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Local Existence, Uniqueness and Regularity for a Class of Degenerate Parabolic Systems Arising in Biological Models.

VINCENZO VESPRI (*)

SUMMARY - We study a class of degenerate nonlinear parabolic systems arising from biological models. We use the linearization method and the analytic semigroup theory in the linear case. For this purpose, we need to study a class of degenerate linear elliptic operators and we prove the generation of analytic semigroups in L^p $(1 < p < \infty)$ and in C_0^α $(0 < \alpha < 1)$.

Introduction.

Many biological models are described by degenerate nonlinear Cauchy problems (see, for instance [12], [14], [23]), and, for this reason they have been studied by several authors, especially in these last few years.

Let us list some examples.

I-st Example: The chemotaxis model.

Alt ([2]) considers a model of chemotaxis (see [15] for a detailed description of this phenomenon) where a motile individual can be attracted by mediators diffused by other individuals. This model is governed

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by the system

$$
(0.1) \quad \begin{cases} u_t(t, x) = \big(\mu_0(u(t, x))\, u_x(t, x) - \chi_0(u(t, x))\, \rho_x(t, x)\big)_x, \\ \hfill t \geq 0, \ \ x \in [0, 1], \\ \rho_t(t, x) = k\rho_{xx}(t, x) + 2ku(t, x), \qquad t \geq 0, \ \ x \in [0, 1], \\ u(0, x) = u_0(x), \qquad \rho(0, x) = \rho_0(x), \qquad x \in [0, 1], \end{cases}
$$

where

u is the cellular density,
ρ is the mediator diffusion,
k is a positive constant,
$\mu_0(u)$, $\chi_0(u)$ are nonlinear functions (see [2], section 1).

Note that if (as in [2]) $\mu_0(u) = \chi_0(u) = u$, system (0.1) is degenerate in the first equation and nondegenerate in the second one. Alt studies this model by approximating system (0.1) by non-degenerate system.

II-nd Example: The model of the distributional pattern formations of two interacting species.

Shigesada-Kawasaki-Teramoto ([23]) studied the model of spatial segregation of two competitive species. The system on which their model is based is the following

$$
(0.2) \quad \begin{cases} u_t(((a_1 + b_{11} u + b_{12} v) u)_x + c_1 U_x u)_x, & t \geq 0, \ \ x \in [0, 1], \\ v_t(((a_2 + b_{21} u + b_{22} v) v)_x + c_2 U_x v)_x, & t \geq 0, \ \ x \in [0, 1], \\ u(0, x) = u_0(x), \qquad v(0, x) = v_0(x), & x \in [0, 1], \end{cases}
$$

where

u, v are population densities,
U is the «favorableness» of the habitat.

Many authors analyzed systems of this kind (see [3], [19]-[22] for a rich bibliography on this argument). For instance, we recall Bertsch-

Gurtin-Hilhorst-Peletier ([4]-[7]) who studied the degenerate Cauchy problem

$$
(0.3)\qquad \begin{cases} u_t = \left(u\left(u + \dfrac{1}{2}v\right)_x\right)_x, & t \geq 0, \quad x \in [0, 1], \\ v_t = \left(v\left(\dfrac{1}{2}u + v\right)_x\right)_x, & t \geq 0, \quad x \in [0, 1], \\ u(0, x) = u_0(x), \qquad v(0, x) = v_0(x), & x \in [0, 1]. \end{cases}
$$

They studied the case of segregate initial data (i.e. $\{\operatorname{supp} u_0\} \cap \cap \{\operatorname{supp} v_0\} = \phi$) reducing the system to a single equation by the introduction of a new auxiliary function.

III-rd Example: The infective disease model.

Capasso-Paveri Fontana ([11]) proposed the following model for a cholera epidemic:

$$
(0.4)\qquad \begin{cases} u_t = \Delta\phi_1(u, v) + f_1(u, v), & t \geq 0, \quad x \in [0, 1], \\ v_t = \Delta\phi_2(u, v) + f_2(u, v), & t \geq 0, \quad x \in [0, 1], \\ u(0, x) = u_0(x), \qquad v(0, x) = v_0(x), & x \in [0, 1], \end{cases}
$$

where ϕ_i $(i = 1, 2)$ are positive functions in $\mathbb{R}^+ \times \mathbb{R}^+$, and $\phi_i(0, 0) = 0$ for $i = 1, 2$. (For the biological meaning of u, v and for more details on ϕ_i, f_i we refer the reader to [11], [13]).

Dal Passo-De Mottoni ([13]) studied this problem under the assumptions that ϕ_i and f_i are regular enough and ϕ_1 and ϕ_2 satisfy a concavity condition.

In order to prove global existence and uniqueness they apply a linearization method based on the semigroup approach.

Moreover they have to assume a supplementary condition:

$$
(0.5)\qquad \det \frac{\partial(\phi_1, \phi_2)}{\partial(u, v)} \neq 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^+,
$$

i.e. they have t_0 assume that the linearized system is non-degenerate.

The aim of this paper is to study a wide class of degenerate nonlinear Cauchy problems (all the examples above belong to this class) following a unitary approach, i.e. the semigroup approach.

More precisely we obtain local existence, uniqueness and regularity results for the solution $u: [0, T] \times \Omega \to \mathbb{R}^N$ of the nonlinear system

$$(0.6)\begin{cases}(u_h(t, x))_t = \phi_h(x, u(t, x))\,\Delta(\theta_h(x, u(t, x))) + \chi_h(x, u(t, x)\,\nabla(u(t, x))),\\ u_h(0, x) = u_{0h}(x), \qquad t \in [0,], \ \ x \in \Omega, \ \ h = 1, \ldots, N,\end{cases}$$

where Ω is a regular domain in $\mathbb{R}^n$; $\phi_h\colon \mathbb{R}^{n+N} \to \mathbb{R}^+$, $\theta_h\colon \mathbb{R}^{n+N} \to \mathbb{R}^+$, $\chi_h\colon \mathbb{R}^{n+N+nN} \to \mathbb{R}$ are smooth functions; u_{0h} are nonnegative and regular enough (exact assumptions on ϕ_h, θ_h, χ_h, u_{0h} and Ω are stated in the last section).

We use here the same linearization procedure and semigroup approach followed in [25] (where a single equation was considered).

Before stating the scheme we follow, let us make some remarks. The unitary approach allows us to study a wide class of degenerate systems; on the other hand, due to its generality, it cannot give sharp estimates about the solutions of special degenerate systems (this can be obtained only by using specific techniques).

The case of degenerate systems such that every equation has the same order of degeneration is an easy consequence of the case of a single degenerate equation (case studied in [25]). However, as shown by the previous examples, this condition is too much restrictive in the applications. Hence, here, we analyze systems where every equation is allowed to have a different order of degeneration.

The scheme of this paper is the following.

Section 1 is devoted to notation, definitions and preliminary results. We state also L^p and C^α generation results for systems where the equations have the same order of degeneration. Since these results are very technical, we prove them in the appendix for the reader's convenience.

In order to study the linearization of system (0.6), in Sections 2 and 3 we prove generation of an analytic semigroup in $[L^p(\Omega)]^N$, $1 < p < \infty$ (Section 2) and in $[C_0^\alpha(\overline{\Omega})]^N$, $0 < \alpha < 1$ (Section 3) by elliptic systems.

More precisely, we prove some estimates (that imply the generation result) for the solution of the system

$$(0.7) \qquad (\lambda - E)u = f,$$

where

λ is a complex number, $u = (u_1, \ldots, u_N)$, $f = (f_1, \ldots, f_N)$,
$Eu = (E_1 u, \ldots, E_N u)$,

(0.8) $$E_h u = \sum_{i,j=1}^{n} \sum_{k=1}^{N} (d(x))^{2a(h,k)} A_{ij}^{hk}(x) D_{ij} u_k(x) +$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{N} (d(x))^{a(h,k)} B_i^{hk}(x) D_i u_k(x) + \sum_{k=1}^{N} C^{hk}(x) u_k(x), \qquad h = 1, \dots, N$$

(0.9) $d(x)$ belongs to $C^\infty(\Omega) \cap C^2(\overline{\Omega})$ and is such that $d(x) = 0$ and $|\nabla d(x)| \neq 0$ for each $x \in \partial\Omega$; $d(x) > 0$ for each $x \in \Omega$,

$a(h,k)$ are nonnegative numbers for each $h, k = 1, \dots, N$. Moreover

(0.10) there is a nondecreasing function $b: \{1, \dots, N\} \to \{0\} \cup [1, \infty[$ such that $a(h, k) = b(\max(h, k),)$

(0.11) $$A_{ij}^{hk}, \quad B_i^{hk}, \quad C^{hk} \in C(\overline{\Omega}),$$

the coefficients A_{ij}^{hk} satisfy an ellipticity condition:

(0.12) $$\begin{cases} \text{there exists } \nu > 0 \text{ such that for each } x \\ \displaystyle\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} A_{ij}^{hk}(x)(d(x))^{2a(h,k)} \xi_h^i \xi_k^j \geq \nu \sum_{s=1}^{N} (d(x))^{2b(s)} |\xi_s|^2 \\ \text{for each choice of } \xi_1, \dots, \xi_N \in \mathbb{R}^n. \end{cases}$$

Let us make some comments on these assumptions.

The function $d(x)$ has the same behaviour, near the boundary, of the function distance between x and the boundary of Ω. We cannot choose $d(x) =$ distance because it is not a regular function in $\overline{\Omega}$.

The function b allows us to analyze degenerate systems where every equation has a different order of degeneration. For instance $B(i) = 0$ means that the i-th equation of system (0.7) is nondegenerate. The case $b(1) = \dots = b(N)$ means that every equation has the same order of degeneration.

We assume Dirichlet boundary conditions on u_i if $b(i) = 0$, no boundary conditions otherwise. We refer the reader to the introduction of [25] for an interpretation of the lackness of standard boundary conditions and for a detailed bibliography on the subject.

In Section 4 we study nonlinear problem (0.6) using the statements of Section 3. Lastly we apply these results to analyze the examples stated above.

Before concluding this section we need to make this remark. Let X be a Banach space and let A: $D(A) \subseteq X \to X$ be a linear operator. Throughout this paper, we say that A generates an analytic semigroup in X if there exists $(1/2)\pi < \pi_0 < \pi$ and $M, R > 0$ such that if $\lambda \in \mathbb{C}$ with $|\arg \lambda| < \pi_0$ and $|\lambda| > R$, then $(\lambda - A)^{-1} \in L(X)$ and $\|(\lambda - A)^{-1}\|_{L(X)} \leq$

$\leq M|\lambda|^{-1}$ ($L(X)$ is the space of the linear continuous operators from X into itself). Note that we do not require $\overline{D(A)} = X$ because also in the case of not dense domain it is possible to apply the semigroup theory to study abstract Cauchy problems ([24]). Lastly we note that all this machinery can be utilized to study systems with complex valued coefficients. But, in our opinion, this analysis is not justified because it needs further technical complications and, in our knowledge, all the applications handle with real valued coefficients.

1. Notation and preliminary results.

Notation.

In this paragraph we introduce weighted Banach spaces related to the function $d(x)$ because in the sequel we deal with suitable function spaces depending on the degeneration of the single equations.

Let $1 < p \leq \infty$ be a real number and $m \geq 0$ be an integer. We denote by $H^{m,p}(\Omega)$ and $H_0^{m,p}(\Omega)$ the well known Sobolev spaces endowed with their usual norm $\|\cdot\|_{m,p,\Omega}$.

Let ψ be a positive function in Ω. We say that u belongs to $H^{1,p}(\Omega, \psi)$ (to $H^{2,p}(\Omega, \psi)$, resp.) if

$$\|u\|_{1,p,\psi} \doteq \|u\|_{0,p,\psi} + \sum_{i=1}^{n} \|\psi D_i u\|_{0,p,\Omega} < +\infty ,$$

$$\left(\|u\|_{2,p,\psi} \doteq \|u\|_{1,p,\psi} + \sum_{i,j=1}^{n} \|\psi^2 D_{ij} u\|_{0,p,\Omega} < +\infty, \text{ resp.} \right).$$

If d and b are the functions defined in (0.9)-(0.10), $u \in [H^{1,p}(\Omega, d(b))]^N$ ($u \in [H^{2,p}(\Omega, d(b))]^N$, resp.) means that $u = (u_1, \ldots, u_N)$ and $u_i \in H^{1,p}(\Omega, d^{b(i)})$ if $b(i) \geq 1$, $u_i \in H_0^{1,p}(\Omega)$ if $b(i) = 0$ ($u_i \in H^{2,p}(\Omega, d^{b(i)})$ if $b(i) \geq 1$, $u_i \in H^{2,p} \cap H_0^{1,p}(\Omega)$ if $b(i) = 0$, resp.).

If $0 < \gamma < 1$, $C_0^{m,\gamma}(\overline{\Omega})$ is the space of all the functions in $C^{m,\gamma}(\overline{\Omega})$ whose derivatives up to order m vanish on the boundary of Ω. We define $C^{m,\gamma}(\Omega, \psi)$ ($m \geq 2$) to be the space of those functions such that

$$\|u\|_{m,\gamma,\psi} \doteq \|u\|_{m-2,\gamma,\overline{\Omega}} + \sum_{\alpha:\ |\alpha| = m-1} \|\psi D^\alpha u\|_{0,\gamma,\overline{\Omega}} + \\ + \sum_{\alpha:\ |\alpha| = m} \|\psi^2 D^\alpha u\|_{0,\gamma,\Omega} + \infty .$$

where $\|\cdot\|_{m,\gamma,\overline{\Omega}}$ is the usual $C^{m,\gamma}$-norm.

Moreover $u \in [C^{m,\gamma}(\Omega, d(b))]^N$ if and only if $u = (u_1, \ldots, u_N)$ and $u_i \in C^{m,\gamma}(\Omega, d^{b(i)})$ for each $i = 1, \ldots, N$.

$u \in h^{m,\gamma}(\overline{\Omega})$ means that $u \in C^{m,\gamma}(\overline{\Omega})$ and for each α: $|\alpha| = m$

$$\lim_{t=0} \left\{ \sup_{\substack{x,\, y \in \Omega \\ |x-y| = t}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} = 0 \right\}.$$

We recall that the space $h^{m,\gamma}$ is called the space of the little Hölder continuous functions and it is the closure of C^∞ in $C^{m,\gamma}$-norm.

Preliminary results.

We state two results that will be useful in the sequel. In the first one we give some generation results that will be proved in the appendix. In the second one we recall a result of local existence and uniqueness of solutions to a class of abstract nonlinear Cauchy problems.

Consider the elliptic operator $\overline{E} = (\overline{E}_1, \dots, \overline{E}_N)$

$$\overline{E}_h = \sum_{i,j=1}^{n} \sum_{k=1}^{N} d^{2b} a_{ij}^{hk} D_{ij} u_k +$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{N} d^b b_i^{hk} D_i u_k + \sum_{k=1}^{N} c^{hk} u_k, \qquad h = 1, \dots, N$$

and assume that

(1.1) $$a_{ij}^{hk}, \quad b_i^{hk}, \quad c^{hk} \in C(\overline{\Omega})$$

(1.2) $$\begin{cases} \text{there exists } \nu > 0 \text{ such that for each } x \in \Omega \text{ and for each} \\ \zeta_1, \dots, \zeta_N \in \mathbb{R}^n \\ \displaystyle\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk} \zeta_h^i \zeta_k^j \geq \nu \sum_{s=1}^{n} |\zeta_s|^2, \end{cases}$$

(1.3) either $b = 0$ or $b \geq 1$,

(1.4) Ω is a bounded domain with C^2 boundary,

(1.5) d belongs to $C^2(\overline{\Omega}) \cap C^\infty(\Omega)$ and satisfies (0.9),

$\overline{E}$ generates an analytic semigroup in suitable Banach spaces as the following theorem shows (we recall that we prove it in the appendix).

THEOREM 1.1. Assume that (1.1)-(1.5) hold, then the operator

$$\begin{cases} E^{(p)}:\ D(E^{(p)}) = [(H^{2,p}(\Omega,\, d^b))]^N \to [L^p(\Omega)]^N, \\ E^{(p)}u = \overline{E}u, \qquad \text{for each } u \in D(E^{(p)}) \end{cases}$$

generates an analytic semigroup.

Moreover there exists $\gamma_1 > 0$ such that for each λ with $\operatorname{Re}\lambda > \gamma_1$ and for each $f \in [L^p(\Omega)]^N$, there exists a unique solution $u \in [H^{2,p}(\Omega,\, d(b))]^N$ of the system

$$(\lambda - \overline{E})u = f \tag{1.6}$$

and

$$\|u\|_{2,p,d(b)} + (|\lambda| - \gamma_1)^{1/2}\|u\|_{1,p,d(b)} + (|\lambda| - \gamma_1)\|u\|_{0,p,\Omega} \leq c_1\|f\|_{0,p,\Omega}, \tag{1.7}$$

where c_1 is independent of λ, f and u.

Furthermore if a_{ij}^{hk}, b_i^{hk}, c^{hk} are α-Hölder continuous; Ω is a bounded domain with $C^{2,\alpha}$ boundary; and d belongs to $C^{2,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$, then the operator

$$\begin{cases} E^{(\alpha)}:\ D(E^{(\alpha)}) = \{u \in [C^{2,\alpha}(\Omega,\, d(b)) \cap C_0^{0,\alpha}(\overline{\Omega})]^N\} \to [C_0^{0,\alpha}(\overline{\Omega})]^N, \\ E^{(\alpha)}u = \overline{E}u, \qquad \text{for each } u \in D(E^{(\alpha)}) \end{cases}$$

generates an analytic semigroup.

Moreover there exists $\gamma_2 > \gamma_1$ such that for each λ with $\operatorname{Re} > \gamma_2$ and for each $f \in [C_0^{0,\alpha}(\overline{\Omega})]^N$, the solution u of system (1.6) belongs to $[C^{2,\alpha}(\Omega,\, d(b))]^N$, and

$$\begin{aligned} &\|u\|_{2,\alpha,d(b)} + (|\lambda| - \gamma_2)^{1/2}\sum_{i=1}^{n}\|d^b D_i u\|_{0,\alpha,\overline{\Omega}} + \\ &\qquad + (|\lambda| - \gamma_2)\|u\|_{0,\alpha,\overline{\Omega}} \leq c_2\|f\|_{0,\alpha,\overline{\Omega}}, \end{aligned} \tag{1.8}$$

where c_2 is independent of λ, f and u.

REMARK 2.2. The previous results can be generalized in this way. Assume that a_{ij}^{hk}, b_i^{hk}, $c^{hk} \in C^{r,\alpha}(\overline{\Omega})$, $b \geq 1$; Ω is an open bounded domain with $C^{r+2,\alpha}$ boundary; and $d \in C^{r+2,\alpha}(\overline{\Omega}) \cap C^{\infty}(\Omega)$, then the operator

$$\begin{cases} E^{r,\alpha}\colon D(E^{r,\alpha}) = \{u \in [C^{r+2,\alpha}(\Omega, d(b)) \cap C_0^{r,\alpha}(\overline{\Omega})]^N\} \to [C_0^{r,\alpha}(\overline{\Omega})]^N, \\ E^{r,\alpha}u = \overline{E}u \qquad \text{for each } u \in D(E^{r,\alpha}), \end{cases}$$

generates an analytic semigroup.

Moreover there exists a constant $\gamma_3 > \gamma_2$ such that for each λ with $\operatorname{Re}\lambda > \gamma_3$ and for each $f \in [C_0^{r,\alpha}(\overline{\Omega})]^N$, the solution u of (1.6) belongs to $[C^{r+2,\alpha}(\Omega, d(b)) \cap C_0^{r,\alpha}(\overline{\Omega})]^N$ and

$$(1.9) \qquad \|u\|_{r+2,\alpha,d(b)} + (|\lambda| - \gamma_3)^{1/2} \sum_{\beta:\ |\beta| = r+1} \|d^b D^\beta u\|_{0,\alpha,\overline{\Omega}} + \\ + (|\lambda| - \gamma_3)\|u\|_{r,\alpha,\overline{\Omega}} \leq c_3 \|f\|_{r,\alpha,\overline{\Omega}},$$

where c_3 is independent of λ, f and u.

Furthermore if $b \in \mathbb{N}$, it is possible to prove ([25]) that the operator

$$\begin{cases} E_*^{\alpha}\colon D(E_*^{\alpha}) = \{u \in [C^{r,\alpha}(\Omega, d(b))]^N \to [C^{r,\alpha}(\overline{\Omega})]^N, \\ E_*^{\alpha}u = \overline{E}u, \qquad \text{for each } u \in D(E_*^{\alpha}), \end{cases}$$

generates an analytic semigroup. Let us recall that this result is not true even in the case of a single equation, $b = 0$ and $r = 2$ ([26]).

The following results will be useful in the last section in order to prove local existence and uniqueness of solutions of problem (0.6).

THEOREM 1.3 ([20]). Consider the following Cauchy problem in a Banach space X

$$(1.10) \qquad \begin{cases} u_t = f(u(t)), & t \geq 0, \\ u(0) = u_0, \end{cases}$$

where f is a C^2 function from D to X, D is a continuously embedded subspace of X and u_0 belongs to D.

Assume that

(1.11) the linear operator $f_x(u_0)\colon D \to X$ generates an analitic semigroup,

$$(1.12) \qquad f(u_0) \in \overline{D},$$

then there exist an interval $[0,T]$ and a strict solution u of (1.10) in $[0,T]$ (i.e. $u \in C^1([0, T]; X) \cap C([0, T]; D)$. Moreover u is the unique solution of (1.10) belonging to $z^{\varepsilon}(0, T; D)$ for any $0 < \varepsilon < 1$), where $z^{\varepsilon}(0, T; D)$ is so defined

$$z^{\varepsilon}(0, T; D) = \{u \in C^0([0, T]; D) \cap C^{0,\varepsilon}([s, T]; D) \text{ for any } 0 < s < T \text{ and } \lim_{t \to 0} t^{\varepsilon}[u]_{C^{0,\varepsilon}((1/2)t,\, t;\, D)} = 0\}.$$

REMARK 1.4. In [20], Theorem 1.3 is proved under the supplementary assumption that

(1.13) there is a neighborhood U_0 of u_0 such that for each $u \in U_0$, the linear operator $f_x(u)$: $D \to X$ generates an analitic semigroup in X.

Now, as we assume that $f: D \to X$ is continuously differentiable, note that $f_x(u)$ belongs to $L(D,X)$ (the space of linear applications from D to X) and depends continuously on $u \in D$ in the operator norm of $L(D,X)$. Since the set $\{A \in L(D, X),\ A$ generates an analytic semigroup$\}$ is open in $L(D,X)$ with the norm topology, we get that (1.13) is a consequence of (1.11)-(1.12).

2. L^p estimates.

The aim of this section is to prove that the operator

$$\begin{cases} E_{(p)}:\ D(E_{(p)}) = \{u \in [H^{2,p}(\Omega, d(b))]^N\} \to [L^p(\overline{\Omega})]^N, \\ E_{(p)}u = Eu, \qquad \text{for each } u \in D(E_{(p)}) \end{cases}$$

(E is the operator defined in (0.8)) generates an analytic semigroup.

We achieve this goal proving a more general result

THEOREM 2.1. Let

(2.1) Ω be a bounded domain with C^2 boundary,

Under assumptions (0.9)-(0.12), there exists a positive constant ω_0 such that for each λ with $\operatorname{Re} \lambda > \omega_0$ and for each $f \in [L^p(\Omega)]^N$, there is a unique solution $u \in [H^{2,p}(\Omega, d(b))]^N$ of the system

$$(\lambda - E)u = f. \tag{2.2}$$

Moreover

$$(2.3)\qquad \|u\|_{2,\,p,\,d(b)} + (|\lambda| - \omega_0)^{1/2}\|u\|_{1,\,p,\,d(b)} + \\ + (|\lambda| - \omega_0)\|u\|_{0,\,p,\,\Omega} \leq k_0\|f\|_{0,\,p,\,\Omega}\,,$$

where k_0 is independent of λ, f and u.

We show this theorem by proving some intermediate lemmas. First, we assume that the coefficients A_{ij}^{hk}, B_i^{hk}, C^{hk} satisfy this differentiability condition:

$$(2.4)\qquad A_{ij}^{hk},\quad B_i^{hk},\quad C^{hk} \in C^1(\overline{\Omega})$$

and we will show

LEMMA 2.2. There are a constant k_1 (independent of the norm of the derivatives of the coefficients) and a constant ω_1 (depending possibly on the C^1 norm of the coefficients) such that for each λ with $\operatorname{Re}\lambda > \omega_1$ and for each $f \in [L^2(\Omega)]^N$ there is a unique variational solution of (2.2). Moreover

$$(2.5)\,(|\lambda| - \omega_1)^{1/2}\|u\|_{1,\,2,\,d(b)} + (|\lambda| - \omega_1)\|u\|_{0,\,2,\,\Omega} \leq k_1\|f\|_{0,\,2,\,\Omega}\,. \qquad \blacksquare$$

At this point we will prove an interior estimate. Let $p \geq 2$ and let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ be open domains with C^2 boundary. Consider a cutoff function $\theta \in C^\infty(\overline{\Omega})$ such that $\theta = 1$ in Ω' and $\theta = 0$ in $\Omega\setminus\Omega''$. Let $f \in [L^p(\Omega)]^N$ and consider the variational solution u of (2.2). It is not difficult to see that θu belongs to $[H_0^{1,2}(\Omega'')]^N$ and solves in Ω'' a nondegenerate variational system. By classical regularity results and by generation results for nondegenerate elliptic systems, we will be able to prove

LEMMA 2.3. For each $p \geq 2$ and for each $\Omega' \subset\subset \Omega$, there are a constant k_2 (independent of the C^1 norm of the coefficients) and $\omega_2 > 0$ (dependent on the C^1 norm of the coefficients) such that for each λ with $\operatorname{Re}\lambda > \omega_2$ and for each $f \in [L^p(\Omega)]^N$, the variational solution u of (2.2) belongs also to $[H^{2,p}(\Omega')]^N$ and satisfies

$$(2.6)\qquad \|u\|_{2,\,p,\,\Omega'} + (|\lambda| - \omega_2)^{1/2}\|u\|_{1,\,p,\,\Omega'} + \\ + (|\lambda| - \omega_2)\|u\|_{0,\,p,\,\Omega'} \leq k_2\|f\|_{0,\,p,\,\Omega}\,. \qquad \blacksquare$$

The next step is to show that the variational solution u belongs also to $[H^{2,p}(\Omega, d(b))]^N$. More precisely we will prove

LEMMA 2.4. For each $p \geqslant 2$, there are a constant k_3 (independent of the C^1 norm of the coefficients) and a constant ω_3 (depending, possibly, on the C^1 norm of the coefficients) such that for each λ with $\operatorname{Re} \lambda > > \omega_3$ and for each $f \in [L^p(\Omega)]^N$, the variational solution u of (2.2) belongs to $[H^{2,p}(\Omega, d(b))]^N$. Moreover

$$(2.7) \qquad \|u\|_{2, p, d(b)} + (|\lambda| - \omega_3)^{1/2} \|u\|_{1, p, d(b)} + \\ + (|\lambda| - \omega_3) \|u\|_{0, p, \Omega} \leqslant k_3 \|f\|_{0, p, \Omega} . \qquad \blacksquare$$

We will show (2.7) by applying a tricky argument of induction and contraction and by using the results of Lemma 2.3 and Theorem 1.1. A similar procedure will be also applied in the next section to prove $C^{0,\alpha}$ estimates.

By applying a duality technique and by repeating the arguments of Lemmas 2.3 and 2.4 (i.e. by proving interior estimates by means of a suitable localization and by proving boundary estimates by an induction and contraction procedure) it is possible to show that (2.7) holds for each $1 < p < \infty$. More precisely

LEMMA 2.5. For each $1 < p < \infty$ there are two constant k_4 (independent of the C^1 norm of the coefficients) and ω_4 (dependent on the C^1 norm of the coefficients) such that for each λ with $\operatorname{Re} \lambda > \omega_4$ and for each $f \in [L^p(\Omega)]^N$ there is a unique solution $[H^{2,p}(\Omega, d(b))]^N$ of (2.2). Moreover

$$(2.8) \qquad \|u\|_{2, p, d(b)} + (|\lambda| - \omega_4)^{1/2} \|u\|_{1, p, d(b)} + \\ + (|\lambda| - \omega_4) \|u\|_{0, p, \Omega} \leqslant k_4 \|f\|_{0, p, \Omega} .$$

For the moment we assume true (2.8) and we show how to drop assumption (2.4) by an approximation and contraction technique

PROOF OF THEOREM 2.1. Let $\widetilde{E} = (\widetilde{E}_1, \ldots, \widetilde{E}_N)$

$$(2.9) \qquad \widetilde{E}_h u = \sum_{i,j=1}^{n} \sum_{k=1}^{N} \widetilde{A}_{ij}^{hk} d^{2a(h,k)} D_{ij} u_k + \\ + \sum_{i=1}^{n} \sum_{k=1}^{N} \widetilde{B}_i^{hk} d^{a(h,k)} D_i u_k + \sum_{k=1}^{N} \widetilde{C}^{hk} u_k , \qquad h = 1, \ldots, N,$$

where $\widetilde{A}_{ij}^{hk}$, $\widetilde{B}_i^{hk}$, $\widetilde{C}^{hk}$ are $C^1(\overline{\Omega})$ functions satysfying the ellipticity condition (0.12) and such that

$$(2.10)\qquad \sum_{i,j=1}^{n}\sum_{h,k=1}^{N}\|\widetilde{A}_{ij}^{hk}-A_{ij}^{hk}\|_{0,\infty,\Omega}+\sum_{i=1}^{n}\sum_{h,k=1}^{N}\|\widetilde{B}_{i}^{hk}-B_{i}^{hk}\|_{0,\infty,\Omega}+$$
$$+\sum_{h,k=1}^{N}\|\widetilde{C}^{hk}-C^{hk}\|_{0,\infty,\Omega}\leqslant(2k_4)^{-1}$$

(k_4 is the constant introduced in (2.8)).

Consider the application

$$T_\lambda\colon\ [H^{2,p}(\Omega, d(b))]^N\to[H^{2,p}(\Omega, d(b))]^N,$$
$$T_\lambda w=u,$$

where u is the solution of the system

$$(\lambda-\widetilde{E})u=f+(E-\widetilde{E})w.$$

Note that, by (2.10)

$$(2.11)\qquad \|(E-\widetilde{E})w\|_{0,p,\Omega}\leqslant(2k_4)^{-1}\|w\|_{2,p,d(b)}.$$

From (2.8) we have that there is ω_4 such that for each λ with $\operatorname{Re}\lambda>\omega_4$

$$(2.12)\qquad \|u\|_{2,p,d(b)}+(|\lambda|-\omega_4)^{1/2}\|u\|_{1,p,d(b)}+$$
$$+(|\lambda|-\omega_4)\|u\|_{0,p,\Omega}\leqslant k_4(\|f\|_{0,p,\Omega}+\|(E-\widetilde{E})w\|_{0,p,\Omega}).$$

Therefore, by (2.11), T_λ is a contraction for each λ with $\operatorname{Re}\lambda>\omega_4$. Hence T_λ has a unique fixed point which is the solution of (2.2). Moreover, from (2.12), we get (2.3) with $k_0=2k_4$ and $\omega_0=\omega_4$. (For more details about this contraction and approximation technique, we refer the reader to Theorem 3.1 of [9] where a similar argument is applied). ■

Now, let us prove the technical lemmas was stated above. Lemma 2.2 can be proved by writing system (2.2) in variational form (and this is possible by (2.4)) and by applying variational estimates

PROOF OF LEMMA 2.2. By the regularity of the coefficients, system (2.2) admits the variational formulation

$$(2.13)\qquad \lambda(u,\phi)_{0,\Omega}+A(u,\phi)_\Omega+B(u,\phi)_\Omega=(f,\phi)_{0,\Omega},$$

for each $\phi \in [H^{1,2}(\Omega, d(b))]^N$, where

$$(2.14) \qquad (u, \phi)_{0,\Omega} = \int_\Omega \sum_{h=1}^{N} u_h \bar{\phi}_h \, dx,$$

$$(2.15) \qquad A(u, \phi)_\Omega = \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} d^{2a(h,k)} A_{ij}^{hk} D_i u_k D_j \bar{\phi}_h \, dx,$$

$$(2.16) \qquad B(u, \phi)_\Omega = \int_\Omega \left(\sum_{i=1}^{n} \sum_{h,k=1}^{N} \left(d^{a(h,k)} B_i^{hk} - \sum_{j=1}^{n} (d^{2a(h,k)} D_j (A_{ij}^{hk}) + \right.\right.$$
$$\left.\left. + 2a(h,k) d^{2a(h,k)-1} D_j(d) A_{ij}^{hk}) \right) D_i(u_k) + \sum_{h,k=1}^{N} C^{hk} u_k \right) \bar{\phi}_h \, dx.$$

Choose $\omega_5 > 0$ such that for each λ with $\operatorname{Re} \lambda > \omega_5$ the form

$$(u, \phi) \to A(u, \phi)_\Omega + B(u, \phi)_\Omega + \lambda(u, \phi)_{0,\Omega}$$

is coercive in $[H^{1,2}(\Omega, d(b))]^N \times [H^{1,2}(\Omega, d(b))]^N$. The, by Lax-Milgram theorem, there is a unique function u belonging to $[H^{1,2}(\Omega, d(b))]^N$ which solves (2.13). Setting $\phi = u$, we get

$$(2.17) \qquad |A(u, u)_\Omega + \lambda(u, u)_{0,\Omega}| \leq |B(u, u)_\Omega| + |(f, u)_{0,\Omega}|.$$

Therfore, from (0.21), (2.13)-(2.17), we have that for each $\varepsilon > 0$

$$(2.18) \qquad \|u\|_{1,2,d(b)} + |\lambda| \|u\|_{0,2,\Omega} \leq \varepsilon \|u\|_{1,2,d(b)} + k_1(\varepsilon) \|u\|_{0,2,\Omega} + k_5 \|f\|_{0,2,\Omega},$$

where k_5 does not depend on ε, λ and on the C^1 norm of the coefficients, and $k_1(\varepsilon)$ does not depend on λ.

Therefore, choosing $\varepsilon = 1/2$ and $\omega_1 = \omega_5 + K_1(1/2)$, we obtain

$$(2.19) \qquad (|\lambda| - \omega_1) \|u\|_{0,2,\Omega} \leq 2k_5 \|f\|_{0,2,\Omega}.$$

Hence from (2.17) and (2.19), we deduce

$$(2.20) \qquad (|\lambda| - \omega_1)^{1/2} \|u\|_{1,2,d(b)} \leq k_6 \|f\|_{0,2,\Omega},$$

where $k_6 > 2k_5$ does not depend on λ and on the C^1 norm of the coefficients.

(2.5) comes from (2.19) and (2.20) with $k_1 = 2k_6$. ■

Now, we are able to prove the interior estimates

PROOF OF LEMMA 2.3. Assume, first, that $2 \leq p \leq 2^*$ $(2^* =$

$= (2n)/(n-2))$ if $n > 2$; $2 \leq p \leq \infty$ if $n \leq 2$ and note that $f \in [L^p(\Omega)]^N$ implies that $f \in [L^2(\Omega)]^N$. Let $\Omega' \subset\subset \Omega''' \subset\subset \Omega'' \subset\subset \Omega$ be open domains with boundary of class C^2. Let θ be a cutoff function such that $\theta = 1$ on Ω''', $\theta = 0$ on $\Omega \setminus \Omega''$. Let $v \in [H^{2,p} \cap H_0^{1,p}(\Omega'')]^N$ be the solution of the non-variational elliptic system

$$(2.21) \qquad (\lambda - E)u = g,$$

where $g = (g_1, \ldots, g_N)$ and

$$(2.22) \qquad g_h = \theta f_h + \sum_{i,j=1}^{n} \sum_{k=1}^{N} A_{ij}^{hk} d^{2a(h,k)} (D_{ij}(\theta) u_k +$$

$$+ D_i(u_k) D_j \theta + D_j(u_k) D_i \theta) + \sum_{i=1}^{n} \sum_{k=1}^{N} B_i^{hk} d^{a(h,k)} (D_i \theta) u_k, \quad h = 1, \ldots, N.$$

Note that, if λ is such that $\operatorname{Re} \lambda > \omega_1 + 1$, by (2.22), (2.5) and by Sobolev embedding theorem

$$(2.23) \qquad \|g\|_{0,p,\Omega} \leq \tilde{k}_7 (\|f\|_{0,p,\Omega} + \|f\|_{0,2,\Omega}) \leq k_7 \|f\|_{0,p,\Omega},$$

where $\tilde{k}_7$, k_7 are constants independent of λ, f, u and of the C^1 norm of the coefficients.

On the other hand, by a well known generation result (see [1]) there are two constants k_8 (independent of λ and of the C^1 norm of the coefficients) and $\omega_5 > 0$ such that for each λ with $\operatorname{Re} \lambda > \omega_5$, there is a unique solution $v \in [H^{2,p}(\Omega'') \cap H_0^{1,p}(\Omega'')]^N$ of (2.21). Moreover

$$(2.24) \quad \|v\|_{2,p,\Omega''} + (|\lambda| - \omega_5)^{1/2} \|v\|_{1,p,\Omega''} + (|\lambda| - \omega_5)\|v\|_{0,p,\Omega''} \leq k_8 \|g\|_{0,p,\Omega}.$$

This implies $v = \theta u$. Actually both the functions solve the variational formulation of (2.21), and, as shown in Lemma 2.2, there is a unique solution of (2.21).

From (2.23) and (2.24) we get

$$(2.25) \quad \|u\|_{2,p,\Omega'''} + (|\lambda| - \omega_6)^{1/2} \|u\|_{1,p,\Omega'''} +$$

$$+ (|\lambda| - \omega_6)\|u\|_{0,p,\Omega'''} \leq k_9 \|f\|_{0,p,\Omega},$$

where $\omega_6 = \omega_5 + \omega_1 + 1$ and $k_9 = k_7 k_8$.

If $n \leq 2$ the proof is finished.

If $n > 2$, assume $2 \leq p \leq 2^{**}$ $(2^{**} = (2^* n)/(n - 2^*))$ if $n > 4$; and $2 \leq p < \infty$ if $n \leq 4$.

Let $\Omega' \subset\subset \Omega'''' \subset\subset \Omega'''$ be an open domain with C^2 boundary and let θ_1 be a cutoff function such that $\theta_1 = 1$ on Ω'''' and $\theta_1 = 0$ on $\Omega \setminus \Omega'''$.

Let $v \in [H^{2,p} \cap H_0^{1,p}(\Omega''')]^N$ be the solution of

$$(\lambda - E)v = g_1,$$

where g_1 is obtained by g substituting θ with θ_1.

Now, repeating the previous argument and using (2.25) (calculated for $p = 2^*$) instead of (2.5) we have that (2.25) holds for each $2 \leq p \leq 2^{**}$ id $n > 4$ and for each $2 \leq p < \infty$ if $n \leq 4$.

If $n \leq 4$ the proof is finished.

If $n > 4$, we iterate this procedure and after a finite number of steps we prove the statement. ■

Let us show, that, for each $p > 2$, the solution of (2.2) belongs to $[H^{2,p}(\Omega, d(b))]^N$.

Proof of Lemma 2.4. We show this statement by induction on N (it holds for $N = 1$ thanks to Theorem 1.1).

By the induction hypothesis, we assume (2.7) true for $N = m$ (m natural number) and we prove that (2.7) holfs for $N = m + 1$. Let us introduce some notation

$$\begin{cases} \text{Let } i_0 \text{ be the natural number such that} \\ b(i_0) \neq b(i_0 + 1), \qquad b(1) = b(i_0). \end{cases} \tag{2.26}$$

(Note that if $i_0 = m + 1$ our statement holds by Theorem 1.1 because it implies that the equations have the same order of degeneration).

For each $\varepsilon > 0$, let $d^\varepsilon(x)$ be a $C^\infty(\Omega) \cap C^2(\overline{\Omega})$ function such that

$$\begin{aligned} &0 \leq d^\varepsilon(x) \leq 2\varepsilon, &&\text{for each } x \in \overline{\Omega}, \\ &d^\varepsilon(x) = d(x), &&\text{if } d(x) \leq \varepsilon, \\ &d^\varepsilon(x) > \varepsilon, &&\text{if } d(x) > \varepsilon. \end{aligned}$$

Let $\Omega_\varepsilon = \{x \in \Omega \colon d(x) > \varepsilon\}$.

Now, we prove (2.7) using a fixed point argument.

Consider the endomorphism T_λ^ε on the space $\prod_{j = i_0 + 1}^{m+1} H^{2,p}(\Omega, d^{b(j)})$ $T_\lambda^\varepsilon v = w$ (T_λ^ε is defined below).

Let $v \in \prod_{j = i_0 + 1}^{m+1} H^{2,p}(\Omega, d^{b(j)})$ and consider the solution z of the system

$$(\lambda - A^1)z = g^1(v), \tag{2.27}$$

where $A^1 = (A_1^1, \ldots, A_{i_0}^1)$, $g^1 = (g_1^1, \ldots, g_{i_0}^1)$

$$(2.28)\quad A_h^1 z = \sum_{k=1}^{i_0} \sum_{i,j=1}^{n} d^{2a(h,k)} A_{ij}^{hk} D_{ij} z_k + \sum_{k=1}^{i_0} \sum_{i=1}^{n} d^{a(h,k)} B_i^{hk} D_i z_k + \sum_{k=1}^{i_0} C^{hk} z_k, \qquad h = 1, \ldots, i_0,$$

$$(2.29)\quad g_h^1(v) = f_h + \sum_{k=i_0+1}^{m+1} \sum_{i,j=1}^{n} d^{2a(h,k)} A_{ij}^{hk} D_{ij} v_k + \sum_{k=i_0+1}^{m+1} \sum_{i=1}^{n} d^{a(h,k)} B_i^{hk} D_i v_k + \sum_{k=i_0+1}^{m+1} C^{hk} v_k, \qquad h = 1, \ldots, i_0.$$

Note that z belongs to $[H^{2,p}(\Omega, d^{b(1)})]^{i_0}$ by Theorem 1.1. Before finishing to define the application T_λ^ε, let us make a remark. If u (the variational solution of system (2.2)) belongs to $[H^{2,p}(\Omega, d(b))]^N$ and v is equal to u^2 (where $u^2 = (u_{i_0+1}, \ldots, u_{m+1})$), then $(z_1, \ldots, z_{i_0}) = (u_1, \ldots, u_{i_0}) = u^1$.

Actually both z than u^1 solve system (2.27) that, by Theorem 1.1, admits of a unique solution.

Let us now define w (the image of v by T_λ^ε): w is the solution of the system

$$(2.30)\qquad (\lambda - A^2) w = g^2(z),$$

where $A^2 = (A_{i_0+1}^2, \ldots, A_{m+1}^2)$, $g^2 = (g_{i_0+1}^2, \ldots, g_{m+1}^2)$

$$(2.31)\quad A_h^2 w = \sum_{k=i_0+1}^{m+1} \sum_{i,j=1}^{n} d^{2a(h,k)} A_{ij}^{hk} D_{ij} w_k + \sum_{k=i_0+1}^{m+1} \sum_{i=1}^{n} d^{a(h,k)} B_i^{hk} D_i w_k + \sum_{k=i_0+1}^{m+1} C^{hk} w_k, \qquad h = i_0 + 1, \ldots, m+1,$$

$$(2.32)\quad g_h^2(z) = f_h + \sum_{k=1}^{i_0} \sum_{i,j=1}^{n} ((d)^{2a(h,k)} - (d^\varepsilon)^{2a(h,k)}) A_{ij}^{hk} D_{ij} u_k + \sum_{k=1}^{i_0} \sum_{i,j=1}^{n} (d^\varepsilon)^{2a(h,k)} A_{ij}^{hk} D_{ij} u_k + \sum_{k=1}^{i_0} \sum_{i=1}^{n} d^{2a(h,k)} B_i^{hk} D_i z_k + \sum_{k=1}^{i_0} C^{hk} z_k, \qquad h = i_0 + 1, \ldots, m+1,$$

where u is the solution of (2.2).

Note that $g^2(z)$ belongs to $[L^p(\Omega)]^{m+1-i_0}$. Therefore T_λ^ε is well defined because $w \in \prod_{j=i_0+1}^{m+1} [H^{2,p}(\Omega, d^{b(j)})]$ by the induction assumption.

Before proving that T_λ^ε is a contraction for ε small and λ large enough, let us make a remark.

Note that if T_λ^ε is a contraction then its fixed point is equal to u^2. Actually let $(v_{i_0+1}, \ldots, v_{m+1})$ be the fixed point of T_λ^ε and let $U \in [H^{2,p}(\Omega, d(b))]^N$ be equal to $(z_1, \ldots, z_{i_0}, v_{i_0+1}, \ldots, v_{m+1})$. Then U solves the system

$$(\lambda - A)\, U = F, \tag{2.33}$$

where $A = (A_1, \ldots, A_{m+1})$, $F = F(F_1, \ldots, F_{m+1})$

$$A_h + \sum_{k=1}^{m+1} \sum_{i,j=1}^{n} d^{2a(h,k)} A_{ij}^{hk} D_{ij} + \\ + \sum_{k=1}^{m+1} \sum_{i=1}^{n} d^{a(h,k)} B_i^{hk} D_i \sum_{k=1}^{m+1} C^{hk}, \qquad h = 1, \ldots, i_0,$$

$$A_h = \sum_{k=1}^{i_0} \sum_{i,j=1}^{n} (d)^{2a(h,k)} A_{ij}^{hk} D_{ij} + \sum_{k=i_0+1}^{m+1} \sum_{i,j=1}^{n} d^{2a(h,k)} A_{ij}^{hk} D_{ij} + \\ + \sum_{k=1}^{m+1} \sum_{i=1}^{n} d^{a(h,k)} B_i^{hk} D_i + \sum_{k=1}^{m+1} C^{hk}, \qquad \text{if } h = i_0 + 1, \ldots, m+1,$$

$$F_h = f_h, \qquad \text{if } h = 1, \ldots, i_0,$$

$$F_h = f_h + \sum_{k=1}^{i_0} \sum_{i,j=1}^{n} ((d)^{2a(h,k)} - (d^\varepsilon)^{2a(h,k)}) A_{ij}^{hk} D_{ij} u_k, \\ \text{if } h = i_0 + 1, \ldots, m+1.$$

Note that also u is a solution (in variational sense) of (2.33). Therefore $U = u$ by the uniqueness of the variational solutions proved in Lemma 2.2.

Let us show that T_λ^ε is a contraction for ε small and λ large enough.

Let $v', v'' \in \prod_{j=i_0+1}^{m+1} H^{2,p}(\Omega, d^{b(j)})$ and let w', w'' be such that $T_\lambda^\varepsilon v' = w'$ and $T_\lambda^\varepsilon v'' = w''$.

Then $w' - w''$ solves the system

$$(\lambda - A^2)(w' - w'') = g^2(z') - g^2(z'') \tag{2.34}$$

and $z' - z''$ is the solution of

$$(\lambda - A^1)(z' - z'') = g^1(v') - g^2(v''). \tag{2.35}$$

By Theorem 1.1, we get

$$\|z' - z''\|_{2, p, d^{b(i_0)}} + (|\lambda| - \gamma_1)^{1/2}\|z' - z''\|_{1, p, d^{b(i_0)}} + \\ + (|\lambda| - \gamma_1)\|z' - z''\|_{0, p, \Omega} \leqslant c_1 \sum_{i = i_0 + 1}^{m+1} \|v_i' - v_i''\|_{2, p, d^{b(i)}}. \tag{2.36}$$

By the induction assumptions, there exists $\omega_7 > \gamma_1$ such that for each λ with $\operatorname{Re}\lambda > \omega_7$ there exists a unique solution of (2.34). Moreover

$$\sum_{i = i_0 + 1}^{m+1} \|w_i' - w_i''\|_{2, p, d^{b(i)}} + \\ +(|\lambda| - \omega_7)^{1/2} \sum_{i = i_0 + 1}^{m+1} \|w_i' - w_i''\|_{1, p, d^{b(i)}} + (|\lambda| - \omega_7)\|w' - w''\|_{0, p, \Omega} \leqslant \\ \leqslant k_{10}(\varepsilon^{(b(i_0 + 1) - b(i_0))})\|z' - z''\|_{2, p, d^{b(i_0)}} + \|z' - z''\|_{2, p, d} b(i_0)) \tag{2.37}$$

where k_{10} is independent of λ and of the C^1 norm of the coefficients. Let $\eta(i_0 + 1) - b(i_0)$.

Summing up, from (2.36) and (2.37) we obtain

$$\sum_{i = i_0 + 1}^{m+1} \|w_i' - w_i''\|_{2, p, d^{b(i)}} \leqslant \\ \leqslant c_1 k_{10}(\varepsilon^{\eta} + (|\lambda| - \omega_7)^{-1/2}) \sum_{i = i_0 + 1}^{m+1} \|v_i' - v_i''\|_{2, p, d^{b(i)}},$$

Hence T_λ^ε is a contraction for $\varepsilon = (4k_{10}c_1)^{-(1/\eta)}$ and for each λ with $\operatorname{Re}\lambda > \omega_7 + 16(k_{10}c_1)^2 + 1 = \omega_8$.

Now, we show (2.7).

Consider systems (2.27) and (2.30). By the induction assumption

$$\sum_{i = i_0 + 1}^{m+1} \|w_i\|_{2, p, d^{b(i)}} + (|\lambda| - \omega_8)^{1/2} \sum_{i = i_0 + 1}^{m+1} \|w_i\|_{1, p, d^{b(i)}} + \\ + (|\lambda| - \omega_8)\|w\|_{0, p, \Omega} \leqslant k_{11}(\|f\|_{0, p, \Omega} + \|u\|_{2, p, \Omega}\,\varepsilon) + \frac{1}{2} \sum_{i = i_0 + 1}^{m+1} \|v_i\|_{2, p, d^{b(i)}}, \tag{2.38}$$

where k_{11} is independent of λ, f, u and of the C^1 norm of the coefficients.

By Lemma 2.3 (u is the fixed point of T_λ^ε)

$$\sum_{i=i_0+1}^{m+1} \|u_i\|_{2,p,d^{b(i)}} + (|\lambda| - \omega_8)^{1/2} \sum_{i=i_0+1}^{m+1} \|u_i\|_{1,p,d^{b(i)}} + \\ + (|\lambda| - \omega_8) \sum_{i=i_0+1}^{m+1} \|u_i\|_{0,p,\Omega} \leq k_{12} \|f\|_{0,p,\Omega}, \tag{2.39}$$

where k_{12} does not depend on λ, u, f and on the C^1 norm of the coefficients.

As u^1 solves the system

$$(\lambda - A^1) u^1 = g^1(u^2),$$

from (1.7) and (2.39) we get

$$\|u^1\|_{2,p,d^{b(i_0)}} + (|\lambda| - \omega_8)^{1/2} \|u^1\|_{1,p,d^{b(i_0)}} + \\ + (|\lambda| - \omega_8) \|u^1\|_{0,p,\Omega} \leq k_{13} \|f\|_{0,p,\Omega}, \tag{2.40}$$

where k_{13} is independent of λ, f, u and of the C^1 norm of the coefficients.

Now (2.39) and (2.40) imply (2.7). ■

In order to complete the proof of Theorem 2.1, we show (2.8). As its proof is based on a duality technique and on argument very close to the ones of the previous lemmas, we only sketch it leaving the details to the readers.

PROOF OF LEMMA 2.5. Write system (2.2) in variational form. By (2.7) and by a standard duality technique (see, for instance [8]; see also Section 4 of [10]) there exist two constants k_{14} (independent of λ and of the C^1 norm of the coefficients) and ω_9 (independent only of λ) such that for each λ with $\operatorname{Re} \lambda > \omega_9$ and for each $f \in [L^p(\Omega)]^N$ ($1 < p \leq 2$), there exists a unique variational solution $u \in [H^{1,p}(\Omega, d(b))]^N$. Moreover

$$(|\lambda| - \omega_9)^{1/2} \|u\|_{1,p,d(b)} + (|\lambda| - \omega_9) \|u\|_{0,p,\Omega} \leq k_{14} \|f\|_{0,p,\Omega}. \tag{2.41}$$

Now, we prove the L^p estimates for the second derivatives of u in the interior of Ω.

Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ be open domains with C^2 boundary and let θ be the cutoff function introduced in Lemma 2.3.

θu solves a nondegenerate Dirichlet problem in Ω''. Therefore, arguing as in Lemma 2.3, there are two constants k_{15} (independent of λ and of the C^1 norm of the coefficients) and ω_{10} (independent of λ) such that for each λ with $\operatorname{Re}\lambda > \omega_{10}$ the variational solution u of (2.2) satisfies also

$$\|u\|_{2,p,\Omega'} \leq k_{15}\|f\|_{0,p,\Omega}. \tag{2.42}$$

Hence (2.8) follows by repeating the same induction and contraction argument of Lemma 2.4 and by using (2.41) and (2.42) instead of (2.5) and (2.6).

3. The $C^{0,\alpha}$ estimate.

Let E be the operator defined in (0.8) and assume that there is $0 < \alpha < 1$ such that

$$A_{ij}^{hk},\quad B_i^{h,k},\quad C^{hk} \in C^{0,\alpha}(\overline{\Omega}), \tag{3.1}$$

(3.2) Ω is an open bounded domain with $C^{2,\alpha}$ boundary,

$$d \in C^{2,\alpha}(\overline{\Omega}) \cap C^{\infty}(\Omega). \tag{3.3}$$

In this section we show that, under assumptions (0.12) and (3.1)-(3.3), the operator

$$E^{\alpha}\colon D(E^{\alpha}) = \{u \in [C^{2,\alpha}(\Omega, d(b)) \cap C_0^0(\overline{\Omega})]^N\colon Eu \in [C_0^0(\overline{\Omega})]^N\} \to$$

$$\to [C_0^{0,\alpha}(\overline{\Omega})]^N, \qquad E^{\alpha}u = Eu \text{ for each } u \in D(E^{\alpha})$$

generates an analytic semigroup.

Also in this case we focus our attention on the system

$$(\lambda - E)u = f \tag{3.4}$$

and we prove a more general result.

As the techniques involved are very similar to the ones applied in the previous section, we prefer to focus our attention only on what is really new (leaving to the reader the proofs analogous to the ones of section 2).

THEOREM 3.1. There exists $\omega_{11} > \omega_0$ such that for each λ with $\operatorname{Re}\lambda > \omega_{11}$ and for each $f \in [C_0^{0,\alpha}(\overline{\Omega})]^N$ there is a unique solution u of

(3.4) that belongs to $[C^{2,\alpha}(\Omega, d(b)) \cap C_0^0(\overline{\Omega})]^N$. Moreover

$$\|u\|_{2,\alpha,d(b)} + (|\lambda| - \omega_{11})^{1/2} \sum_{j=1}^{n} \sum_{i=1}^{N} \|d^{b(i)} D_j u_i\|_{0,\alpha,\overline{\Omega}} + \\ + (|\lambda| - \omega_{11})\|u\|_{0,\alpha,\overline{\Omega}} \leq k_{16}\|f\|_{0,\alpha,\overline{\Omega}}, \tag{3.5}$$

where k_{16} does not depend on λ, f and u.

Before proving this theorem, we state a preliminary result concerned with interior estimates

LEMMA 3.2. Let $\Omega' \subset\subset \Omega$ open domain with $C^{2,\alpha}$ boundary. Then, there is $\omega_{12} > \omega_0$ such that, for each λ with $\operatorname{Re}\lambda > \omega_{12}$ and for each $f \in [C_0^{0,\alpha}(\overline{\Omega})]^N$, the solution u of (3.4) satisfies

$$\|u\|_{2,\alpha,\overline{\Omega}'} + (|\lambda| - \omega_{12})^{1/2}\|u\|_{1,\alpha,\overline{\Omega}'} + \\ + (|\lambda| - \omega_{12})\|u\|_{0,\alpha,\overline{\Omega}'} \leq k_{17}\|f\|_{0,\alpha,\overline{\Omega}}, \tag{3.6}$$

where k_{17} is a constant independent of λ, f and u.

PROOF. This lemma can be proved using the same argument of Lemma 2.3 to which we refer for more details.

If $f \in [C_0^0(\overline{\Omega})]^N$, then $f \in [L^q(\Omega)]^N$ where $q = n/(1-\alpha)$.

By Theorem 2.1

$$\|u\|_{2,q,d(b)} \leq k_0\|f\|_{0,q,\Omega}. \tag{3.7}$$

On the other hand (using the same notation of Lemma 2.3) $\theta u \in [H^{2,q} \cap H_0^{1,q}(\Omega'')]^N$ and solves the system

$$(\lambda - E)(\theta u) = g. \tag{2.21}$$

By (3.7) and by Sobolev embedding theorem $g \in [C_0^{0,\alpha}(\overline{\Omega}'')]^N$ and

$$\|g\|_{0,\alpha,\overline{\Omega}''} \leq k_{18}\|f\|_{0,\alpha,\overline{\Omega}}, \tag{3.8}$$

where k_{18} is a constant independent of λ, f and u.

Hence (3.6) is a consequence of (3.8) and of the $C_0^{0,\alpha}$ generation result for nondegenerate elliptic system (see [9], for instance). ∎

PROOF OF THEOREM 3.1. As the proof is analogous to the one of Lemma 2.4 (to which we refer for more details) we only sketch it.

Also in this case we apply an induction-contraction argument. By Theorem 1.1 (3.5) holds for $N = 1$.

Assume that (3.5) holds for $N = m$ and prove it for $N = m + 1$. Consider the application from

$$\sum_{j=i_0+1}^{m+1} C^{2,\alpha}(\Omega, d^{b(j)}) \cap C_0^{0,\alpha}(\overline{\Omega})$$

into itself $T_\lambda^\varepsilon v = w$ where w is the solution of system (2.30). Reasoning as in Lemma 2.4 and using (1.8) and (3.6) instead of (1.7) and (2.6), it is possible to show that T_λ^ε is a contraction for λ large enough and ε small enough.

Noting that u^2 is the fixed point of T_λ^ε, repeating the argument of Lemma 2.4 and using Theorem 1.1 and Lemma 3.2 (instead of Lemma 2.3), one can deduce (3.5). ■

REMARK 3.3. If u is the solution of (3.4) then, for each $\eta < \alpha$, for each $i,j = 1, \dots, n$, and for each h such that $b(h) \geq 1$

$$d^{2b(h)-\eta} D_{ij} u_h \in L^\infty(\Omega). \tag{3.9}$$

Since this result is very technical we sketch only the proof and we refer the reader to [25] where this statement is proved in detail in the case of a single equation.

Assume that, for each $i = 1, \dots, N$, $b(i) \geq 1$.

Then it is possible to show that $d^{-\eta} u$ solves a degenerate system satisfying (0.12) and (3.1)-(3.3) (substituing α with $\alpha - \eta$). Hence (3.9) comes by Theorem 3.1.

If $b(1) = \dots = b(i_0) = 0$ and $b(i_0 + 1) \geq 1$, then it is possible to prove (3.9) by focussing the attention on the last $N - i_0$ equation, noting that $d^{-\eta} u^2$ solves a degenerate system and repeating the previous argument. ■

In order to apply these results to nonlinear problems arising from biological models, it is necessary to extend these generation results to the spaces $[C_0^{r,\alpha}(\overline{\Omega})]^N$, with $r \geq 1$.

For the sake of simplicity, here, we consider the case $r = 1$.

Let us introduce a new function space:

$$\begin{aligned}
&u \in [\widetilde{C}^{1,\alpha}(\Omega, d(b))]^N, && \text{if } u = (u_1, \dots, u_N),\\
&u_i \in C^{1,\alpha} \cap C_0^0(\overline{\Omega}), && \text{if } b(i) = 0,\\
&u_i \in C_0^{1,\alpha}(\overline{\Omega}), && \text{if } b(i) \geq 1.
\end{aligned}$$

THEOREM 3.4. Assume that (0.12) holds and that

$$A_{ij}^{hk},\ B_i^{h,k},\ C^{hk} \in C^{1,\alpha}(\overline{\Omega}), \tag{3.10}$$

$$\Omega \text{ is a bounded doma with } C^{3,\alpha} \text{ boundary}, \tag{3.11}$$

$$d \in C^{3,\alpha}(\overline{\Omega}) \cap C^{\infty}(\Omega). \tag{3.12}$$

The then operator

$$E_{1,\alpha}\colon D(E_{1,\alpha}) =$$

$$= \{u \in [C^{3,\alpha}(\Omega, d(b))]^N \cap [\widetilde{C}^{1,\alpha}(\Omega, d(b))]^N \colon Eu \in [C_0^0(\overline{\Omega})]^N\} \to$$

$$\to [\widetilde{C}^{1,\alpha}(\Omega, d(b))]^N, \qquad E_{1,\alpha}u = Eu \text{ for each } u \in D(E_{1,\alpha})$$

generates an analytic semigroup.

PROOF. Also in this case we show a more general result.

We prove that there exists a constant $\omega_{13} > \omega_{11}$ such that for each λ with $\operatorname{Re}\lambda > \omega_{13}$ and for each $f \in [\widetilde{C}_0^{1,\alpha}(\Omega, d(b))]^N$ there is a unique $u \in \in D(E_{1,\alpha})$ that solves (3.4). Moreover

$$\|u\|_{3,\alpha,d(b)} + (|\lambda| - \omega_{13})\|u\|_{1,\alpha,\overline{\Omega}} \leq k_{19}\|f\|_{1,\alpha,\overline{\Omega}}, \tag{3.13}$$

where k_{19} does not depend on λ, f and u.

This theorem can be proved following the scheme of Theorems 2.1 and 3.1 (to which we refer for more details).

Also in this case the first step consists in proving interior estimates. Let θ be the cutoff function introduced in Lemma 2.3. θu belongs to $[C^{2,\alpha}(\overline{\Omega}'')]^N$ (by Theorem 3.1) and solves the system

$$(\lambda - E)(\theta u) = g, \tag{2.21}$$

where $g \in [C_0^{1,\alpha}(\overline{\Omega}'')]^N$ (again by Theorem 3.1).

Hence, by well known results for nondegenerate elliptic systems, there is $\omega_{14} > \omega_{11}$ such that for each λ with $\operatorname{Re}\lambda > \omega_{14}$ the solution u of (3.4) belongs to $[C^{3,\alpha}(\overline{\Omega}')]^N$ and satisfies

$$\|u\|_{3,\alpha,\overline{\Omega}'} \leq k_{20}\|f\|_{1,\alpha,\overline{\Omega}}, \tag{3.14}$$

where k_{20} is a constant independent of λ, f and u.

The second step consists in applying an induction-contraction argument very close to the one of Lemma 2.4.

Note that (3.13) holds for $N = 1$ (If $b \geq 1$ by Remark 1.2, if $b = 0$ by classical generation results).

Assume that (3.13) holds for $N = m$ and prove it for $N = m + 1$. Consider the application T_λ^ε from

$$\prod_{j = i_0 + 1}^{m+1} C^{3,\alpha}(\Omega, d^{b(j)}) \cap C_0^{1,\alpha}(\overline{\Omega})$$

into itself $T_\lambda^\varepsilon v = w$ where w is the solution of system (2.30). Then (3.13) follows reasoning as in Lemma 2.4, using (1.9) and (3.14) instead of (1.7) and (2.6) and proving that T_λ^ε is a contraction for λ large enough and ε small enough.

Now we state other generation results. Since the proof is very close to the ones of Theorems 2.1 and 3.1 (i.e. one, first, obtains some interior estimates by a suitable localization, and, then, proves the statement using an induction-contraction argument) we omit it.

REMARK 3.5. Assume that $b(i) \in \mathbb{N}$ for each $i = 1, \dots, N$. Then the operator

$$E_\alpha^*: D(E^*) = u \in [C^{2,\alpha}(\Omega, d(b))]^N \to [C^{0,\alpha}\overline{\Omega})]^N,$$

$$E_\alpha^* u = Eu \text{ for each } u \in D(E^*)$$

generates an analytic semigroup.

Assume that $b(1) = b(2) = \dots = b(i_0) = 0$; $b(i) \in \mathbb{N}$ for each $i = i_0 + 1, \dots, N$; and that A_{ij}^{hk}, B_i^{hk}, C^{hk} vanish on $\partial\Omega$ for each $i, j = 1, \dots, n$; $h = 1, \dots, i_0$; $k = i_0 + 1. \dots, N$. Then the operator

$$\underline{E}_\alpha: D(\underline{E}_\alpha) = \left\{ u \in \prod_{i=1}^{i_0} (C^{2,\alpha} \cap C_0^0(\overline{\Omega})) \times \prod_{i = i_0 + 1}^{N} C^{2,\alpha}(\Omega, d^{b(i)}): \right.$$

$$\left. E_i u \in C_0^0(\overline{\Omega}) \text{ for each } i = 1, \dots, i_0 \right\} \to [[C_0^{0,\alpha}(\overline{\Omega})^{i_0} \times$$

$$\times [C^{0,\alpha}(\overline{\Omega})]^{N - i_0}] \text{ for each } \underline{E}_\alpha u \in D(\underline{E}_\alpha)$$

generates an analytic semigroup.

Before concluding this section, we note that the generation results stated for the operators E^α, E_α^*, $\underline{E}_\alpha$ can be extended (with the obvious modifications) to the operator $E_{r,\alpha}$, $E_{r,\alpha}^*$, $\underline{E}_{r,\alpha}$ (r positive integer) repeating the argument of Theorem 3.4.

4. Nonlinear Cauchy problems and applications.

In this section we show how the prove existence, uniqueness and local regularity for the solution u of the nonlinear Cauchy problem (0.6) by applying the results of the previous sections. Moreover we analyze the examples stated in the introduction.

Before studying (0.6), we remark that we consider here the Laplace operator Δ for the sake of simplicity, but one can replace Δ with any elliptic operator with smooth coefficients. Solving (0.6) by linearization, we shall consider the operator $E = (E_1, \ldots, E_N)$ where for each $h = 1, \ldots, N$

$$E_h u = \sum_{i,j=1}^{n} \sum_{k=1}^{N} A_{ij}^{hk}(u_0) D_{ij} u_k + \sum_{i=1}^{n} \sum_{k=1}^{N} B_i^{hk}(u_0) D_i u_k + \sum_{k=1}^{N} C^{hk}(u_0) u_k,$$

$$A_{ij}^{hk}(u_0) = \phi_h(x, u_0(x)) \frac{\partial}{\partial u_k} \theta_h(x, u_0(x)) \delta_{ij},$$

$$B_i^{hk}(u_0) = 2\phi_h(x, u_0(x) D_i) \left(\frac{\partial}{\partial u_k} \theta_h(x, u_0(x)) \right) + \frac{\partial}{\partial p_{k,i}} \chi_h(x, u_0(x), \nabla u_0(x)),$$

$$C^{hk}(u_0) = \phi_h(x, u_0(x)) \Delta \frac{\partial}{\partial u_k} \theta_h(x, u_0(x)) +$$

$$+ \frac{\partial}{\partial u_k} \phi_h(x, u_0(x)) \Delta \theta_h(x, u_0(x)) + \frac{\partial}{\partial u_k} \chi_h(x, u_0(x), \nabla u_0(x))$$

(E is the linear part of the second member of system (0.6) at $t = 0$).

At this point, in order to apply Theorem 1.3, one first fixes the function d and the spaces D and X, and then verifies that assumptions (1.11)-(1.12) are satisfied.

Here, as an example of applications of Theorems 3.1 and 1.3 to problem (0.6), we consider the particular case $\theta = id$ and $\chi = 0$ because, for the generality of system (0.6), to state all the assumptions on ϕ, θ and χ does not involve new mathematical ideas, but it is extremely more complicated technically.

At the end of this section, however, we apply Theorems 1.3, 3.1 and 3.4 to the examples stated in the introduction showing how this semigroup approach works also in other situations. (We refer also to section 7 of [25] where several different applications of this method are given.)

Fix $r \in \mathbb{N}$, $r > 2$. Let $0 < \alpha < 1$, $N = 2$ and b: $\{1,2\} \to \mathbb{R}$ such that $b(1) = (r + \alpha)/2$, $b(2) = r + \alpha$.

Here

(4.1) Ω is a bounded set with $C^{r+2,\alpha}$ boundary,

$$\begin{cases} d \in C^{\infty}(\Omega) \cap C^{r+2,\alpha}(\overline{\Omega}) \text{ and satisfies, (0.9)} \\ X = [C_0^{r,\alpha}(\overline{\Omega})]^2 \quad \text{and} \quad D = [C^{r+2,\alpha}(\Omega, d(b))]^2, \end{cases} \tag{4.2}$$

$$u_0 = (u_{01}, u_{02}) \in D, \tag{4.3}$$

$$(x, u) \to \phi(x, u) \in C^{r+3}(\mathbb{R}^{n+2}; \mathbb{R}), \tag{4.4}$$

$$\theta = id, \qquad \chi = 0. \tag{4.5}$$

Moreover, assume that there is a positive constant $k > 0$ such that

$$k \leqslant u_{0i}/d^{2b(i)} \leqslant k^{-1}, \tag{4.6}$$

$$\phi_i(x, u_0(x)) \geqslant kd^{2b(i)}(x). \tag{4.7}$$

Furthermore, assume that for each $v \in D$ there is a constant $k(v)$ such that

$$\phi_i(x, v(x)) \leqslant k(v)\, d^{2b(i)}(x) \tag{4.8}$$

(conditions (4.7)-(4.8) are satisfied in many cases. For instance if, for each $i,j = 1,2$,

$$\phi_i(x, u) = \phi_i(u), \qquad \phi_i(0) = 0, \qquad \frac{\partial}{\partial u_i}\phi_1(0) > 0,$$

$$\frac{\partial}{\partial u_i}\phi_2(0) = 0, \qquad \frac{\partial^2}{\partial u_i \partial u_j}\phi_2(0) > 0.$$

Actually these conditions imply that near the boundary

$$\phi_1(v(x)) \sim v_1(x) + v_2(x), \qquad \phi_2(v(x)) \sim v_1^2(x) + v_2^2(x).$$

Note that the last line implies (4.7)-(4.8) because $v \in X$ and therefore $|v|$ has the same behaviour of $d^{b(2)}$).

Moreover we ask that for each $i = 1,2$ and for each $s = 1,\dots,N$

$$\frac{\partial^2}{\partial x_s \partial u_i}\phi_2(x, u)\, d^{-b(2)}(x) \in C^{0,\alpha}(\overline{\Omega}), \tag{4.9}$$

$$\frac{\partial}{\partial u_i}\phi_2(x, u)\, d^{-b(2)}(x) \in C^{0,\alpha}(\overline{\Omega}). \tag{4.10}$$

(Note that (4.10) is satisfied, for instance, if $\phi_2(x, y)$ has a quadratic growth in u).

Lastly we assume that

$$\phi(x, u_0(x))\Delta u_0(x) \in [h_0^{r,\alpha}(\overline{\Omega})]^2. \tag{4.11}$$

Now, we can apply Theorem 3.1 to problem (0.6).

THEOREM 4.1. Under assumptions (4.1)-(4.11) there are t_0 and a solution $u \in C^1([0, t_0]; X) \cap C^0([0, t_0]; D)$ of problem (0.6) in $[0, t_0]$. Moreover u is the unique solution of (0.6) belonging to $z_\alpha(0, t_0; D)$.

PROOF. Denote by $f\colon D \to X$ the map $f(u) = \phi(x, u)\Delta u$, and by f_u and f_u^2 the first and the second Frechet derivatives of f.

To begin with, we show that the operator $f_u(u_0)$ generates an analytic semigroup in X.

We cannot apply directly Theorem 3.4 because $C^{hk}(u_0)$ does not belong to $C^{r,\alpha}(\overline{\Omega})$. We overcome this difficulty by a contraction argument.

Let g belong to X and let τ_λ be the application from D into itself, so defined $\tau_\lambda v = w$ where w is the solution of the system

$$(\lambda - \phi_i(x, u_0)\Delta) w_i = \sum_{k=1}^{2} C^{ik}(u_0) v_k + g_i, \qquad i = 1, 2. \tag{4.12}$$

Now, as $v \in D$, v has the same behaviour of $d^{b(2)}$ near the boundary. Therefore, by (4.9)-(4.10) we get that

$$C^{hk}(u_0) v_k \in C^{r,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|C^{hk}(u_0) v_k\|_X \leq K\|v\|_X. \tag{4.13}$$

By (4.13) and Remark 3.3 we get that $C^{hk}(u_0) v_k \in X$.

Therefore, by Theorem 3.4 there is $\omega > 0$ such that for each λ with $\operatorname{Re}\lambda > \omega$ there is a unique solution $w \in D$ of (4.12). Moreover

$$(|\lambda| - \omega)\|w\|_X \leq \bar{k}(\|g\|_X + K\|v\|_X).$$

Now, for each λ: $\operatorname{Re}\lambda \geq \omega + \bar{k}K + 1 = \bar{\omega}$, τ_λ is a contraction and its fixed point satisfies the estimate

$$(|\lambda| - \bar{\omega})\|w\|_X \leq \bar{k}\|g\|_X,$$

that implies the generation of an analytic semigroup in X. In order to apply Theorem 3.1, let us verify that

$$f(u_0) \in \overline{D}, \tag{4.14}$$

$$f_w^2 \in C^0(D \times D; X) \quad \text{for each } w \in D, \tag{4.15}$$

(4.14) is a consequence of (4.11). Actually

$$\overline{D} \supseteq [C_0^\infty(\overline{\Omega})]^2 = [h_0^{r,\alpha}(\overline{\Omega})]^2$$

(it is possible to show that $\overline{D} = [C_0^{3,\alpha}(\overline{\Omega}) \cap h_0^{r,\alpha}(\overline{\Omega})]^2$). In order to prove (4.15), we calculate explicitly $f_w^2(z, v)$ for each $z, v \in D$

$$f_w^2(z, v) = \sum_{i=1}^{2} \frac{\partial}{\partial u_i} \phi_h(x, w)(\Delta v_h z_i + \Delta z_h v_i) +$$

$$+ \sum_{i,j=1}^{2} \frac{\partial^2}{\partial u_i \partial u_j} \phi_h(x, w) \Delta w_h (z_i v_j + v_i z_j), \qquad h = 1, 2.$$

Now, for each $w, w', v, z \in D$

$$\|f_w^2(v, z) - f_{w'}^2(v, z)\|_X \leq k\|w - w'\|_D \|v\|_D \|z\|_D$$

and this implies that f_w^2 is continuous in D (more precisely is locally Lipschitz continuous in D).

Now, the statement follows applying Theorem 1.3 to problem (0.6) since all its assumptions are satisfied.

Before analyzing the examples stated in the introduction we refer the reader to Section 7 of [25] for a rich bibliography about examples of nonexistence, nonregularity and nonuniqueness in the case of a single degenerate equation.

I-st Example.

Here we assume

$$\mu_0(u) = \chi_0(u) = \varepsilon^{-1} u. \tag{4.16}$$

In order to apply Theorems 4.3 and 3.4, we assume that there are $k > 0$, r positive integers greater than 1, and $0 < \alpha < \gamma < 1$ such that

$$k(\sin x)^{r+\alpha} \leq u_0(x) \leq k^{-1}(\sin \pi x)^{r+\alpha}, \qquad 0 \leq x \leq 1, \tag{4.17}$$

$$u_0 \in C^{r+2,\alpha}([0, 1], (\sin \pi x)^{(r+\alpha)/2}) \cap C_0^r([0, 1]), \tag{4.18}$$

$$u_0(u_0)_{xx} + (u_0)_x^2 \in C^{r,\alpha}([0, 2]), \tag{4.19}$$

$$\rho_0 \in C^{r+2,\alpha}([0, 1]) \cap C_0^r([0, 1]), \tag{4.20}$$

$$\rho_0(0)^{(r+2)} = \rho_0(1)^{(r+2)} = 0 \tag{4.21}$$

(before applying Theorem 1.3, we remark that (4.19) is not as restrictive as it may seem: for instance it is verified if $u_0(x) = (\sin \pi x)^{r+\alpha}$).

Note that all the assumptions of Theorem 1.3 are fulfilled: indeed, let $d(x) = \sin \pi x$, b: $\{1, 2\} \to \mathbb{R}$ such that $b(1) = 0$, $b(2) = (r + \alpha)/2$, $\Omega = (0, 1)$, $D = C^{r+2,\alpha}(\Omega, d(b))$, $X = \widetilde{C}^{r,\alpha}(\Omega, d(b))$ and $\overline{u}_0^{\delta} = (\rho_0, u_0)$, then $\overline{u}_0 \in D$ by (4.18), (4.20)-(4.21), $f(\overline{u}_0) \in \overline{D}$ by (4.19), f: $D \to X$ is of class C^2 by (4.16).

Hence, by Theorem 1.3 we get the existence of $t_0 > 0$ and of a strict solution $(u(t), \rho(t))$ of (0.1) in $[0, t_0]$.

II-nd Example.

Here we consider system (0.2). Imposing a few different conditions on a_i, b_{ij} and c_i $(i, j = 1, 2)$ and applying Theoren 1.3 we are able to say «something» about the local existence, uniqueness and regularity of the solutions. For simplicity, we assume $\Omega = (0,1)$, but, as the reader may easily check, the case Ω smooth domain of $\mathbb{R}^n$ can be studied analogously.

We consider, first, the case $a_i(x) > 0$ for $i = 1,2$ and $x \in (0,1)$. More precisely we assume that there are k_1, k_2, $k_3 > 0$, $0 < \alpha < 1$, and b: $\{1, 2\} \to \mathbb{R}$ such that $b(1) = 0$ or ≥ 1 and $b(2) \geq \max(1, b(1))$. Moreover we assume that for each $j, i = 1,2$

$$k_i (\sin \pi x)^{2b(i)} \leq a_i(x) \leq k_i^{-1} (\sin \pi x)^{2b(i)},$$

$$b_{ij}(x) \leq k_3 (\sin \pi x)^{2b(k)}, \qquad \text{where } k = \max(i, j),$$

$$c_i(x) \leq k_3 (\sin x)^{b(i)},$$

$$b_{ii}(x) \geq 0, \qquad \text{for each } x \in (0, 1),$$

$$u_0 \in C^{2,\alpha}([0, 1], \ (\sin \pi x)^{b(1)}) \cap C_0^{0,\alpha}([0, 1]),$$

$$v_0 \in C^{2,\alpha}([0, 1], \ (\sin \pi x)^{b(2)}) \cap C_0^{0,\alpha}([0, 1]).$$

Moreover assume that the ellipticity condition (0.12) holds. Then by Theorem 1.3 and 3.4 deduce the existence of a positive constant $t_0 > 0$ such that there is a unique strict solution $(u(t), v(t))$ in $[0, t_0]$. Moreover for each $0 < \gamma < \alpha$

$$u \in C^1([0, t_0]); \ \ C_0^{0,\gamma}([0, 1]) \cap C^0([0, t_0]); \ \ C^{2,\gamma}([0, 1], \ (\sin \pi x)^{b(1)}),$$

$$v \in C^1([0, t_0]); \ \ C_0^{0,\gamma}([0, 1]) \cap C^0([0, t_0]); \ \ C^{2,\gamma}([0, 1], \ (\sin \pi x)^{b(2)}).$$

Consider now the case

$$a_1(x) = a_2(x) = 0.$$

For simplicity, we consider system (0.3) (i.e. we assume $c_1 = c_2 = 0$, $b_{11} = 2b_{12} = 2b_{21} = b_{22} = 1$).

Assume that there are $s \geqslant 1$, $r \geqslant s + 3$, $0 < \alpha < 1$ such that

$$u_0,\ v_0 \in C^{r,\alpha}([0, 1]) \cap C_0^s([0, 1]),$$

$$u_0^{(s+1)}(0), \quad u_0^{(s+1)}(1), \quad v_0^{(s+1)}(0), \quad v_0^{(s+1)}(1) > 0,$$

$$u_0(x),\ v_0(x) > 0 \quad \text{for each } 0 < x < 1.$$

Lastly choose

$$X = [C^{r-2,\alpha}([0, 1]) \cap C_0^s([0, 1])]^2,$$

$$D = [C^{r,\alpha}([0, 1]), (\sin \pi x)^{(s+1)/2}]^2 \cap X.$$

Repeating the same argument used to prove Theorem 4.1 and using Remark 3.5 instead of Theorem 3.4, we deduce the existence of a positive constant t_0 and a strict solution $(u(t), v(t))$ of (0.3) in $[0, t_0]$. Moreover

$$u, v \in C^0([0, t_0]);\ \ C^{r,\alpha}([0, 1], (\sin \pi x)^{(s+1)/2}) \cap C^1([0, t_0]);\ \ C^{r-2,\alpha}([0, 1]).$$

Note that this result implies that the waiting time is strictly positive for such initial data. We recall that the waiting time is the time until which the support of the solution of a free boundary problem is contained in the initial support. We recall also that in the case of a single equation the waiting time phenomenon was studied by many authors (see [14] for a rich bibliography on this argument).

Lastly we consider the system:

$$\begin{cases} u_t = v(u)_{xx}, & 0 \leqslant x \leqslant,\ t \geqslant 0, \\ v_t = u(v)_{xx}, & 0 \leqslant x \leqslant 1,\ t \geqslant 0, \\ u(0) = u_0, \quad v(0) = v_0, & 0 \leqslant x \leqslant 1 \end{cases}$$

(it is a simple modification of the case $a_1 = a_2 = b_{11} = b_{22} = c_1 = c_2 = 0$, $b_{12} = b_{12} = 1$). Also this case may be studied as the previous one assuming $u, v \in C^{r,\alpha}([0, 1]) \cap C_0^s([0, 1])$, $r - s \geqslant 3$, $s \geqslant 1$; and proving that the waiting time phenomenon occurs. But now, we assume different conditions on the initial value (u_0, v_0). We consider the case of a species diffused in all the enviroment and of another species localized in a specified region. This case can be represented assuming

$$\begin{cases} u_0(x) > 0, & \text{for each } 0 < x < 1, \ u_0(0) = u_0(1) = 0, \\ v_0(x) > 0, & \text{for each } 0 < x < 1. \end{cases}$$

More precisely we assume that there are $s \geqslant 2$, $0 < \alpha < 1$ such that

$$\begin{cases} u_0 \in C^{s+2,\alpha}([0,1]) \cap C_0^s([0,1]); \qquad u_0^{(s+1)}(0) = 0, \ u_0^{(s+1)}(1) = 0, \\ v_0 \in C^{s+2,\alpha}([0,t_0]). \end{cases}$$

Using Remark 3.5 and arguing as in the proof of Theorem 4.1, it is possible to prove that there are $t_0 > 0$ and a strict solution (u, v) in $[0, t_0]$. Moreover

$$\begin{cases} u \in C^1([0,t_0]); \ C^{s,\alpha}([0,1]) \cap C^0([0,t_0]); \ C^{s+2,\alpha}([0,1]), \\ v \in C^1([0,t_0]); \ C^{s,\alpha}([0,1]) \cap C^0([0,t_0]); \ C^{s+2,\alpha}([0,1], (\sin \pi x)^{(s+1)/2}). \end{cases}$$

III-rd Example.

Assuming suitable hypotheses on ϕ_i, f_i, u_0, v_0 (in order to satisfy (4.11)-(4.12)), Theorem 1.3 allows us to obtain local existence and uniqueness of solutions of (0.4) dropping (0.5).

Appendix.

PROOF OF THEOREM 1.2. Theorem 1.2 was proved in [25] in the case of a single equation under the assumption

(A.1) $\qquad \Omega$ is a bounded set with C^∞ boundary.

Here we prove that (1.7) holds also if (A.1) does not hold. We use an argument of local maps: by means of local maps of class C^2 we make C^∞ the boundary of Ω. Then we apply the results proved in [25] and we obtain the L^p estimates for a localized solution. Lastly, by the reverse transformations and by the interior estimates proved in Lemma 2.3, we get (1.7). Before proving (1.7), let us make some remarks.

We show only (1.7) because the same argument (with the obvious modifications) works also for (1.8) and (1.9) (also these results are proved in [25] under assumption (A.1)).

Moreover, for the sake of simplicity, we restrict ourselves to consider only the case of a single equation, but, as the reader can easily check, either Theorem 4.1 of [25] that this argument of local maps may be extended to the case of systems.

Lastly, the assumption that Ω has boundary of class C^2 ($C^{2,\alpha}$, $C^{r,\alpha}$, resp.) implies that $d(x)$ cannot be more regular than $C^2(\overline{\Omega})$ ($C^{2,\alpha}(\overline{\Omega})$, $C^{r,\alpha}(\overline{\Omega})$, resp.). Actually as $d(x) = 0$ on $\partial\Omega$ and $\nabla d(x) \neq 0$ on $\partial\Omega$, than $\partial\Omega$ has the same regularity of $d(x)$.

PROOF OF (1.7). Let us introduce some notation.

In $s > 0$, set

$$B(s) = \{x \in \mathbb{R}^n : \ |x| < s\},$$
$$B^+(s) = \{x \in B(s) : \ x_n > 0\},$$
$$G(s) = \{x \in B(s) : \ x_n = 0\}.$$

First we focus our attention on $p \geq 2$.

Let $2 \leq p \leq 2^*$ if $n > 0$ ($2 \leq p < \infty$ if $n \leq 2$). As $\partial\Omega$ is of class C^2 there exists a finite open covering $\{U_j\}_{j=1}^M$ of $\partial\Omega$ such that for each j there exists a one-to-one transformation $\phi_j : U_j \to B(1)$ with the following properties: ϕ_j and ϕ_j^{-1} have continuous derivatives up to order 2 on $\overline{U_j}$ and $\overline{B(1)}$ resp.

$$\text{(A.2)} \qquad \begin{cases} \phi_j(U_j \cap \Omega) = B^+(1), \\ \phi_j(U_j \cap \partial\Omega) = G(1), \\ \bigcup_{j=1}^{M} \phi_j^{-1}(B^+(1/2)) \supset \partial\Omega. \end{cases}$$

By Theorem 3.1 of [25], there exists a constant $\tilde{c}_0 > 0$ such that for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \tilde{c}_0$ and for each $f \in L^2(\Omega)$ there is a solution $u \in H^{2,2}(\Omega, d(b))$ of (1.7).

Let λ be such that $\operatorname{Re}\lambda > \tilde{c}_0$ and, for each $s = 1, 2, \ldots, M$ consider $v_s = u \circ \phi_s^{-1} : B^+(1) \to \mathbb{C}$.

$v_s \in H^{2,2}(B^+(1), x_n^b)$ and solves in $B^+(1)$ the equation

$$(\lambda - E_s)\, v_s = f_s,$$

where

$$E_s v_s = \sum_{i,j=1}^{n} (x_n)^{2b} A_{ij}^s D_{ij} v_s + \sum_{i=1}^{n} (x_n)^b B_i^s D_i v_s + C^s v_s,$$

$$f_s = f \circ \phi_s^{-1},$$

$$A_{ij}^s = (d \circ \phi_s^{-1}/x_n)^{2b} A_{ij} \circ \phi_s^{-1} D_i \phi_s D_j \phi_s,$$

$$B_i^s = (d \circ \phi_s^{-1}/x_n)^b B_i \circ \phi_s^{-1} D_i \phi_s + \sum_{j=1}^{n} ((d \circ \phi_s^{-1})^2/x_n)^b A_{ij} \circ \phi_s^{-1} D_{ij} \phi_s,$$

$$C^s = C \circ \phi_s^{-1}.$$

Let $\theta \in C^\infty(\overline{B(1)})$ be such that $\theta = 1$ in $B(1/2)$, $\theta = 0$ in $B(1)\setminus B(2/3)$ and let $\widetilde{\Omega}$ be a domain of class C^∞ such that $B^+(1) \supset \widetilde{\Omega} \supset B^+(3/4)$ and fix a function $\widetilde{d}(x) \in C^\infty(\widetilde{\Omega})$ satisfying (0.9) and such that $\widetilde{d}(x) = x_n$ on $B^+(2/3)$.

Now we show that θv_s solves a suitable degenerate equation in $\widetilde{\Omega}$ so that we are able to apply the estimates proved in Theorem 4.1 of [25].

Actually $\theta v_s \in H^{2,2}(\widetilde{\Omega}, \widetilde{d}))$ and solves in $\widetilde{\Omega}$ the equation

$$(\lambda - \widetilde{E}_s)\,\theta v_s = g_s\,,$$

where

$$g_s = f_s + \sum_{i=1}^{n} (x_n)^b B_i^s v_s D_i \theta + \sum_{i,j=1}^{n} (x_n)^b A_{ij}^s (D_i \theta D_j v_s + D_j \theta D_i v_s + v_s D_{ij} \theta)\,.$$

and $\widetilde{E}_s$ is determined E_s substituting x_n with $\widetilde{d}(x)$.

By Sobolev embedding theorem $g \in L^p(\Omega)$ and by Theorem 4.1 of [25], $\theta v_s \in (H^{2,p}(\widetilde{\Omega}, \widetilde{d}(b)))$ and satisfies (1.7) with $\widetilde{\Omega}$, $\widetilde{d}$ instead of Ω and d. Therefore, by the reverse transformation, there exists $\widetilde{c}_0^s$ such that for each λ with $\operatorname{Re}\lambda > \widetilde{c}_0^s$

$$\text{(A.3)}\qquad \sum_{i,j=1}^{n} \|d^{2b} D_{ij} u\|_{0,p,\Omega s} + (|\lambda| - \widetilde{c}_0^s)^{1/2} \sum_{i=1}^{n} d^b \|D_i u\|_{0,p,\Omega s} + \\ + (|\lambda| - \widetilde{c}_0^s)\|u\|_{0,p,\Omega s} \leqslant \widetilde{c}_1^s \|f\|_{0,p,\Omega}\,,$$

where $\Omega s = \phi_s^{-1}(B^+(1/2))$ and $\widetilde{c}_1^s$ is a constant independent of λ, f and u.

Hence (1.7) follows from (A.2), (A.3) and (2.6).

If $n = 2$ the statement is proved.

If $n > 2$, let us consider the case $2^* \leqslant p \leqslant 2^{**}$ if $n \geqslant 4$, $2^* \leqslant p < \infty$ if $n \leqslant 4$. Using again the above argument of local coordinates, it is possible to prove (1.7).

If $n \leqslant 4$ the proof is finished.

If $n > 4$, we repeat the above argument and after a finite number of steps we show (1.7) for each $p > 2$.

The case $1 < p < 2$ can be studied by a standard duality argument.

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