LUISE-CHARLOTTE KAPPE
JOHN H. YING

On exact power margin groups

Rendiconti del Seminario Matematico della Università di Padova, tome 87 (1992), p. 245-265
<http://www.numdam.org/item?id=RSMUP_1992__87__245_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 87 (1992)

On Exact Power Margin Groups.

LUISE-CHARLOTTE KAPPE - JOHN H. YING (*)

1. Introduction.

The investigation of groups with special conditions on their power structure is the topic of this paper. Research in this area started with regular p-groups, introduced by P. Hall in [3]. The guiding principle in Hall's investigations has been to look at groups which have properties similar to abelian ones in as far as their power structure is concerned. We will follows the same principle by investigating groups which have an abelian-like structure of their power margins.

In 1940, P. Hall in [4] introduced the margin or marginal subgroup of a word. In the special case of a single variable word $f(x) = x^n$, n an integer, the n-power margin of a group G is defined as

$$M_n(G) = \{a \in G \mid (ag)^n = g^n \text{ for all } g \in G\}.$$

The n-power margin, as all margins, forms a characteristic subgroup of G. The embedding of the n-power margin in a group for various values of n was the topic of a recent paper by the first author [7].

We start with some observations on power margins of abelian groups. Let a be a torsion element of order $|a|$ in an abelian group G. Then $(ag)^{|a|} = g^{|a|}$ for all $g \in G$, hence $a \in M_{|a|}(G)$. This leads to the following definition in general:

DEFINITION 1.1. A torsion element a of a group G is called an exact power margin element if $a \in M_{|a|}(G)$.

The set of exact power margin elements in a group G is denoted by

$$EPM(G) = \{a \in G \mid |a| < \infty \text{ and } (ag)^{|a|} = g^{|a|} \text{ for all } g \in G\}.$$

(*) Indirizzo degli AA.: L.-C. KAPPE: SUNY at Binghamton, Binghamton, N.Y. 13902-6000; J. H. YING: Montrose, PA 18801.

We will show that this set can be a nontrivial proper subgroup of G, but it does not have to be a subgroup at all, even in the case of finite p-groups (Example 4.2 and Example 6.3).

Observing $M_n(G) = M_{-n}(G)$ and $M_0(G) = G$, we will assume from now on that $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. Setting $g = 1$, we obtain $a^n = 1$ for $a \in M_n(G)$. Thus all elements in $M_n(G)$ have an order dividing n. Denoting with

$$G[n] = \{a \in G \mid a^n = 1\}$$

the set of elements in G of order dividing n, we can reformulate this as $M_n(G) \subseteq G[n]$ for any group G. In an abelian group the other inclusion holds too. This leads to the following definition:

DEFINITION 1.2. Let G be a group and $n \in \mathbb{N}$. We say G has an exact n-power margin if $M_n(G) = G[n]$. The class of exact n-power margin groups is denoted by $\mathfrak{M}_n$.

It can be easily seen that every n-abelian group, i.e. a group in which $(ab)^n = a^n b^n$ for all a, b in the group, is in $\mathfrak{M}_n$. An abelian group has an exact n-power margin for all $n \in \mathbb{N}$. Thus we define:

DEFINITION 1.3. A group G is an exact power margin group if $G \in \mathfrak{M}_n$ for all $n \in \mathbb{N}$. The class of exact power margin groups is denoted by $\mathfrak{M}$.

We note that, by Theorem 4.4, a finite group G is an exact power margin group if and only if $EPM(G) = G$. Moreover, a finite exact power margin group is nilpotent. Thus we can focus our attention on finite p-groups when studying finite exact power margin groups.

The class $\mathfrak{M}$ properly contains all regular p-groups. Other such properties have been investigated by A. Mann in [8] and M. Y. Xu in [9]. Two special classes defined in [9], namely the class of groups with a regular power structure and the class of strongly semi-p-abelian groups, are of particular interest, because they are both subclasses of $\mathfrak{M}$.

To make our notions more precise, we introduce the following notations. For a group G and $n \in \mathbb{N}$ we define

$G_n = \langle G[n] \rangle$, the subgroup generated by the elements of order dividing n,

$G^{[n]} = \{g^n \mid g \in G\}$, the set of n-th powers of elements in G,

$G^n = \langle G^{[n]} \rangle$, the subgroup generated by the n-th powers of elements in G.

For the convenience of the reader we list here the notational equivalents for a p-group G and $n = p^i$, a p-power. We have $V_i(G) = G^{[p^i]}$ and $\mho_i(G) = G^{p^i}$, as well as $\Lambda_i(G) = G[p^i]$ and $\Omega_i(G) = G_{p^i}$.

In [9], Xu lists the following conditions as the main properties of the power structure of a regular p-group:

(i) $G^{[p^i]} = G^{p^i}$ for all i;

(ii) $G[p^i] = G_{p^i}$ for all i;

(iii) $\pi_i\colon gG_{p^i} \to g^{p^i}$ is a well-defined bijection from G/G_{p^i} onto $G^{[p^i]}$ for all i.

We note that (iii) slightly differs from what appears in [9], namely:

(iii′) $\pi_i\colon gG_{p^i} \to g^{p^i}$ is a well-defined bijection from G/G_{p^i} onto G^{p^i} for all i.

Condition (iii′) has (i) as an immediate consequence. Our conditions (i) and (iii) turn out to be independent. We include two more conditions into our discussion, also satisfied by a regular p-group:

(iv) For all $a, b \in G$, $(ab^{-1})^{p^i} = 1$ if and only if $a^{p^i} = b^{p^i}$ for all i;

(v) $M_{p^i}(G) = G[p^i]$ for all i.

In [9], Xu calls a finite p-group a *strongly semi-p-abelian group* if it satisfies (iv). On the other hand, a finite p-group satisfies (v) if and only if it is an exact power margin group (Proposition 5.1).

Xu calls a finite p-group satisfying (i), (ii) and (iii) a group with a regular power structure. In [9, Theorem 1], he states that a finite p-group G satisfies (i), (ii) and (iii) if and only if G satisfies (i) and (iv). A stronger and more general version of this theorem is provided in Theorem 5.3. It turns out that (ii) is redundant (Theorem 5.2). More precisely, we show that the following equivalencies and implications hold:

$$\text{(iii)} \leftrightarrow \text{(iv)} \rightarrow \text{(v)} \rightarrow \text{(ii)}. \tag{1.0.1}$$

Xu's definitions, as well as the equivalencies and implications of (1.0.1), bring up a natural question: what is the complete classification of all the combinations of the above five basic conditions? A main result of our paper is the following graph that answers this question in the case of finite p-groups:

$$
\begin{tikzcd}
\text{Regularity} \arrow[r] & (i)+(iv) \arrow[r] \arrow[dr] & (iv) \arrow[r] & (v) \arrow[r] & (ii) \\
 & & (i)+(v) \arrow[r] \arrow[ur] & (i)+(ii) \arrow[r] \arrow[ur] & (i)
\end{tikzcd}
\tag{1.0.2}
$$

Each of the $2^4 - 1$ non-empty subsets of the four basic conditions (i), (ii), (iv) and (v) belongs to a node in (1.0.2). That is, each subset is equivalent to the representative conditions of a certain node in that graph. On the other hand, Theorem 5.5 shows that the reverse of each implication in (1.0.2) is *not* true. Hence the graph of (1.0.2) cannot be further reduced, and so it represents a complete classification.

Like (v), each of the other conditions has an analogue power condition for any $n \in \mathrm{N}$ in the class of all groups. In the following section we will introduce some of these general power conditions and investigate their intrdependencies and closure properties. Though of interest in their own right, these results will become essential tools for the above-mentioned complete classification of the five conditions for finite p-groups, as discussed above.

2. Groups with a special power structure.

As an analogue to (i) and (ii) for p-groups we define the following general conditions for a group G.

DEFINITION 2.1. For a group G and $n \in \mathrm{N}$ we say

(*a*) G is n-power closed if $G^{[n]} = G^n$;

(*b*) G is n-exponent closed if $G_n = G[n]$.

The classes of n-power closed and n-exponent closed groups are denoted by $\mathfrak{P}_n$ and $\mathfrak{E}_n$.

As an analogue of (iv) we define:

DEFINITION 2.2. For a group G and $n \in \mathrm{N}$, we say

(*a*) G is semi-n-abelian, provided for $a, b \in G$ we have $(ab^{-1})^n = 1$ if and only if $a^n = b^n$;

(*b*) G has a regular n-power structure if G is n-power closed and semi-n-abelian.

The classes of semi-n-abelian groups and those with a regular n-power structure are denoted by $\mathfrak{S}_n$ and $\mathfrak{R}_n$, respectively.

We observe that n-abelian groups satisfy all of the above conditions and abelian groups satisfy them for all n. This prompts the following definitions.

DEFINITION 2.3. A group G is said to be power closed if $G \in \mathfrak{P}_n$ for all $n \in \mathrm{N}$, and exponent closed if $G \in \mathfrak{E}_n$ for all $n \in \mathrm{N}$. The respective classes are denoted by $\mathfrak{P}$ and $\mathfrak{E}$.

DEFINITION 2.4. (*a*) A group G is said to be strongly semi-abelian if $G \in \mathfrak{S}_n$ for all $n \in \mathrm{N}$. The class of strongly semi-abelian groups is denoted by $\mathfrak{S}$.

(*b*) A group G is said to have regular power structure if $G \in \mathfrak{R}_n$ for all $n \in \mathrm{N}$. The class of groups with regular power structure is denoted by $\mathfrak{R}$.

We now investigate the interdependence of the above conditions and their relationship to exact n-power margin groups. We start with acharacterization of groups with an exact n-power margin.

THEOREM 2.5. *Let G be a group and $n \in \mathrm{N}$. Then G has an exact n-power margin if and only if $(ab^{-1})^n = 1$ implies $a^n = b^n$ for $a, b \in G$.*

PROOF. Assume that $(ab^{-1})^n = 1$ implies $a^n = b^n$. Consider $u \in G[n]$ and $b \in G$ arbitrary. Setting $a = ub$, it follows that $(ub)^n = b^n$ for all $b \in G$, hence $u \in M_n(G)$, and therefore $G[n] \subseteq M_n(G)$. Our claim follows, since $M_n(G) \subseteq G[n]$ holds always.

Conversely, assume $G[n] = M_n(G)$. Thus $u^n = 1$ implies $(ux)^n = x^n$ for all $x \in G$. Setting $ux = a$ and $x = b$, we have $a^n = b^n$ whenever $(ab^{-1})^n = 1$. ∎

The following corollary, stated without proof, connects semi-n-abelian and exact n-power margin groups.

COROLLARY 2.6. *Any semi-n-abelian group has an exact n-power margin.*

The next result shows that for groups in general and arbitrary $n \in \mathrm{N}$, the analogue of (iii) always implies the corresponding condition (ii).

PROPOSITION 2.7. *Let G be a group and $n \in \mathbb{N}$, such that $\pi_n: gG_n \to g^n$ is a well-defined bijection of G/G_n onto $G^{[n]}$. Then $G_n = G[n]$.*

PROOF. Suppose $a, b \in G[n]$. We have to show $ab^{-1} \in G[n]$. Now $a, b \in G[n]$ implies $b^{-1} \in G[n] \subseteq G_n$. Therefore $b^{-1}G_n = G_n$, and hence $ab^{-1}G_n = aG_n = 1 \cdot G_n$. Thus by our assumption we have $(ab^{-1})^n = 1^n$. We conclude $ab^{-1} \in G[n]$, and hence $G_n = G[n]$. ∎

The following theorem characterizes the property semi-n-abelian.

THEOREM 2.8. *For a group G and $n \in \mathbb{N}$, the following conditions are equivalent:*

(a) G is semi-n-abelian;

(b) G has an exact n-power margin and $\mu_n: gM_n(G) \to g^n$ is a well-defined bijection of $G/M_n(G)$ onto $G^{[n]}$;

(c) $\pi_n: gG_n \to g^n$ is a well-defined bijection of G/G_n onto $G^{[n]}$.

PROOF. Assume (a). By Corollary 2.6 it follows immediately that $M_n(G) = G[n]$. This together with our assumption implies that $a \equiv b \bmod M_n(G)$ if and only if $(ab^{-1})^n = 1$ if and only if $a^n = b^n$. We conclude that μ_n is well-defined and one-one. Since μ_n maps trivially onto $G^{[n]}$, it follows that (a) implies (b).

Next assume (b). We always have $M_n(G) \subseteq G_n$. Thus, G having an exact n-power margin implies $G_n \subseteq M_n(G)$, hence $G_n = M_n(G)$. Therefore $\pi_n = \mu_n$, i.e. π_n is a well-defined bijection of G/G_n onto $G^{[n]}$. Thus (b) implies (c).

Finally assume (c); π_n well-defined and one-one means $a \equiv b \bmod G_n(G)$ if and only if $a^n = b^n$. But by Proposition 2.7 we have $G_n = G[n]$. This implies $a \equiv b \bmod G_n$ if and only if $(ab^{-1})^n = 1$. We conclude $(ab^{-1})^n = 1$ if and only if $a^n = b^n$. Thus G is semi-n-abelian, and (c) implies (a). ∎

The following corollary is an immediate consequence of the above theorems and the definitions.

COROLLARY 2.9. *For the classes $\mathfrak{R}_n$, $\mathfrak{S}_n$, $\mathfrak{M}_n$ and $\mathfrak{E}_n$ the following inclusions hold:*

$$\mathfrak{R}_n \subseteq \mathfrak{S}_n \subseteq \mathfrak{M}_n \subseteq \mathfrak{E}_n.$$

In Section 5 we will show that the inclusions are strict. The concluding result of this section is a discussion of the closure properties of the various classes under consideration.

THEOREM 2.10. *The classes $\mathfrak{S}_n$, $\mathfrak{M}_n$ and $\mathfrak{E}_n$ are subgroup and direct product closed, but not quotient closed. The class $\mathfrak{P}_n$ is quotient and direct product closed, but not subgroup closed. The class $\mathfrak{R}_n$ is only direct product closed.*

PROOF. The proofs of the subgroup and direct product closure of $\mathfrak{S}_n$, $\mathfrak{M}_n$ and $\mathfrak{E}_n$, as well as the quotient and direct product closure of $\mathfrak{P}_n$ are straight forward and omitted here. The p-group G, p odd, of Example 6.1 is semi-p-abelian, thus in $\mathfrak{M}_p$ as well as $\mathfrak{E}_p$. But the group K of Example 6.4, a homomorphic image of G, is not in $\mathfrak{E}_p$, thus not in $\mathfrak{M}_p$ and $\mathfrak{S}_p$ as well. Thus $\mathfrak{E}_n$, $\mathfrak{S}_n$ and $\mathfrak{M}_n$ are not quotient-closed. The group H of Example 6.2 is p-power closed. Now H contains an isomorphic copy of the group G of Example 6.1 as a subgroup. But G is not p-power closed. It follows that $\mathfrak{P}_n$ is not subgroup closed. Finally, observing that $\mathfrak{R}_n = \mathfrak{P}_n \cap \mathfrak{S}_n$ leads to the conclusion that $\mathfrak{R}_n$ is direct product closed. To show that $\mathfrak{R}_n$ is neither subgroup closed nor quotient closed, we note that H of Example 6.2 has a regular power structure, but it has a subgroup and a quotient group not in $\mathfrak{R}_p$. ■

3. Elementary properties of exact power margin elements.

The first three lemmas of this section state some general facts about power margins.

LEMMA 3.1. *Let G be a group and $n, m \in \mathbb{N}$.*

(*a*) *If $m \mid n$, then $M_m(G) \subseteq M_n(G)$;*

(*b*) *Let $a \in M_n(G)$ and $g \in G$. Then $[a, g^n] = [a^n, g] = [a, g]^n = 1$;*

(*c*) *Suppose $(m, n) = t$. Then for any $a \in M_m(G)$ and $b \in M_n(G)$*

$$[a, b^t] = [a^t, b] = [a, b]^t = 1.$$

In particular, if $(m, n) = 1$, then $[M_m(G), M_n(G)] = 1$.

PROOF. The proof of (*a*) is omitted here. To prove (*b*), we note that from the definition of $M_n(G)$ and the fact that it is a characteristic subgroup, it follows $a^{-1}g^n a = (a^{-1}ga)^n = (ga)^n = g^n$, hence $[a, g^n] = 1$. The facts that $[a, g] \in M_n(G)$ and $u^n = 1$ for all $u \in M_n(G)$ yield the other two identities.

To prove (c), let i, j be integers such that $t = im + jn$. Then $b^t = b^{im}$. Because $a \in M_{im}(G)$, we obtain $[a, b^t] = [a, b^{im}] = 1$ by (b), and similarly $[a^t, b] = 1$. Now $[a, b] \in M_m(G) \cap M_n(G)$ together with (b) implies $[a, b]^t = [a, b]^{im} \cdot [a, b]^{jn} = 1$. ■

Without proof we state the following result for finite groups.

LEMMA 3.2. *Let G be a finite group and $(n, |G|) = t$. Then $M_n(G) = M_t(G)$ and $G[n] = G[t]$.*

This lemma suggests that, for finite p-groups, only those n-power margins, where n is a p-power, need to be considered. This will be established in Proposition 5.1. The following lemma specifies a boundary for such power margins in finite p-groups.

LEMMA 3.3. *Let G be a finite p-group, and $n = p^i$, an arbitrary positive power of p. Then*

$$\Omega_i(Z_{p-1}(G)) \subseteq M_{p^i}(G) \subseteq \Omega_i(G).$$

Moreover, if G is a regular p-group, then $M_{p^i}(G) = \Omega_i(G)$.

PROOF. Obviously, $M_{p^i}(G) \subseteq \Omega_i(G)$. For $H = \langle Z_{p-1}(G), g\rangle$, $g \in G$, we observe that $H' \subseteq Z_{p-2}(G)$. Therefore H has nilpotency class $\leqslant p-1$, and hence is a regular p-group. Consider $a \in \Omega_i(Z_{p-1}(G))$. Then $(ag)^{p^i} = a^{p^i} g^{p^i} c^{p^i}$ for some $c \in \langle a, g\rangle'$. But both a and c belong to $\Omega_i(H)$, which has exponent p^i due to the regularity of H. So $(ag)^{p^i} = g^{p^i}$. This shows $a \in M_{p^i}(G)$. Hence $\Omega_i(Z_{p-1}(G)) \subseteq M_{p^i}(G)$.

If G itself is regular and $a \in \Omega_i(G)$, then for any $g \in G$, we conclude again $(ag)^{p^i} = g^{p^i}$. This shows $M_{p^i}(G) = \Omega_i(G)$. ■

The next two lemmas present a few operations which serve to derive new exact power margin elements from old ones.

LEMMA 3.4. *For any group G we have:*

(a) If $a \in EPM(G)$ and $\sigma \in \mathrm{Aut}(G)$, then a^σ, a^{-1} and, more generally a^t for $(t, |a|) = 1$ are elements in $EPM(G)$.

(b) If $a, b \in EPM(G)$ and $(|a|, |b|) = 1$, then $a \cdot b \in EPM(G)$ and $|ab| = |a|\,|b|$.

(c) Suppose $a, b \in EPM(G)$ and $|a| = p^i < p^j = |b|$, where p is a prime and $i, j \in \mathbb{N}$. Then $ab, ba \in EPM(G)$, and $|ab| = |ba| = p^j$.

PROOF. We omit the proof of (a). To prove (b), let $k = |a| \cdot |b|$. Because $a \in EPM(G)$, $(ab)^k = b^k = 1$. Hence $|ab|$ divides k. On the other hand, since $a \in M_{|a|}(G)$, $b \in M_{|b|}(G)$ and $(|a|, |b|) = 1$, by Lemma 3.1 (*c*) we have $[a, b] = 1$. Thus $1 = (ab)^{|ab|} = a^{|ab|} \cdot b^{|ab|}$. But $\langle a \rangle \cap$ $\cap \langle b \rangle = 1$, because $(|a|, |b|) = 1$. Hence $a^{|ab|} = 1$ and $b^{|ab|} = 1$, and so k divides $|ab|$. This shows that $|ab| = |a| \cdot |b|$. Thus, by Lemma 3.1 (*a*), $a, b \in M_{|ab|}(G)$, and hence $ab \in M_{|ab|}(G)$.

To prove (*c*), let $n = p^j$ and $m = p^{j-1}$. Then, by (*a*) of Lemma 3.1, $a \in M_m(G) \subseteq M_n(G)$. Thus both ab and ba are elements in $M_n(G)$, and their orders divide n. On the other hand, $(ab)^m = b^m \neq 1$. Hence $|ab| =$ $= n$, thus $ab \in EPM(G)$, and similarly, $ba \in EPM(G)$.

For finite groups we have in addition:

LEMMA 3.5. *Let G be a finite group.*

(*a*) *If $n = \exp(G)$, and $g \in G$ with $|g| = n$, then $g \in EPM(G)$. In particular if G is a p-group, then $EPM(G)$ contains all elements of maximal order, thus $EPM(G) \neq 1$.*

(*b*) *$Z(G) \subseteq EPM(G)$.*

(*c*) *Let p be a prime. If $1 \neq g \in Z_{p-1}(G)$ with $|g| = p^i$, then $g \in$ $\in EPM(G)$. In particular, if G is a p-group, $Z_{p-1}(G) \subseteq EPM(G)$.*

(*d*) *If G is a p-group, $N \lhd G$ and $|N| < p^p$, then $N \subseteq EPM(G)$.*

PROOF. The proofs of (*a*) and (*b*) are trivial and omitted here. For (*c*), we note that, for an arbitrary element x in G, $H = \langle Z_{p-1}(G), x \rangle$ is a nilpotent group of class at most $p - 1$. Let P be the Sylow p-subgroup of H, and let $x = u \cdot v$, where $u \in P$ and $(|v|, p) = 1$. Because H is nilpotent, $(gx)^{p^i} = (gu)^{p^i} \cdot v^{p^i}$. Since $g \in Z_{p-1}(G) \cap P \subseteq Z_{p-1}(P)$, Lemma 3.3 implies $(gu)^{p^i} = u^{p^i}$. Consequently, $(gx)^{p^i} = u^{p^i} \cdot v^{p^i} = (uv)^{p^i} = x^{p^i}$. To prove (*d*), we note that $N \subseteq Z_{p-1}(G)$. Hence by (*c*), $N \subseteq$ $\subseteq EPM(G)$. ■

We conclude this section with an expansion formula for powers and commutators in metabelian groups which can be found in [5, Lemma 1].

LEMMA 3.6. *Let G be a metabelian group, $n \in \mathbb{N}$, and $v, w \in G$. Then*

$$(3.6.1) \qquad (vw^{-1})^n = v^n \cdot \left(\prod_{0 < i+j < n} [v, {}_i w, {}_j v]^{\binom{n}{i+j+1}} \right) \cdot w^{-n}.$$

4. The set of exact power margin elements.

In this section we investigate the set of exact power margin elements, and in particular torsion groups in which every element is an exact power margin element. Our first theorem describes the behavior of exact power margin elements in case of a direct product.

THEOREM 4.1. *Let A, B be groups and $G = A \times B$ their direct product. Then*:

(*a*) $EPM(A) \times EPM(B) \subseteq EPM(A \times B)$.

(*b*) *There exist groups A, B such that $EPM(A) \times EPM(B)$ is a proper subset of $EPM(A \times B)$.*

(*c*) *Let A and B be finite groups of relatively prime order. Then*

$$EPM(A) \times EPM(B) = EPM(A \times B).$$

PROOF. The proofs of (*a*) and (*c*) are straightforward and omitted here. To show (*b*), let $A = B = S_3$, the symmetric group on three letters. It is easy to see that $EPM(S_3)$ consists only of the identiy. However, as shown in Example 4.2, $EPM(S_3 \times S_3)$ is nontrivial. Thus, by (*a*), $EPM(A) \times EPM(B)$ is a proper subset of $EPM(A \times B)$. ■

The question may be rasied on whether the set of exact power margin elements always forms a subgroup. The following example answers this question in the negative.

EXAMPLE 4.2. *Let $G = S_3 \times S_3$. Then $EPM(G) \neq 1$, whereas $EPM(S_3) \times EPM(S_3) = 1$ and $EPM(G)$ is not a subgroup of G.*

PROOF. Since $EPM(S_3) = 1$, it follows that

$$EPM(S_3) \times EPM(S_3) = 1.$$

Consider $g \in G$ with $|g| = 6$. Since $\exp(G) = 6$, Lemma 3.5(*a*) implies that $g \in EPM(G)$, thus $EPM(G) \neq 1$. Assume $EPM(G)$ is a subgroup. Then $1 \neq g^2 \in EPM(G)$. However, g^2 is completely contained in one of the components, and thus should be an exact power margin element with respect to the elements in this component. Since $EPM(S_3) = 1$, this is a contradiction, hence $EPM(G)$ is not a subgroup. ■

Even if G is a finite p-group, $EPM(G)$ does not have to be a subgroup, as Example 6.3 demonstrates. In the next example we give a finite p-group, p any prime, in which $EPM(G)$ is a proper subgroup of index p. For p odd, these examples originate in a paper by Cody [1, Theorem 2], where they occur as the generic groups for the lowest values of the parameters, in which the H_p-subgroup has index p. Note that for any group G with $G^p \neq 1$ we have $H_p(G) = \langle g \in G \mid g^p \neq 1 \rangle$, and $H_p(G) = 1$, if $G^p = 1$. These groups could likewise be obtained as homomorphic images of the p-groups in Example 6.1.

EXAMPLE 4.3. *Let p be a prime. For $p = 2$, define $K = D_4$, the dihedral group of order 8, and for p odd we define K as follows: Let $B = \langle b \rangle \times A$, where $|b| = p^2$ and $A = \langle a_1 \rangle \times \ldots \times \langle a_{p-2} \rangle$, an elementary abelian p-group of rank $p - 2$. Moreover, define $K = [B] \cdot \langle c \rangle$ to be a semidirect product of B by a cyclic group $\langle c \rangle$, where*

$$c^p = 1, \qquad [b, c] = a_1,$$

$$[a_i, c] = a_{i+1} \quad \text{for } i = 1, \ldots, p-3, \qquad \text{and} \qquad [a_{p-2}, c] = b^{-p}.$$

Then for every prime p, the group K is a finite metabelian p-group of order p^{p+1}, in which $EPM(K)$ is a proper subgroup of index p.

PROOF. First let $p = 2$. It follows from (a) in Lemma 3.5 and the fact that $M_2(G) \subseteq Z(G)$ for any group G that $EPM(D_4) = \langle c \rangle$, where $\langle c \rangle$ is the cyclic subgroup of order 4 in D_4.

Now let p be an odd prime. According to [1] we have $\exp(K) = p^2$ and $c(K) = p$. Moreover, $H_p(K) = B$ and $K' = \langle b^p \rangle \times A$ thus $K' \subseteq H_p(K)$ and K' consists of all elements of order dividing p in $H_p(K)$.

We claim $EPM(K) = H_p(K)$, and thus $EPM(K)$ is a subgroup of index p in K. We first show that $H_p(K) \subseteq EPM(K)$. Let $h \in H_p(K)$. If $|h| = p^2$, then $h \in EPM(K)$ by (a) in Lemma 3.5. Thus. we can assume $|h| = p$, hence $h \in K'$ by the above. But $|K'| = p^{p-1}$. Thus, by Lemma 3.5(d), $K' \subseteq EPM(K)$. Hence $H_p(K) \subseteq EPM(K)$, as claimed. Let $g \notin H_p(K)$. Then $gb^{-1} \notin H_p(K)$, and hence $1 = (gb^{-1})^p$. Thus $(gb^{-1})^p \neq b^{-p}$ for all such g. This implies $EPM(K) \subseteq H_p(K)$, and our claim follows. ∎

We conclude this section with a characterization of those torsion groups, in which every element is an exact power margin element.

THEOREM 4.4. *Let G be a torsion group. Then $G = EPM(G)$ if and only if G is an exact power margin group. Moreover, if $G = EPM(G)$, then G is the direct product of its Sylow subgroups. In particular, if G is finite, then G is nilpotent.*

PROOF. Suppose $G = EPM(G)$. First we show that G is the direct product of its Sylow subgroups. Let $x, y \in G$ with $(|x|, |y|) = 1$. Since $x \in M_{|x|}(G)$ and $y \in M_{|y|}(G)$, it follows by (c) of Lemma 3.1 that $[x, y] = 1$. Hence elements of relatively prime order commute in G, and our claim follows.

To show that G is an exact power margin group, we need to show $G[n] = M_n(G)$ for all $n \in \mathbb{N}$. Since $M_n(G) \subseteq G[n]$ holds always, it suffices to prove the other inclusion. Suppose $x \in G[n]$. Obviously $|x|$ divides n. Since x is an exact power margin element, we obtain by (a) of Lemma 3.1 that $x \in M_{|x|}(G) \subseteq M_n(G)$. Thus $G[n] \subseteq M_n(G)$.

Conversely, if G is an exact power margin group, then for any $x \in G$, $x \in G[|x|] = M_{|x|}(G)$. Hence $x \in EPM(G)$. This shows $G = EPM(G)$. ■

The following theorem is an immediate consequence of Lemma 3.2 and Lemma 3.3:

THEOREM 4.5. *A regular p-group is an exact power margin group.*

Thus, being an exact power margin group is one of the consequences of regularity. But the class of regular p-groups is properly contained in the class of the exact power margin groups, as can be seen from Example 6.1. On the other hand, since, for any group G and any integer n, $M_n(G)$ is a subgroup of G, we conclude:

THEOREM 4.6. *An exact power margin group is exponent closed. That is,* $\mathfrak{M} \subseteq \mathfrak{E}$.

Theorems 4.5 and 4.6 specify boundaries for exact power margin groups. These boundaries will be further elaborated in the following sections.

5. Comparison of power structure conditions for finite p-groups.

In this section we give a complete analysis of the interrelationships of the classes $\mathfrak{R}$, $\mathfrak{S}$, $\mathfrak{M}$, $\mathfrak{E}$, and $\mathfrak{P}$ in the case of finite p-groups. For a class $\mathfrak{X}$ of groups and p a prime we denote by $\mathfrak{X}(p)$ the sublcass of finite p-groups in $\mathfrak{X}$. Furthermore, $\mathfrak{R}^*(p)$ denotes the class of finite regular p-groups. The following proposition is crucial for the investigation. It allows us to establish membership in the classes $\mathfrak{S}(p)$, $\mathfrak{M}(p)$, $\mathfrak{E}(p)$ and $\mathfrak{P}(p)$ by just checking the relevant p-powers.

PROPOSITION 5.1. *Let G be a finite p-group and let $\mathfrak{X} \in \in \{\mathfrak{M}, \mathfrak{S}, \mathfrak{E}, \mathfrak{P}\}$. Then $G \in \mathfrak{X}$ if and only if $G \in \mathfrak{X}_{p^i}$ for all i.*

PROOF. One of the directions in each case follows immediately from the definitions, namely if $G \in \mathfrak{X}$, then obviously $G \in \mathfrak{X}_{p^i}$ for all i. Thus it suffices to establish in each case that $G \in \mathfrak{X}_{p^i}$ for all i implies $G \in \mathfrak{X}$. In case $G \in \mathfrak{M}_{p^i}$ for all i, we obtain as an immediate consequence of Lemma 3.2 that $G \in \mathfrak{M}$. Assume $G \in \mathfrak{S}_{p^i}$ for all i. By Theorem 2.5 and Definition 2.2 (a) it follows that $G \in \mathfrak{M}_{p^i}$ for all i, and thus, by the above, $G \in \mathfrak{M}$. Therefore, it suffices to show that $a^n = b^n$ implies $(ab^{-1})^n = 1$ for $a, b \in \in G$ and $n \in \mathbb{N}$. Let $n = p^i k$ with $(k, p) = 1$. Then there exist integers α, β such that $1 = \alpha k + \beta |G|$. Assume $a^n = b^n$, then $(a^{\alpha k})^{p^i} = (b^{\alpha k})^{p^i}$. By our assumption, this yields $(a^{\alpha k} b^{-\alpha k})^{p^i} = 1$. Since $\alpha k = 1 - \beta |G|$, this implies $(ab^{-1})^n = 1$, the desired result. In case $G \in \mathfrak{E}_{p^i}$ for all i, the proof follows in a similar manner as in the preceding case.

Finally, we have to show that $G \in \mathfrak{P}_{p^i}$ for all i implies $G \in \mathfrak{P}_n$ for all $n \in \mathbb{N}$. Again let $n = kp^i$ with $(k, p) = 1$ and integers α, β such that $\alpha k + + \beta |G| = 1$. By our assumption we obtain $a^n \cdot b^n = (a^k)^{p^i} \cdot (b^k)^{p^i} = c^{p^i}$ for some $c \in G$. Now $c \in \langle c^{\alpha k} \rangle$ by the above, and hence $a^n \cdot b^n = (c^{\alpha})^n$, the desired result. ■

With the help of Proposition 5.1 and the results of the second section, we establish now a stronger and more general version of Xu's results, quoted in the introduction. It also shows that Xu's definition of regular power structure and our definition coincide in the case of finite p-groups. The following theorem shows that condition (ii) in Xu's definition is redundant.

THEOREM 5.2. *Let G be a finite p-group with $\pi_i \colon gG_{p^i} \to g^{p^i}$ a well-defined bijection of G/G_{p^i} onto $G^{[p^i]}$ for all i. Then G is exponent closed.*

PROOF. Proposition 2.7 implies that $G \in \mathfrak{E}_{p^i}$ for all i. By Proposition 5.1 it follows that $G \in \mathfrak{E}$, the desired result. ■

The above theorem and the following one justify to define regular power structure in terms of semi-n-abelian.

THEOREM 5.3. *Let G be a finite p-group. Then G is strongly semi-abelian if and only if $\pi_i \colon gG_{p^i} \to g^{p^i}$ is a well-defined bijection of G/G_{p^i} onto $G^{[p^i]}$.*

PROOF. By Theorem 2.8, $G \in \mathfrak{S}_{p^i}$ if and only if $\pi_i \colon gG_{p^i} \to g^{p^i}$ is a well-defined bijection of G/G_{p^i} onto $G^{[p^i]}$. Obviously if $G \in \mathfrak{S}$, then

$G \in \mathfrak{S}_{p^i}$ for all i. Conversely, if $G \in \mathfrak{S}_{p^i}$ for all i, then $G \in \mathfrak{S}$ by Proposition 5.1. ■

We conclude our investigations with a complete comparison of power structure conditions for finite p-groups. First, we present the following:

THEOREM 5.4. *Let p be a prime. Then*

$$\mathfrak{R}^*(p) \subseteq \mathfrak{R}(p) \subseteq \mathfrak{S}(p) \subseteq \mathfrak{M}(p) \subseteq \mathfrak{E}(p)$$

and

$$\mathfrak{R}(p) \subseteq \mathfrak{M}(p) \cap \mathfrak{P}(p) \subseteq \mathfrak{E}(p) \cap \mathfrak{P}(p) \subseteq \mathfrak{P}(p).$$

PROOF. Let G be a finite regular p-group. If $p = 2$, then G is abelian, and it is trivial to show that $G \in \mathfrak{R}(p)$. If p is odd, then Hauptsatz 10.5(b) and Satz 10.6(a) of [6] show that $G \in \mathfrak{P}_{p^i} \cap \mathfrak{S}_{p^i} = \mathfrak{R}_{p^i}$ for all i. Thus, Proposition 5.1 yields $G \in \mathfrak{R}(p)$. Hence $\mathfrak{R}^*(p) \subseteq \mathfrak{R}(p)$. The rest of the proof follows trivially from Corollary 2.9. ■

REMARK. By Proposition 5.1 we can claim that the set-inclusion order of Theorem 5.4 is equivalent to the order of implications shown in (1.0.2). Each arrow in this graph means that the conditions of the source node imply the conditions of the target node. There are $2^4 - 1$ non-empty combinations for the four basic conditions (i), (ii), (iv) and (v) specified in Section 1. It can be easily seen that each combination is equivalent to the representative conditions of a particulr node in the graph.

We now show that the set inclusions of Theorem 5.4 are all proper inclusions, i.e., the graph (1.0.2) cannot be further reduced. Here strict inclusion is denoted by «$\subset$».

THEOREM 5.5. *Let p be a prime. Then*

$$\begin{array}{ccccccccc}
\mathfrak{R}^*(p) \subset \mathfrak{R}(p) \subset \mathfrak{S}(p) \subset & & \mathfrak{M}(p) & \subset & \mathfrak{E}(p) & & \\
& & \cup & & \cup & & \\
& \mathfrak{R}(p) \subset & \mathfrak{M}(p) \cap \mathfrak{P}(p) & \subset & \mathfrak{E}(p) \cap \mathfrak{P}(p) & \subset & \mathfrak{P}(p).
\end{array}$$

PROOF. That the inclusions hold is an immediate consequence of Theorem 5.4. It suffices to show here that the inclusions are strict.

Case 1: $p > 2$. In [2], Groves gives a regular p-group $H(p)$, $p \geqslant 3$, such that, for any regular p-group G, which is not p-abelian, the direct product $H(p) \times G$ is not regular. However, by Theorem 5.4 and Theorem 2.10, $H(p) \times G$ has a regular power structure. Thus $\mathfrak{R}^*(p) \subset \mathfrak{R}(p)$. To show that the other inclusions are strict, we turn to the examples of Section 6. The group G in Example 6.1 is strongly semi-abelian but not power closed. Hence $\mathfrak{R}(p) \subset \mathfrak{S}(p)$, $\mathfrak{M}(p) \cap \mathfrak{P}(p) \subset \mathfrak{M}(p)$, $\mathfrak{E}(p) \cap \mathfrak{P}(p) \subset \mathfrak{E}(p)$ and $\mathfrak{E}(p) \not\subset \mathfrak{P}(p)$. To show that $\mathfrak{R}(p) \subset \mathfrak{M}(p) \cap \mathfrak{P}(p)$, we use the group of Example 6.6, which is power closed and an exact power margin group, but not strongly semi-abelian. This same group also shows $\mathfrak{S}(p) \subset \mathfrak{M}(p)$.

Next, consider the group F of Example 6.5. We have $F \in \mathfrak{E} \cap \mathfrak{P}$, but $F \notin \mathfrak{M}$. Thus $\mathfrak{M}(p) \subset \mathfrak{E}(p)$ and $\mathfrak{M}(p) \cap \mathfrak{P}(p) \subset \mathfrak{E}(p) \cap \mathfrak{P}(p)$. Finally, we consider the group K of Example 6.4, which is power closed but not exponent closed. Hence $\mathfrak{E}(p) \cap \mathfrak{P}(p) \subset \mathfrak{P}(p)$ and $\mathfrak{P}(p) \not\subset \mathfrak{E}(p)$.

Case 2: $p = 2$. In [9, Theorem 3], Xu gives an example of a non-abelian 2-group with a regular power structure. This proves $\mathfrak{R}^*(2) \subset \mathfrak{R}(2)$, because regular 2-groups are abelian. Now consider the group U of Example 6.7, which is strongly semi-abelian but not power-closed. Hence $\mathfrak{R}(2) \subset \mathfrak{S}(2)$, $\mathfrak{M}(2) \cap \mathfrak{P}(2) \subset \mathfrak{M}(2)$, $\mathfrak{E}(2) \cap \mathfrak{P}(2) \subset \mathfrak{E}(2)$ and $\mathfrak{E}(2) \not\subset \mathfrak{P}(2)$.

Now we consider the quaternion group Q of order 8. Obviously, Q is power closed and not strongly semi-abelian. Moreover, by (a) and (b) of Lemma 3.5, Q is an exact power margin group. Hence, $\mathfrak{R}(2) \subset \mathfrak{M}(2) \cap \mathfrak{P}(2)$ and $\mathfrak{S}(2) \subset \mathfrak{M}(2)$.

Next we consider the group V of Example 6.8, which is shown to be exponent closed, power closed, but not an exact power margin group. Thus $\mathfrak{M}(2) \subset \mathfrak{E}(2)$ and $\mathfrak{M}(2) \cap \mathfrak{P}(2) \subset \mathfrak{E}(2) \cap \mathfrak{P}(2)$.

Finally, to show that $\mathfrak{E}(2) \cap \mathfrak{P}(2) \subset \mathfrak{P}(2)$, we observe that the dihedral group D of order 8 is power closed but not exponent closed. Thus it also shows that $\mathfrak{P}(2) \not\subset \mathfrak{E}(2)$. ■

6. Examples.

This section is comprised of two parts, the first assuming an odd prime p. The group exhibited in the first example of this part can be considered generic, since all the groups presented in the other examples of this part are either homomorphic images, or in one case an extension of the group at hand.

EXAMPLE 6.1. *Let* $X = \langle x \rangle \times \langle y_1 \rangle \times \ldots \times \langle y_{p-1} \rangle$, *where* $|x| = p^2$ *and* $|y_i| = p$ *for* $i = 1, \ldots, p-1$. *Moreover, define* $G = [X] \cdot \langle b \rangle$ *to be a* *semidirect product of* X *by a cyclic group* $\langle b\rangle$, *where* $|b| = p^2$ *and*

$$[x, b] = y_1, \quad [y_i, b] = y_{i+1} \text{ for } i = 1, \ldots, p-2, \quad \text{and} \quad [y_{p-1}, b] = 1.$$

Then G *is a strongly semi-abelian group of order* p^{p+3} *and nilpotency class* p *and exponent* p^2, *which is not power closed. Hence* G *has not a regular power structure, in particular* G *is not regular.*

PROOF. We note that the mapping induced by b on X is an automorphism of order p on X, hence $b^p \in Z(G)$. In fact,

$$Z(G) = \langle b^p\rangle \times \langle x^p\rangle \times \langle y_{p-1}\rangle. \tag{6.1.1}$$

It is also easy to see that $G' = \langle y_1\rangle \times \ldots \times \langle y_{p-1}\rangle$, with $\exp(G') = p$, and $c(G) = p$ as well as $|G| = p^{p+3}$.

We first show that $G \in \mathfrak{M}$. By Proposition 5.1 and the fact that $\exp(G) = p^2$, it suffices to show $G[p] = M_p(G)$. For an arbitrary element u in G the class of $\langle G', u\rangle$ is at most $p-1$. Hence $\langle G', u\rangle$ is regular, and so by Theorem 4.5 it is an exact power margin group. This shows that $G' \subset EPM(G)$. For an arbitrary element g in G, there exist integers i, j and $y \in G'$ such that $g = x^i b^{-j} y$. By the above we obtain $g^p = (x^i b^{-j})^p$. Since G is metabelian, we may apply (3.6.1). This yields

$$g^p = (x^i b^{-j})^p = x^{ip}\left(\prod_{0 < s+t < p} [x^i,{}_s b^j,{}_t x^i]^{\binom{p}{s+t+1}}\right) b^{-jp}.$$

Observing $\exp(G') = p$ and using the relations of G, we obtain

$$g^p = (x^i b^{-j})^p = x^{ip} b^{-jp} y_{p-1}^{i \cdot j^{p-1}}. \tag{6.1.2}$$

Now consider $g \in G[p] \setminus G'$. Then (6.1.2) implies $i \equiv j \equiv 0 \bmod p$, thus $x^i b^{-j} \in Z(G)$. So, for any element $w \in G$, this together with the fact that $G' \subseteq EPM(G)$ yields

$$(gw)^p = (x^i b^{-j} yw)^p = (yw)^p x^{ip} b^{jp} = (yw)^p = w^p.$$

Hence $g \in EPM(G)$. It follows that $G \in \mathfrak{M}$.

Next we show that G is strongly semi-abelian. By Proposition 5.1 and the fact that $\exp(G) = p^2$, it suffices to show that $G \in \mathfrak{S}_p$. Using Theorem 2.5, Definition 2.4 (a) and the fact that G is an exact power margin group, it suffices to show that $g^p = h^p$ implies $(gh^{-1})^p = 1$ for $g, h \in G$. There exist integers i, j, m, n and $u, v \in G'$ such that $g = x^i b^j u$ and $h = x^m b^n v$. If $g^p = h^p$, then $x^{ip} b^{jp} \equiv x^{mp} b^{np} \bmod G'$. Hence p divides $i - m$ and $j - n$, and so x^{i-m}, $b^{j-n} \in Z(G) \cap G[p]$. But $gh^{-1} =$

$= x^{i-m} b^{j-n} w$ for some $w \in G'$. This implies $(gh^{-1})^p = w^p (x^{i-m})^p \cdot \cdot (b^{j-n})^p = w^p = 1$. It follows that $G \in \mathfrak{S}$.

Finally we show that $G^{[p]}$ is not a subgroup of G, hence $G \notin \mathfrak{P}$. Assume to the contrary, and consider $(xb^{-1})^p$. By (6.1.2) we obtain $(xb^{-1})^p = x^p y_{p-1} b^{-p}$. This implies $y_{p-1} = g^p$ for some $g = x^i b^j u$, $u \in \in G'$. Again, by (6.1.2), it follows $y_{p-1} = x^{ip} b^{jp} \cdot y_{p-1}^{ij^{p-1}}$. This together with (6.1.1) yields $i \equiv j \equiv 0 \bmod p$. Thus $y_{p-1} = 1$, a contradiction. Since $\mathfrak{R} = \mathfrak{S} \cap \mathfrak{P}$, it follows now that G does not have a regular power structure, and, in particular, G is irregular. ■

To show that being power closed is not inherited to subgroups, we use an extension of the group in the preceding example.

EXAMPLE 6.2. *Let $H = G \cdot \langle z \rangle$, where G is the group of Example* 6.1 *with $[g, z] = 1$ for all $g \in G$ and $z^p = y_{p-1}$. Then H is power closed but the subgroup $G \subsetneq H$ is not. Moreover, H has a regular power structure. But it contains a subgroup and a quotient group which do not have a regular power structure.*

PROOF. It was shown in Example 6.1 that G is not p-power closed. In view of Proposition 5.1 and the fact that $\exp(H) = p^2$, it suffices to show that $H^{[p]} = T$, where $T = \langle x^p, b^p, z^p \rangle \subsetneq Z(H)$. For $h \in H$ we have $h = gz^k$, where $g \in G$ and k an integer. Thus $h^p = g^p z^{pi}$. Now $g = = x^i b^{-j} u$ for some $u \in G'$ and i, j integers. By (6.1.2) it follows $h^p = = y_{p-1}^{ij^{p-1}} x^{ip} b^{-jp} z^{kp} = x^{ip} b^{-jp} z^{(ij^{p-1}+k)p}$. Hence $H^{[p]} \subsetneq H^p \subsetneq T$. Conversely, let $t = x^{\alpha p} b^{-\beta p} z^{\gamma p}$ with integers α, β, γ be an arbitrary element in T. If $\beta \equiv 0 \bmod p$, then $t = x^{\alpha p} z^{\gamma p} = (x^\alpha \cdot z^\gamma)^p \in H^{[p]}$. If $\beta \not\equiv 0 \bmod p$, then let $h = x^\alpha b^{-\beta} z^{\gamma - \alpha}$. It follows that $h^p = x^{\alpha p} z^{\alpha p} b^{-\beta p} z^{p(\gamma - \alpha)} = t$. Therefore $T \subsetneq H^{[p]}$, hence $H^{[p]} = H^p$, and we conclude that H is power closed.

Next we prove that $H \in \mathfrak{S}$. Again, it suffices to prove that for all $g, h \in H$ $g^p = h^p$ if and only if $(gh^{-1})^p = 1$. Let $g = x^i \cdot b^j \cdot u \cdot z^k$ and $h = = x^r \cdot b^s \cdot v \cdot z^t$, where $u, v \in G'$ and i, j, k, r, s, t are integers. Then $g^p = = x^{ip} \cdot b^{jp} \cdot y_{p-1}^{(k+i(-j)^{p-1})}$, $h^p = x^{rp} \cdot b^{sp} \cdot y_{p-1}^{(t+r(-s)^{p-1})}$, and $gh^{-1} = x^{i-r} \cdot b^{j-s} \cdot \cdot z^{k-t} \cdot w$ for some $w \in G'$.

If $g^p = h^p$, then $i \equiv r \bmod p$, $j \equiv s \bmod p$, and $k \equiv t \bmod p$. Hence $x^{i-r} \cdot b^{j-s} \cdot z^{k-t} \in \langle x^p \rangle \times \langle b^p \rangle \times \langle y_{p-1} \rangle = \Omega_1(Z(G))$, and therefore $(gh^{-1})^p = w^p = 1$. On the other hand, if $(gh^{-1})^p = 1$, then $x^{(i-r)p} \cdot \cdot b^{(j-s)p} \equiv 1 \bmod G'$. Hence $p \mid i - r$ and $p \mid j - s$, and so $1 = (gh^{-1})^p = = z^{(k-t)p}$. Thus $p \mid k - t$. In other words, $i \equiv r \bmod p$, $j \equiv s \bmod p$ and $k \equiv \equiv r \bmod p$. So $g^p = h^p$.

Thus, being strongly semi-abelian and power closed, H has a regular power structure. However, G, as a subgroup of H, does not have a regular power structure. Moreover, $H/\langle x^p \rangle$ does not have a regular power structure. This is because, if $H/\langle x^p\rangle$ has a regular power structure, then $H/\langle x^p\rangle$ is an exact power margin group, and so is $G/\langle x^p\rangle$, contradicting the claims of Example 6.3. ∎

The group W in the next example is exponent closed, but not an exact power margin group. Since W is a homomorphic image of the strongly semi-abelian group G in Example 6.1, this shows that the classes $\mathfrak{S}$ and $\mathfrak{M}$ are not quotient closed.

EXAMPLE 6.3. *Let* $W = G/\langle x^p\rangle$, *where* G *is the group defined in Example* 6.1. *Then* W *is exponent closed but not power closed and not an exact power margin group, and* $EPM(W)$ *is not a subgroup of* W.

PROOF. We first note that $x^p \in Z(G)$ by (6.1.1). To prove $W \in \mathfrak{E}$, we note that $\exp(W) = p^2$. Thus by Proposition 5.1, it suffices to show that $W[p]$ is a subgroup. Let $N = \langle x^p\rangle$ and $u = gN \in W[p]$, where $g \in G$. Since $G = [X]\cdot\langle b\rangle$, $g = hb^i$ for some element $h \in X$ and some integer i. But $(gN)^p = 1$, and $N \subseteq X$. Hence $b^{ip} = 1$. In other words, $b^i \in \langle b^p\rangle$. This shows that $W[p]$ is a subset of $(X \times \langle b^p\rangle)/N$, which is an elementary abelian subgroup of W. Hence $W[p]$ is a subgroup of W.

To prove $W \notin \mathfrak{P}$ and $W \notin \mathfrak{M}$, we note that (6.1.2) implies

$$(xN\cdot b^{-1}N)^p = b^{-p}N\cdot y_{p-1}N. \tag{6.3.1}$$

Similarly, as in Example 6.1 it can be shown that $y_{p-1}N$ is not a p-power in W, and we conclude that $W \notin \mathfrak{P}$. To show $W \notin \mathfrak{M}$ set $u = xN$ and $v = b^{-1}N$. Then $|u| = p$ and $|v| = |uv| = p^2$. By (6.3.1), we obtain $(uv)^p = v^p(y_{p-1}N) \neq v^p$. Hence u is not an exact power margin element, and thus $W \notin \mathfrak{M}$. Finally, we note that by (a) of Lemma 3.5, both uv and v^{-1} are elements of $EPM(W)$. But $u = (uv)\cdot v^{-1}$ is not an exact power margin element, showing that $EPM(W)$ is not a subgroup of W. ∎

It can be shown that the group in the next example is isomorphic to the wreath product of a cyclic group of order p with another such group ([6, Satz 10.3.d]). We present this group as a homomorphic image of G in Example 6.1. In addition to facilitating our calculations, this shows that the class $\mathfrak{E}$ is not quotient closed.

EXAMPLE 6.4. *Let* $K = G/N_1$, *where* G *is the group defined in Example* 6.1 *and* $N_1 = \langle x^p\rangle \times \langle b^{-p}y_{p-1}\rangle$. *Then* K *is power closed but not exponent closed.*

PROOF. We first note that $N_1 \subseteq Z(G)$ by (6.1.1). By Proposition 5.1 and the fact that $\exp(K) = p^2$, it suffices to show that $K \in \mathfrak{P}_p$. Consider $(gN_1)^p \in K^{[p]}$, where $g \in G$ with $g = x^i b^{-j} y$, i, j integers and $y \in G'$. If $j \equiv 0 \bmod p$, then $1 = (gN_1)^p \in \langle b^p N_1 \rangle$. Suppose $j \not\equiv 0 \bmod p$. Then by (6.1.2) we obtain $(gN_1)^p = g^p N_1 = (x^{ip} b^{-jp} y_{p-1}^i) N_1 = b^{-(j-i)p} N_1$. Thus $K^{[p]} \subseteq \langle b^p N_1 \rangle$. Since obviously $\langle b^p N_1 \rangle \subseteq K^{[p]}$, it follows that $K^{[p]} = \langle b^p N_1 \rangle$, and thus $K^{[p]}$ is a subgroup of K. To prove that $K \notin \mathfrak{E}$, it suffices to show that $K[p]$ is not a subgroup. By (6.1.2) we have $xN_1, xb^{-1}N_1 \in K[p]$ and

$$(xN_1 \cdot xb^{-1} N_1)^p = (x^2 b^{-1} N_1)^p = x^{2p} b^{-p} y_{p-1}^2 N_1 = y_{p-1} N_1 \neq N_1.$$

Thus $xN_1 \cdot xb^{-1} N_1 \notin K[p]$, and $K[p]$ is not a subgroup. ■

The group in the next example, again a homomorphic image of the group in Example 6.1, shows that a group which is exponent and power closed is not necessarily an exact power margin group.

EXAMPLE 6.5. *Let* $F = G/N_2$ *with* $N_2 = \langle y_{p-1}^{-1} x^p \rangle \times \langle y_{p-1}^{-1} b^p \rangle$, *where* G *is the group of Example* 6.1. *Then* F *is power closed and exponent closed, but not an exact power margin group.*

PROOF. We note that $N_2 \subseteq Z(G)$ by (6.1.1). To show that $F \in \mathfrak{P}$ and $F \in \mathfrak{E}$, it suffices to show that $F \in \mathfrak{P}_p$ and $F \in \mathfrak{E}_p$. This follows by Proposition 5.1 and the fact that $\exp(F) = p^2$. By (6.1.2) we have $g^p \in Z(G)$ for all $g \in G$, and hence $(gN_2)^p \in Z(G)/N_2$, which is a central cyclic subgroup of F of order p. Thus $F^{[p]} = Z(G)/N_2$, and $F \in \mathfrak{P}_p$.

To show $F \in \mathfrak{E}_p$, consider $L = G' \cdot Z(G) \cdot \langle xb^{-2} \rangle$. Because $c(L) < p$, L/N_2 is regular. Moreover, since by (6.1.2)

$$(xb^{-2})^p = x^p \cdot b^{-2p} \cdot y_{p-1} = y_{p-1}^{-1} \cdot x^p \cdot (y_{p-1}^{-1} \cdot b^p)^{-2} \in N_2,$$

we have $\exp(L/N_2) = p$. Now we show that $F[p] = L/N_2$. Suppose $(gN_2)^p = 1$ for $g = x^i b^{-j} y \in G$, i, j integers and $y \in G'$. This means $g^p \in N_2$, and, by (6.1.2), we obtain $g^p = x^{ip} \cdot b^{-jp} \cdot y_{p-1}^{i \cdot j^{p-1}} = y_{p-1}^{-t} x^{pt} y_{p-1}^{-s} \cdot b^{ps}$, for some integers s, t. If $j \equiv 0 \bmod p$, then $s \equiv 0 \bmod p$, $s + t \equiv 0 \bmod p$ and $i \equiv t \bmod p$. Hence $g \in Z(G) \cdot G'$. If $j \not\equiv 0 \bmod p$, then $i \equiv -t - s \bmod p$, $i \equiv t \bmod p$ and $s \equiv -j \bmod p$. Consequently $j \equiv 2i \bmod p$, and so $g \equiv (xb^{-2})^i \bmod G' \cdot Z(G)$. Thus in either case $g \in L$, and hence $F[p] = L/N_2$, showing that $F \in \mathfrak{E}_p$.

Finally, we exhibit an element in F which is not in $EPM(F)$. Consider $xb^{-2} \in G$. Then $(xb^{-2})^p \in N_2$ by the above. If $xb^{-2} N_2 \in M_p(F)$, then $(x^{-1} N_2 xb^{-2} N_2)^p = x^{-p} N_2$. On the other hand $(x^{-1} N_2 xb^{-2} N_2)^p = b^{-2p} N_2$. Thus $x^p b^{-2p} \in N_2$, i.e. there exist integers

s, t such that $x^p b^{-2p} = y_{p-1}^{-s} x^{ps} y_{p-1}^{-t} b^{tp}$. Hence $s \equiv 1 \bmod p$, $t \equiv -2 \bmod p$ and $s + t \equiv 0 \bmod p$. It follows $1 \equiv 0 \bmod p$, a contradiction. Thus F is not in $\mathfrak{M}$. ■

The final example of this part is yet another homomorphic image of the group in Example 6.1, serving to demonstrate a different set of power structure conditions:

EXAMPLE 6.6. *Let* $E = G/N_3$ *with* $N_3 = \langle y_{p-1} x^p \rangle \times \langle y_{p-1}^{-1} b^p \rangle$, *where* G *is the group of Example* 6.1. *Then* E *is power closed and an exact power margin group, but it is not strongly semi-abelian.*

PROOF. Note that $N_3 \subseteq Z(G)$. Let gN_3 be an arbitrary element in E. Then there are integers i, j, and $y \in G'$ such that $g = x^i b^{-j} y$. By (6.1.2) we obtain $g^p = x^{ip} b^{-jp} y_{p-1}^{ij^{p-1}}$. If $j \equiv 0 \bmod p$, then $(gN_3)^p = x^{ip} N_3$. If $j \not\equiv 0 \bmod p$, then $(gN_3)^p = b^{-jp} N_3 = x^{jp} N_3$ because $x^p b^p \in N_3$. Thus $E^{[p]} = \langle x^p N_3 \rangle$, and $E \in \mathfrak{P}$. Moreover, if $(gN_3)^p = 1$, then $i \equiv j \equiv 0 \bmod p$, and $g \in G' Z(G)$. Since $|G' Z(G)/N_3| = p^{p-1}$, by Lemma 3.5 (*d*), $gN_3 \in EPM(E)$. This shows $E \in \mathfrak{M}$.

Finally, we show $E \notin \mathfrak{S}$. Because $x^p b^p \in N_3$, $(x^{-1} N_3)^p = (bN_3)^p$. However, $(bx)^p = (xb)^p = x^p b^p y_{p-1}$ by (6.1.2). Thus $(bN_3 \cdot xN_3)^p \neq N_3$, and $E \notin \mathfrak{S}$. ■

The second part of this section is comprised of the following two examples of 2-groups, which are needed for proving Theorem 5.5 in the case $p = 2$.

EXAMPLE 6.7. *Let* $U = [\langle x \rangle \times \langle y \rangle] \cdot \langle b \rangle$, *where* $|x| = |b| = 4$, $|y| = 2$, $[x, b] = y$ *and* $[y, b] = 1$. *Then* U *is a strongly semi-abelian* 2*-group of order* 32, *and* U *is not power closed.*

PROOF. We note that U and G of Example 6.1 are actually defined in the same way, where G is defined for $p > 2$ and U for $p = 2$. It is trivial to see that U is a 2-group of order 32 with $\exp(U) = 4$, $U' = \langle y \rangle$ and $Z(U) = \langle x^2 \rangle \times \langle y \rangle \times \langle b^2 \rangle$. Moreover, if $g = x^i \cdot y^j \cdot b^k \in U[2]$, then $1 = g^2 \equiv x^{2i} \cdot b^{2k} \bmod U'$. Hence $2 | i$ and $2 | k$, and so $g \in Z(U)$. In fact, $U[2] = Z(U) = M_2(U)$. By (*a*) and (*b*) of Lemma 3,5, $U \in \mathfrak{M}$.

To show that $U \in \mathfrak{S}$, we only need to prove that, for any $g, y \in U$, $(gh^{-1})^2 = 1$ if $g^2 = h^2$. Suppose $g^2 = h^2$, and let $g = x^{i_1} y^{j_1} b^{k_1}$ and $h = x^{i_2} \cdot y^{j_2} \cdot b^{k_2}$. Then $x^{2i_1} \cdot b^{2k_1} \equiv x^{2i_2} \cdot b^{2k_2} \bmod U'$, and hence $2 | i_1 - i_2$ and $2 | k_1 - k_2$. This implies that $g \equiv h \bmod Z(U)$. Thus $gh^{-1} \in Z(U)$, and therefore $(gh^{-1})^2 = 1$.

To show that $U \notin \mathfrak{P}$, we note that $(xb)^2 = xb^2 \cdot b^{-1} xb = x^2 \cdot b^2 \cdot y$. If $U \in \mathfrak{P}$ then there exists $g \in U$ such that $y = g^2$. Let $g = x^{i_1} y^{j_1} b^{k_1}$. Then

$1 \equiv x^{2i_1} \cdot b^{2k_1} \bmod U'$. This implies that $2|i_1$ and $2|k_1$. So $y = g^2 = 1$, a contradiction. ■

EXAMPLE 6.8. *Let* $V = [\langle u \rangle \times \langle z \rangle] \cdot \langle c \rangle$, *where* $|u| = 2$, $|z| = |c| = 4$, $[u, c] = z^2$ *and* $[z, c] = 1$. *Then* V *is an exponent closed and power closed* 2*-group of order* 32, *and* V *is not an exact power margin group.*

PROOF. It is easy to see that V is a 2-group of order 32, with $\exp(V) = 4$, $V' = \langle z^2 \rangle$ and $Z(V) = \langle z \rangle \times \langle c^2 \rangle$. To show that $V \in \mathfrak{E}$, we need to prove that $V[2]$ is a subgroup of V. Because $\langle z^2 \rangle \times \langle c^2 \rangle$ is a subgroup of $Z(V)$, clearly $\langle c^2, u, z^2 \rangle = \langle u \rangle \times \langle z^2 \rangle \times \langle c^2 \rangle \subseteq V[2]$. For $g \in V[2]$ there exist integers i, j, k such that $g = u^i \cdot z^j \cdot c^k$. Thus $1 \equiv u^{2i} \cdot c^{2k} \bmod Z'$, and so $2|k$. Consequently, $c^k \in Z(G)$ and $1 = g^2 = u^{2i} \cdot z^{2j} \cdot c^{2k} = z^{2j}$. Hence $2|j$ and $g \in \langle c^2, u, z^2 \rangle$. Thus $V[2] = \langle c^2, u, z^2 \rangle$ is a subgroup of V.

To show that $V \in \mathfrak{P}$, we only need to prove that $V^{[2]}$ is a subgroup of V. Let $D = \langle z^2 \rangle \times \langle c^2 \rangle$. Then $V' \subseteq D \subseteq Z(V)$, and V/D clearly has exponent 2. Thus $V^{[2]} \subseteq D$. But $z^2 \cdot c^2 = (zc)^2 \in V^{[2]}$. Hence $V^{[2]} = D$ is a subgroup of V. Finally, to show that $V \notin \mathfrak{M}$, we observe that the element u has order 2 but is not central. Hence $u \notin M_2(G)$, and our claim follows. ■

REFERENCES

[1] W. F. CODY, *Classification of 2-generator finite metabelian p-groups having a proper H_p-subgroup*, Arch. Math., **52** (1989), pp. 105-116.

[2] J. R. J. GROVES, *On direct products of regular p-groups*, Proc. Amer. Math. Soc., **37** (1973), pp. 377-379.

[3] P. HALL, *A contribution to the theory of groups of prime-power order*, Proc. London Math. Soc., **36** (1934), pp. 29-95.

[4] P. HALL, *Verbal and marginal subgroups*, J. Reine Angew. Math., **182** (1940), pp. 156-167.

[5] G. T. HOGAN - W. P. KAPPE, *On the H_p-problem for finite p-groups*, Proc. Amer. Math. Soc., **20** (1969), pp. 450-454.

[6] B. HUPPERT, *Endliche Gruppen I*, Springer-Verlag, Berlin (1967).

[7] L.-C. KAPPE, *On power margins*, J. Algebra, **122** (1989), pp. 337-344.

[8] A. MANN, *The power structure of p-groups - I*, J. Algebra, **42** (1976), pp. 121-135.

[9] M. Y. XU, *The power structure of finite p-groups*, Bull. Austral. Math. Soc., **36** (1987), pp. 1-10.

Manoscritto pervenuto in redazione il 6 marzo 1991.