $G$ and being $G/F(G)$ cyclic, then $S/P$ is cyclic and $S$ is cyclic too, by Theorem 3. This implies that $G$ is supersoluble. Clearly, $G$ is noncyclic, and so we must have $q > p$ and $q \equiv 1 \pmod p$. Certainly $P = Z(G) = \Phi(G)$ and moreover $G' = Q$. This concludes the proof.

When $F(G)$ is a $p$-group there are some more cases. We have in fact:

PROPOSITION 9. Let $w_C(G) = 2$ and assume that $G$ is non-nilpotent and $F(G)$ is a $p$-subgroup. Then $F(G)$ is the Sylow $p$-subgroup of $G$. Moreover:

*a*) If $F(G)$ is a quaternion group, then $G$ is isomorphic to $SL(2,3)$.

*b*) If $F(G)$ is cyclic then $|G| = p^a q$ with $p \equiv 1 \pmod q$ or $|G| = pq^b$, where $q^b$ is a divisor of $p-1$.

*c*) If $F(G)$ is elementary abelian, its order must be 4 (and then $G$ is isomorphic to $A_4$) or $p^3$ and in this case $|G| = p^3 q^b$, with $q^b = p^2 + p + 1$.

PROOF. Since $F(G)$ is a normal $p$-subgroup and $G/F(G)$ is cyclic, we have $G' \leqslant F(G)$, so the Sylow $p$-subgroup of $G$ is normal in $G$ and therefore coincides with $F(G)$. Let $Q$ be a Sylow $q$-subgroup of $G$. Then $Q$ is isomorphic to $G/F(G)$, it is cyclic and it acts transitively on the subgroups of the same order in $F(G)$. So:

*a*) If $F(G)$ is a quaternion group, $\mathrm{Aut}\, F(G) \cong S_4$ implies that we must have $q = 3$ and $|Q| = 3$, therefore $G$ is isomorphic to $SL(2,3)$.

*b*) If $F(G)$ is cyclic and $|F(G)| = p^a$ with $a > 1$ then $|Q| = q$, because otherwise $G$ would have subgroups of order $p^2$, $pq$, $q^2$. If $|F(G)| = p$ then $q^b$ must divide $p-1$, since no $q$-element can centralize $F(G)$.

*c)* Let $F(G)$ be elementary abelian. First of all $|F(G)| \neq 2^5$, since in this case the order of $G/F(G)$ would be a multiple of $31 \cdot 5$, as $G/F(G)$ must act transitively on subgroups of order 2 and 4 respectively. If $|G| = p^2$ then $F(G)$ contains $p+1$ subgroups of order $p$, on which $Q$ must act transitively. Then $p+1 = q^n$. If $p > 2$, we must have $q = 2$, so $n \geq 2$ but since every element of order 2 of $GL(2,p)$ fixes at least one subgroup of order $p$, $G$ contains one subgroup of order $p^2$, one of order 4 and a dihedral one of order $2p$, a contradiction. Thus $p = 2$ and then $G$ is isomorphic to $A_4$. Instead if $|F(G)| = p^3$ then $Q$must have order $q^b$ and must act transitively on $p^2 + p + 1$ subgroups of order $p$. Thus $q^b = = p^2 + p + 1$. This is possible for example, for $b = 1$, if $p = 2$ and $q = 7$, or $p = 3$ and $q = 13$.

Concerning the last case of the previous proposition, we have:

PROPOSITION 10. Let $G = [V]C$, with $|V| = p^3$, $|C| = q^n$, $V$ elementary abelian, $C$ cyclic, and $1 + p + p^2 = q^n$. If $Z(G) = 1$ then $w_C(G) = 2$.

PROOF. Let us consider the equation $1 + p + p^2 = q^n$, with $p$, $q$ primes. If $n = 1$, as known, there are some solutions: $(p,q) = (2,7)$, $(3,13)$, $(5,31)$, .... We have:

*a)* $n$ is odd. In fact the equation $x^2 + x + (1 - q^n) = 0$ has discriminant $4q^n - 3$ and it must be a square $d^2$ in $\mathbb{Z}$, therefore if $n = 2m$ then $(2q^m) - d^2 = 3$, so $d = 1$ and $q = 1$, a contradiction.

*b)* $q \neq 3$. In fact if $q = 3$ then $p^2 + p + 1 \equiv 0 \pmod 3$, so $p \equiv 1 \pmod 3$, i.e. $p = 3r + 1$. But then $9r^2 + 6r + 1 + 3r + 1 + 1 = 3(3r^2 + + 3r + 1) = 3^{2m+1}$ and 3 would divide $3r^2 + 3r + 1$.

*c)* If an element $y$ of order $q$ fixes a subgroup $P$ of order $p$ of $V$ then $Z(G) \neq 1$. In fact let's suppose first that $y$ does not centralize $P$. Then $q$ divides $p - 1$, so $q$ divides also $3p = 1 + p + p^2 - (p-1)^2$. Thus $q = 3$, a contradiction by *b)*. Therefore $y$ centralizes $P$. Moreover by Maschke's theorem there is a complement $W$ in $V$ which is fixed by $y$. Since $W$ contains $p + 1$ subgroups of order $p$, the same argument shows that $W$ is centralized by $y$. Then $[y, V] = 1$ and $y \in Z(G)$.

*d)* If $G$ is as before and $Z(G) = 1$ then $w_C(G) = 2$. In fact an element $u$ of order $q^n$ must permute cyclically the subgroups of order $p$ of $V$: otherwise $N_G(P)$, where $|P| = p$, would be greater than $V$, so at least one element $y$ of order $q$ would fix $P$ and belong to $Z(G)$. Likewise $u$ permutes cyclically the subgroups of order $p^2$. So we have that in $C(G)$ there are only two maximal chains: $[G], [\langle u \rangle], [\langle u^q \rangle], \ldots, [1]$ and $[G], [\langle V, u^q \rangle], \ldots, [V], [W], [P], [1]$, where $|W| = p^2$; thus $w_C(G) = 2$.

REMARK. If $n > 1$ then $q \equiv 1 \pmod 3$ and $p \equiv 2 \pmod 3$. In fact from $4q^{2m+1} - 3 = d^2$ it follows $q^{2m+1} \equiv 1 \pmod 3$, as $q \neq 3$, so $q \equiv 1 \pmod 3$. Next, from $1 + p + p^2 = q^{2m+1}$ it follows $p(p+1) \equiv 0 \pmod 3$, thus if $p = 3$ then $q = 13$ and $n = 1$: otherwise $p \equiv 3 \pmod 3$.

Now let $w_C(G) = |\Pi(G)| = 3$, let $G$ be soluble and $|G| = p^a q^b r^c$. We distinguish three cases depending on $|\Pi(F(G))|$. First of all, we need the following lemma:

LEMMA 11. Let $w_C(G) = |\Pi(G)| = 3$ and let $G$ be soluble. Then:

*a*) $G$ contains no proper subgroup $H$ such that $|\Pi(H)| = 3$.

*b*) All the minimal normal subgroups of $G$ are Sylow subgroups, and just one of them can have order not a prime.

*c*) If the Sylow $p$-subgroup is normal in $G$, $a > 1$ and there is a subgroup of order $pq$ then there is none of order $p^h r$, $0 < h < a$.

PROOF. *a*) Let $0 < h < a$ and let $|H| = p^h q^k r^m$: then $G$ contains subgroups of order $p^a$, $p^h q^k$, $p^h r^m$, $q^k r^m$, a contradiction.

*b*) If $P$ is a minimal normal subgroup, $|P| = p^h$ with $0 < h < a$, then $G$ would contain subgroups of order $p^a$, $p^h q$, $p^h r$, $q^b r^c$, a contradiction. If the Sylow $q$-subgroup $Q$ also is normal in $G$ and $a, b > 1$ then $G$ would contain subgroups of order $p^a q$, $p^a r$, $q^b r$, $pq^b$, another contradiction. Likewise we can prove the last assertion.

COROLLARY 12. Let $G$ be soluble. Then $w_C(G) = |\Pi(G)| = 3$ and $|\Pi(F((G))| = 3$ if and only if $|G| = pqr$ and $G$ is cyclic.

PROPOSITION 13. Let $G$ be soluble. Then $w_C(G) = |\Pi(G)| = 3$ and $|\Pi(F(G))| = 2$ if and only if $G$ is one of the following types:

*a*) $G = [\mathbb{Z}_p \times \mathbb{Z}_q]\mathbb{Z}_r$.

*b*) $G = [(\mathbb{Z}_2)^2 \times \mathbb{Z}_q]\mathbb{Z}_3$ and the 2-Sylow subgroup is a minimal normal subgroup of $G$. (Thus $G$ has a subgroup isomorphic to $A_4$).

*c*) $G = [(\mathbb{Z}_p)^3 \times \mathbb{Z}_q]\mathbb{Z}_r$ and the Sylow $p$-subgroup is a minimal normal subgroup. (Thus $p^2 + p + 1 = r$ and $G$ has no subgroups of order $pr$ or $p^2 r$).

PROOF. Let $w_C(G) = |\Pi(G)| = 3$ and let $|F(G)| = p^h q^k$, where $h, k > 0$. By Lemma 11 we have $h = a$, $k = b = c = 1$, the Sylow $p$-subgroup $P$ of $G$ is a minimal normal subgroup of $G$ and hence elementary abelian. If $a = 1$ we are in Case *a*). If $a = 2$ then the subgroups of order $p$ must be all conjugate. Therefore $p + 1 = r$, which implies $p = 2$,

$r = 3$, and we are in Case *b*). Notice that there are no subgroups of order 6, therefore $G$ contains a subgroup isomorphic to $A_4$. If $a = 3$, we are in Case *c*). Also we must have $p^2 + p + 1 = r$ and there are no subgroups of order $pr$ or $p^2 r$. It is impossible that $p^a = 2^5$ because, as observed in Theorem 3, in this case $|G| = 32 \cdot 31 \cdot 5$, but $|F(G)| = 32$.

Conversely, a straightforward inspection of the poset $C(G)$ shows that if $G$ is as in *a*), *b*), *c*) then $w_C(G) = |\Pi(G)| = 3$. Case *a*) is obvious. In Case *b*), since the Sylow 2-subgroup is a minimal normal subgroup, $G$ cannot have subgroups of order 6. It has subgroups of orders 1, 2, 3, 4, 12, $q$, $2q$, $4q$, $3q$, $12q$, and for each of these orders the subgroups are conjugate in both cases of $A_4 \times \mathbb{Z}_q$ and when the $q$-Sylow subgroup is not central. In Case *c*), likewise, $G$ cannot have subgroups of order $pr$ or $p^2 r$. It has subgroup of orders $1, p, p^2, p^3, q, pq, p^2 q, p^3 q, r, p^3 r, qr$; subgroups of the same order are conjugate. Indeed, as $G$ does not contain subgroups of order $pr$, an element of order $r$ permutes cyclically the $r = p^2 + p + 1$ subgroups of order $p$ and $p^2$ respectively, so it also acts transitively on the set of the (abelian) subgroups of order $pq$ and $p^2 q$ respectively. The remaining subgroups are Hall subgroups.

We say that a group $A$ acts fixed point freely (f.p.f.) on a vector space $V$ if for every $1 \neq a \in A$ and $0 \neq v \in V$ we have $v^a \neq v$.

PROPOSITION 14. Let $G$ be soluble. If $w_C(G) = |\Pi(G)| = 3$ and $|\Pi(F(G))| = 1$ then:

i) $F(G)$ is the Sylow $p$-subgroup of $G$ and it is a minimal normal subgroup.

ii) $G/F(G)$ is cyclic and it acts. f.p.f. on $F(G)$.

iii) $|G| = p^a qr$, with $a = 1$ or $a = 3$.

iv) If $a = 3$ then $p^2 + p + 1 = qr$, $(qr, 6) = 1$, or $p^2 + p + 1 = q$, $r | p - 1$, $r \neq 1$, and an element of order $r$ induces a power automorphism on $F(G)$.

PROOF. Let $|(G)| = p^a q^b r^c$ and let $Q$ be a Hall subgroup of order $q^b r^c$ of $G$.

i) Lemma 11 implies that $F(G)$ is a minimal normal subgroup of $G$, so it is elementary abelian, and $|F(G)| = p^a$.

ii) Since $F(G)$ is abelian, we have $C_G(F(G)) = F(G)$. We show that $Q$ acts f.p.f. on $F(G)$. Let $x \in Q$ be of order $q$. If $x$ would centralize an element $y$ of order $p$ then $a > 1$ (because otherwise $x \in F(G)$) and $C = C_{F(G)}(\langle x \rangle)$ is a proper subgroup of $F(G)$ and the subgroup $M\langle x \rangle$ would have no elements of order $pq$. Therefore $G$ would have subgroups of order $p^a$, $pq$ (cyclic), $|M|q$, $q^b r^c$, a contradiction. Likewise if $x$ has order $r$ or $pq$. Thus, $Q$ acts f.p.f. on $F(G)$.

iii) First we show that $|Q| = qr$ and $Q$ is cyclic. By [5], p. 502 and Theorem 3, for each odd prime dividing $|Q|$ the Sylow subgroup must be cyclic, while the Sylow 2-subgroup must be also cyclic or quaternion. In the first case $Q$ is supersoluble and, if $q > r$, all of its $q$-subgroups are normal in $Q$. If $b > 1$ then $G$ would have subgroups of order $p^a q$, $p^a r$, $q^b$, $qr$, a contradiction. Thus $b = 1$. A similar argument shows that $c = 1$. Therefore $|Q| = qr$ and by [5], p. 502, $Q$ is cyclic. When $q = 2$ the same argument shows that the Sylow 2-subgroup of $Q$ cannot be isomorphic to the quaternion group $Q_8$.

Next, we cannot have $p^a = 2^5$, otherwise $|G| = 2^5 \cdot 31 \cdot 5$ and $F(G)$ would have 31 elements of order 2, so $G$ would have an element of order 10.

By Theorem 3, $|F(G)| \leq p^3$. The subgroups of order $p$ must form just one conjugacy class of $G$, so $G/F(G)$ must act transitively on them. Let $2 \leq a \leq 3$ and let $g$ be of order $qr$, $x = g^q$ and $y = g^r$. First of all we need to consider the case when $x$ fixes a nontrivial subgroup of $F(G)$. Then $x$ fixes a subgroup of order $p$ and has an eigenvalue $k \neq 0$ in $GF(p)$. Being $xy = yx$, the eigenspace $P$ of $k$ is also fixed by $y$. Therefore $P^g = P$, and because $P \lhd F(G)$, we have $P \lhd G$, so i) implies $P = F(G)$ and $x$ must act as a power automorphism on $F(G)$. In this case the number of subgroups of order $p$ must be $q$.

If $a = 2$, $p$ must be odd (otherwise $\mathrm{Aut}(F(G))$ would be noncyclic); then $p + 1$ is even, so we can assume $r = 2$ and $p + 1 = 2q$. But since an element of order 2 has always eigenvalues, by the previous argument we would also have $p + 1 = q$, a contradiction. Thus $a \neq 2$.

iv) If $a = 3$ then either $p^2 + p + 1 = qr$ or $p^2 + p + 1 = q$ and $r$ divides $p - 1$. In the first case we have $r \neq 2, 3$. In fact, if $r = 3$ and $x = g^q$, as a basis of $F(G)$ one could choose three vectors $u$, $v$, $w$ such that $u^x = v$, $v^x = w$, $w^x = u$, but then $(uvw)^x = (uvw)$, a contradiction, because $x$ cannot fix any subgroup of order $p$. Similar arguments can be used for $r = 2$. So $(qr, 6) = 1$. In the second case, when $p^2 + p + 1 = q$ and $r$ divides $p - 1$, we must have $r \neq 3$; in fact if $r = 3$ then $p = 3t + 1$, so $9t^2 + 9t + 3 = q$ and $3|q$, a contradiction.

Proposition 15. Let $G$ be a soluble group. Then $w_C(G) = 3 = |\Pi(G)|$ and $F(G)$ is a $p$-group if and only if:

i) $|G| = p^a qr$, with $a = 1$ or $a = 3$;

ii) $|F(G)| = p^a$ and $F(G)$ is elementary abelian;

iii) in the case $a = 3$ we have:

iii*a*) $p^2 + p + 1 = qr$, $(qr, 6) = 1$, $G/F(G)$ cyclic, or

iii*b*) $p^2 + p + 1 = q$, $r|p - 1$, and an element of order $r$ acts on $F(G)$ as a power automorphism.

PROOF. The necessity follows by Proposition 14. If $G$ satisfies i), ii), iii), the case $a = 1$ is obvious. Let $a = 3$, let $x$ be of order $q$ and let $y$ be of order $r$. If $p^2 + p + 1 = q$, no $p$-subgroup $P$ or order $p$ can be fixed by $x$. In fact otherwise, since $q$ does not divide $p - 1$, we would have $[P, x] = 1$, and on the other hand $x$ would fix a complement $W$ of $P$ in $F(G)$ and permute its $p + 1$ onedimensional subspaces. But $q$ neither divides $p + 1$ nor $p$, therefore $x$ must fix at least two of them, so it would fix them pointwise. Therefore $[x, F(G)] = 1$, a contradiction. Thus $F(G)$ is a minimal normal subgroup, the subgroups of order $p$ are all conjugate and also those of order $p^2$. Moreover there are no subgroups of order $p^h q$, with $0 < h < a$. Since $y$ induces a power automorphism on $F(G)$, then $xy = yx$, so $G/F(G)$ is cyclic. Thus the subgroups of order $pr$ are all conjugate and also these of order $p^2 r$. Then $w_C(G) = 3$. In the case $p^2 + p + 1 = qr$, let $g$ be of order $qr$ and let $x = g^r$ and $y = g^q$. If $x$ should fix some subgroup $P$ of order $p$, then $q$ would divide $p - 1$, so $q = 3$, a contradiction, or $[x, P] = 1$, so $[x, F(G)] = 1$ as before, another contradiction. Likewise, $y$ fixes no subgroup of order $p$. Thus there are no subgroups of order $p^h q$ or $p^h r$, with $0 < h < 3$, $F(G)$ is a minimal normal subgroup, the subgroups of order $p$ are all conjugate and also those of order $p^2$. This implies $w_C(G) = 3$.

EXAMPLES. Let $p = 41$, let $V$ be the elementary abelian $p$-group of order $p^3$ and let $x$ and $y$ be the automorphisms of $V$ whose matrices are

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

respectively. They commute and we have $x^5 = 1$, $y^{1723} = 1$. If $G$ is the split extension of $V$ with respect to $\langle xy \rangle$, it satisfies i), ii), iii*b*) of Proposition 15 and we have $w_C(G) = 3$.

One can construct a similar example, of even order, if $p = 3$, $q = 13$, $r = 2$.

We have an example of the other type, that satisfies iii*a*) of Proposition 15, if $p = 11$, $q = 29$, $r = 7$.

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