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On the Planar Translation Structures.

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Summary - We investigate the planar translation André structures with a finite number of points. A description of all these structures in terms of planar translation structures over an abelian group is obtained.

1. Introduction.

A planar translation structure (p.t.structure) is an André translation structure (t.structure) Σ, endowed with a set ε of subspaces such that any three non-collinear points of Σ lie in exactly one element of ε.

Herzer [9] investigated these structures, obtaining several examples and some results about the structure of the translation group. Subsequently, Schulz [12] proved that, in the finite case, the translation group of Σ must be either an elementary abelian or a Frobenius group.

In this paper, we carry on the study of finite p.t.structures. When the translation group T of Σ is a Frobenius group, we prove that T is isomorphic to a dilatation group of a p.t.structure Σ' with an abelian translation group. Furthermore, the points of Σ are the points of a regular orbit Ξ of T on Σ' and the line-plane structure of Σ is that induced by the line-plane structure of Σ' in Ξ. Thus, all finite p.t.structures can be described in terms of p.t.structures over an abelian group.

The rest of the paper is devoted to the p.t.structures over an abelian group, mainly for what concerns their representation in an affine space. In particular, we introduce the concept of «minimal dimension of an affine representation» and we obtain some results in this connection. Some examples are also given.

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2. Preliminary results.

A *partition (of a finite group T)* is a set $\Pi(T)$ of non-identical sub-groups of T, such that $T \notin \Pi(T)$ and each $g \in T$, with $g \neq 1$, belongs to exactly a subgroup of $\Pi(T)$. When $T \in \Pi(T)$ we shall call of a *trivial partition*. The elements of $\Pi(T)$ are called the *components* of $\Pi(T)$. To any partition $\Pi(T)$ one can associate a geometric structure with parallelism $(\mathscr{P}, \mathscr{L}, \|)$ by taking (see [3])

— *the elements of T as the point-set $\mathscr{P}$,*

— *the right cosets of the components of $\Pi(T)$ as the line-set $\mathscr{L}$,*

— *the binary relation «$\|$» on $\mathscr{L}$ defined as follows*

$$Ha \| Kb \Leftrightarrow H = K, \quad a, b \in T, \ H, K \in \Pi(T)$$

as the parallelism relation.

The triple $(\mathscr{P}, \mathscr{L}, \|)$ is called the *t.structure* $\Sigma(\Pi(T))$. Let $A = \{\alpha \in \text{Aut}(T): H^\alpha = H, \ \forall H \in \Pi(T)\}$ For $a \in T$ and $\alpha \in A$, the map $(a, \alpha): x \to x^\alpha a$ of T onto itself is a collineation of $\Sigma(\Pi(T))$ preserving the parallelism. It is called a *dilatation*. The dilatations make a group D. The map $\bar{a} = (a, 1)$ is a *translation* and acts f.p.f on $\mathscr{P}$—when $a \neq 1$. $\bar{T} = \{\bar{a}: a \in T\}$ is a subgroup of D which is transitive on the points of $\Sigma(\Pi(T))$ and $\bar{T} \simeq T$. An element of $D - \bar{T}$ is a *proper dilatation* and fixes exactly one point. The dilatations fixing the point 1 are the elements of A. $D = \bar{T}A$ is a Frobenius group and A is cyclic (see [2] and [7]).

When T is an abelian group, the dilatations of $\Sigma(\Pi(T))$ fixing the point 0—we use the additive notation for abelian groups—, together with the null endomorphism of T, make a subfield F of $\text{End}(T)$ (see [3]). F is called the *kernel* of $\Sigma(\Pi(T))$ and the dimension of T, regarded as a vector space over F, is called here the *global dimension* of $\Sigma(\Pi(T))$ (over the kernel F). It will be denoted by $g(\Sigma(\Pi(T)))$.

A subgroup S of T is $\Pi(T)$-*admissible* [4] if $H \cap S = \langle 1 \rangle$ or $H \cap S = H$ for each $H \in \Pi(T)$. A right coset Sa of T, for S a $\Pi(T)$-admissible subgroup of T, is called a *subspace* of $\Sigma(\Pi(T))$.

According to Herzer [9] the partition $\Pi(T)$ is *linear* if it satisfies the following condition:

(1)   *There exists a set $\varepsilon(T)$ of $\Pi(T)$-admissible subgroups of T such that every element of $\varepsilon(T)$ contains more than one component of $\Pi(T)$ and is different from T and any pair of distinct components of $\Pi(T)$ is contained in exactly one element of $\varepsilon(T)$.*

The cosets Sa, where $S \in \varepsilon(T)$, are the *planes* of $\Sigma(\Pi(T))$, and $\Sigma(\Pi(T))$, together with the plane-set $\varepsilon(T)$, is called a *planar transla-*

tion structure (p.t.structure). It is easily seen that any three non-collinear points of $\Sigma(\Pi(T))$ lie in exactly one plane.

To each linear partition $(\Pi(T), \varepsilon(T))$ is associated a linear space $\mathcal{T}(\Pi(T), \varepsilon(T))$ having $\Pi(T)$ as the point-set and $\varepsilon(T)$ as the line-set. $\mathcal{T}(\Pi(T), \varepsilon(T))$ can be regarded as the «geometry at infinity» of the t.structure $\Sigma(\Pi(T), \varepsilon(T))$.

RESULT 2.1 (Herzer [9], Schulz [12]). *A group with a linear partition is either an elementary abelian or a Frobenius group.*

3. Some more on groups admitting linear partitions.

The following result extends Prop. 2 of [9].

PROPOSITION 3.1. *Let $(\Pi(T), \varepsilon(T))$ be a linear partition and let H be a subgroup of T. Put*

$$\Pi(H) = \{F \cap H \colon F \in \Pi(T), \ F \cap H \neq \langle 1 \rangle\}$$

$$\varepsilon(H) = \{N \cap H \colon N \in \varepsilon(T), \ N \cap H \nsubseteq F \cap H \ \forall F \in \Pi(T)\}.$$

If $H \nsubseteq F \in \varepsilon(T)$, then $(\Pi(H), \varepsilon(H))$ is a linear partition.

PROOF. $\Pi(H)$ is non-trivial since $H \nsubseteq F \in \varepsilon(T)$. Let $H_1, H_2 \in \Pi(H)$ with $H_1 \neq H_2$. There exist $T_1, T_2 \in \Pi(T)$ with $H_1 \subseteq T_1$ and $H_2 \subseteq T_2$. Let N be the unique element of $\varepsilon(T)$ containing both T_1 and T_2. Then $H_1, H_2 \subseteq N \cap H \in \varepsilon(H)$. Clearly, $N \cap H$ is the unique element of $\varepsilon(H)$ containing both H_1 and H_2.

Now, we can prove the following.

PROPOSITION 3.2. *Let $(\Pi(T), \varepsilon(T))$ be a linear partition of a Frobenius group T with Frobenius kernel K. Then the following hold:*

(2) $\Pi(K)$ *is a non-trivial partition of K and K is an elementary abelian group.*

(3) *Every component of $\Pi(K)$ is normal in T.*

PROOF. Suppose that $\Pi(K)$ is a trivial partition. If $K \leqslant H \in \Pi(T)$, then $H = T$, since a Frobenius complement admits only trivial partitions. So, $K \in \Pi(T)$. Nevertheless, $K \leqslant E$ for some $E \in \varepsilon(T)$. As before, this yields $E = T$, contrary to (1). Therefore, $\Pi(K)$ is non-trivial. If $K \subseteq N$ for some $N \in \varepsilon(T)$, then $N = K$ since $N \neq K$ yields $N = T$. Hence K is elementary abelian

by [9], Remark 2. Otherwise, $(\Pi(K), \varepsilon(K))$ is a linear partition by Prop. 3.1. Again, K is elementary abelian by Result 2.1. Indeed, K is a nilpotent group (see [10]) and any nilpotent group with a non-trivial partition is a p-group. This proves (2).

We notice that in any subgroup of T of the form $K_0 \cdot H$, where $K_0 \leqslant K$ and H is a Frobenius complement, the subgroup K_0 is normal in T. Indeed if $t \in T$, then $t = kh$ for some $k \in K$, $h \in H$. So, $K_0^t = K_0^{kh} = K_0^h$ since K is abelian and $K_0^h = K_0$ because H normalizes K_0. Now, we shall prove (3). Let $K_1 \in \Pi(K)$, where $\langle 1 \rangle \neq K_1 = T_1 \cap K$ for some $T_1 \in \Pi(T)$. If $K_1 \neq T_1$, then $T_1 = K_1 H$ for a suitable Frobenius complement H of T and hence $K_1 \trianglelefteq T$.

Suppose that $K_1 = T_1$. There exist at least two distinct elements M, N in $\varepsilon(T)$ such that $K_1 \subseteq N \not\subseteq K$ and $K_1 \subseteq M \not\subseteq K$. Indeed, an element of $\varepsilon(T)$ cannot contain all Frobenius complements since it would coincide with T. So, $M = K'H'$ and $N = K''H''$, where K', $K'' \leqslant K$ and H', H'' are Frobenius complements. We have that K', $K'' \trianglelefteq T$ and $M \cap N = T_1 = K' \cap K''$ since $T_1 \leqslant K$. Therefore, $T_1 \trianglelefteq T$ and the proof is complete.

4. On the representation of p.t.structures.

In this section we shall see as a p.t.structure can be represented in an affine space. Among other things, we shall prove that each p.t.structure over a Frobenius group arises from a p.t.structure over an abelian group.

Let $\mathcal{C}$ be any affine space over a field F and 0 a point of $\mathcal{C}$. A 0-*partition* of $\mathcal{C}$ is a set $\mathcal{O}(\mathcal{C}) = \{\mathcal{C}_i\}_{i \in I}$ of proper subspaces of $\mathcal{C}$ containing 0, such that $\mathcal{C}_i \cap \mathcal{C}_j = 0$ for each $\mathcal{C}_i \neq \mathcal{C}_j$ and $\bigcup_{i \in I} \mathcal{C}_i = \mathcal{C}$. The geometric structure $\Sigma(\mathcal{O}(\mathcal{C}))$ having the same point-set of $\mathcal{C}$ and whose lines are the translate of the elements of $\mathcal{O}(\mathcal{C})$ is a t.structure. Each t.structure $\Sigma(\Pi(T))$, with T an abelian group, is isomorphic to a suitable structure $\Sigma(\mathcal{O}(\mathcal{C}))$ (see [6]). $\mathcal{C} = \mathcal{C}(T, F_0)$ is the affine space associated to T, when T is regarded as a vector space over any subfield F_0 of the kernel of $\Sigma(\Pi(T))$ and the $\mathcal{C}_i$ correspond to the vector subspaces $T_i \in \Pi(T)$. Likewise, a p.t.structure $\Sigma(\Pi(T), \varepsilon(T))$ is isomorphic to $\Sigma(\mathcal{O}(\mathcal{C}), \varepsilon(\mathcal{C}))$, for a suitable set $\varepsilon(\mathcal{C})$ of subspaces of $\mathcal{C}$.

Let $\Sigma = \Sigma(\mathcal{O}(\mathcal{C}), \varepsilon(\mathcal{C}))$ and let G be any dilatation group of $\mathcal{C}$—and hence of Σ. If G contains only translations, then G acts regularly on each point-orbit. Suppose that G contains both non-identical translations and proper dilatations. Then G is a Frobenius group of the form $K \rtimes H$ with kernel K and complement H. K consists of the translations of G,

while H consists of the dilatations of G fixing a given point of $\mathcal{A}$. By a well known result of Frobenius (see [12]), G acts regularly on each point-orbit, but one, of $\mathcal{A}$. In the exceptional orbit $\Gamma(G)$, the group G acts as a Frobenius group in its usual representation.

Let $\varXi$ be an orbit of G and assume that $\varXi \neq \Gamma(G)$ when G is a Frobenius group. Denote by

— $\mathscr{L}(\varXi)$ *the set of lines of* Σ *with at least two distinct points in common with* $\varXi$,

— $\varepsilon(\varXi)$ *the set of planes in* $\varepsilon(\Sigma)$ *intersecting* $\varXi$ *in at least two distinct lines.*

Then it is not difficult to see that $\mathcal{A}(\Sigma, G, \varXi) = (\varXi, \mathscr{L}(\varXi), \varepsilon(\varXi))$ is a p.t.structure with respect to the same parallelism relation of $\mathcal{A}$ or, which is the same, of $\Sigma(\mathcal{O}(\mathcal{A}), \varepsilon(\mathcal{A}))$. G is the translation group of $\mathcal{A}(\Sigma, G, \varXi)$ and $\mathcal{A}(\Sigma, G, \varXi)$ is called *the structure induced by* Σ *in the orbit* $\varXi$ *of* G. When G is elementary abelian, a point-orbit Ω of G on $\mathcal{A} = \mathcal{A}(T, F_0)$ is a subspace of $\mathcal{A}$ if and only if F_0 is contained in the kernel of $\mathcal{A}(\Sigma, G, \Omega)$. When F_0 is contained in the kernel of $\mathcal{A}(\Sigma, G, \Omega)$ for some orbit Ω, then F_0 is contained in the kernel of $\mathcal{A}(\Sigma, G, \Omega')$ for all orbits Ω' of G. Indeed Ω may be mapped onto Ω' by a translation of $\mathcal{A}$.

PROPOSITION 4.1. *For each p.t.structure* $\Sigma = \Sigma(\Pi(T), \varepsilon(T))$ *there exist a triple* $(\mathcal{A}, \mathcal{O}(\mathcal{A}), \varepsilon(\mathcal{A}))$, *a p.t.structure* $\Sigma' = \Sigma(\mathcal{O}(\mathcal{A}), \varepsilon(\mathcal{A}))$, *a dilatation group* $G \simeq T$ *of* $\mathcal{A}$ *and a point-orbit* $\varXi$ *of* G *on* $\mathcal{A}$ *such that* $\Sigma \simeq \mathcal{A}(\Sigma', G, \varXi)$.

PROOF. If T is abelian, the assertion holds because $\Sigma(\Pi(T), \varepsilon(T)) \simeq \Sigma(\mathcal{O}(\mathcal{A}), \varepsilon(\mathcal{A}))$ for a suitable 0-partition of the affine space $\mathcal{A} = \mathcal{A}(t, F_0)$, for F_0 in the kernel of $\Sigma(\Pi(T), \varepsilon(T))$.

Suppose that $T = K \rtimes H$ with kernel K and complement H. Let $F \simeq GF(q)$ be the kernel of $\Sigma(\Pi(K))$ and $m = g(\Sigma(\Pi(K)))$. As we have seen, we may represent $\Sigma(\Pi(K))$ in the affine space $\mathscr{B} = AG(m, q)$ by means of a 0-partition $\mathcal{O}(\mathscr{B})$.

By Prop. 3.2, we may think of H as a group of (f.p.f.) automorphisms of K mapping each component of $\Pi(K)$ into itself, that is as a dilatation group of $\Sigma(\mathcal{O}(\mathscr{B}))$. Since F is the kernel of $\Sigma(\mathcal{O}(\mathscr{B}))$, H is also a dilatation group of $\mathscr{B}$ and hence we may suppose that H is isomorphic to a subgroup of the dilatation group D_0 of $\mathscr{B}$ fixing the point 0. As it is well known, $D_0 \simeq F^*$ and the Frobenius group $D \simeq KD_0$ is the whole dilatation group of $\mathscr{B}$.

For each $T_i \in \Pi(T)$ let D_i be the subgroup of D defined as follows

— if $T_i \subseteq K$, then $D_i = T_i$,

— if $T_i = K_0 \smallsetminus H^x$ with $K_0 \leq K$ and $x \in K$, then $D_i = K_0 \smallsetminus D_0^x$.

Likewise, we define the subgroups D_j for each $T_j \in \varepsilon(T)$.

Put $\Pi(D) = \{D_i\}_{i \in I}$ and $\varepsilon(D) = \{D_j\}_{j \in J}$. Then it is easily seen that $(\Pi(D), \varepsilon(D))$ is a linear partition of D which induces the linear partition $(\Pi(T), \varepsilon(T))$ in T.

Now, suppose we regard $\mathcal{B}$ as a hyperplane of a $m + 1$-dimensional affine space $\mathcal{A}$ over the same field F. Then D may be regarded as the subgroup of the dilatation group of $\mathcal{A}$ which leaves $\mathcal{B}$ invariant. D splits the points of $\mathcal{A}$ into two orbits. $\mathcal{B}$ is the exceptional orbit. The other orbit will be denoted by $\mathcal{E}$.

Let q be a point of $\mathcal{E}$ and for each $D_i \in \Pi(D)$ let Q_i be the orbit of q under D_i. We have that

— if $D_i \subseteq K$, then Q_i is a subspace of $\mathcal{A}$,

— if $D_i = K_0 \smallsetminus D_0^x$ with $K_0 \leq K$ and $x \in K$, then

$$Q_i \cup \{\operatorname{Fix} D_0^{xk} : \ k \in K_0\},$$

where $\operatorname{Fix} D_0^{xk}$ denotes the point of $\mathcal{B}$ which is fixed by the group D_0^{xk}, is a subspace of $\mathcal{A}$.

In the same manner, we define $\varepsilon(\mathcal{A})$ starting from $\varepsilon(D)$.

It is strightforward to show that $Q(\mathcal{A}) = \{Q_i\}_{i \in I}$ is a q-partition of $\mathcal{A}$ and that the structure $\mathcal{A}(\Sigma', D, \mathcal{E})$ induced by $\Sigma' = \Sigma(Q(\mathcal{A}), \varepsilon(\mathcal{A}))$ in the orbit $\mathcal{E}$ of D is isomorphic to $\Sigma(\Pi(D), \varepsilon(D))$. Since the lines and the planes of $\Sigma(\Pi(T), \varepsilon(T))$ are represented by the intersections of the lines and the planes of $\Sigma(\Pi(D), \varepsilon(D))$ with the suborbit of $\overline{KH} \simeq T$ on $\mathcal{E}$, our assertion is proved.

5. On the dimension of the affine representations of a p.t.structure.

Let Σ be a p.t.structure over an abelian group. As we have seen, if Σ has the global dimension $g(\Sigma) = m$ over its kernel F then Σ may be represented in the form $\Sigma(\mathcal{O}(\mathcal{A}), \varepsilon(\mathcal{A}))$ for a suitable 0-partition $\mathcal{O}(\mathcal{A})$ of the m-dimensional affine space $\mathcal{A}$ over F. In other words, Σ may be trivially represented in the form $\mathcal{A}(\Sigma, T, \mathcal{E})$ for $\mathcal{A}$ a m-dimensional affine space, by assuming as T the whole translation group of $\mathcal{A}$ (and as $\mathcal{E}$ the point-set of $\mathcal{A}$). Nevertheless, it can happen that Σ may be also represented in the form $\mathcal{A}(\overline{\Sigma}, T, \mathcal{E})$ for $\mathcal{A}$ an affine space of dimension

smaller than m, as we shall see in the following. Put

$$a(\Sigma) = \min \left\{ \dim \mathcal{Q} \colon \Sigma \simeq \mathcal{Q}(\bar{\Sigma}, T, \Xi) \text{ for some quadruple } \mathcal{Q}, \bar{\Sigma}, T, \Xi \right\}.$$

Here, the question arises whether $\mathcal{Q}$ ranges over all affine spaces or, rather, over all finite affine spaces. Following the line of this paper, we are inclined to assume that $\mathcal{Q}$ ranges only over the finite affine spaces.

We call $a(\Sigma)$ *the minimal dimension of the affine representation of* Σ or, briefly, *the minimal affine dimension of* Σ.

Why, this definition? One may start by observing that from a given p.t.structure Σ with $g(\Sigma) = m$ and kernel F, we obtain many p.t.structure in the following way. We put Σ in the form $\Sigma(\mathcal{O}(\mathcal{Q}), \varepsilon(\mathcal{Q}))$ for a suitable 0-partition $\mathcal{O}(\mathcal{Q})$ of the m-dimensional affine space $\mathcal{Q}$ over F and then we construct all p.t.structures of the form $\mathcal{Q}(\Sigma, G, \Xi)$, where G is any proper translation subgroup of $\mathcal{Q}$ and Ξ is a point-orbit of G. If we attempt to classify all p.t.structures over an abelian group, the structures of the form $\mathcal{Q}(\Sigma, G, \Xi)$ originate from Σ in a rather natural way.

Actually, a p.t.structure Σ is *essentially new* when $a(\Sigma)$ and $g(\Sigma)$ coincide. Indeed, such a structure does not apper, as an induced structure, in any affine space of dimension smaller than $g(\Sigma)$. It seem natural to call a p.t.structure Σ such that $a(\Sigma) = g(\Sigma)$ a *basic p.t.structure*. We have the following.

PROPOSITION 5.1. *If* Σ *is a p.t.structure then* $\Sigma \simeq \mathcal{Q}(\bar{\Sigma}, G, \Xi)$ *for some basic p.t.structure* $\bar{\Sigma}$.

PROOF. Let $a(\Sigma) = n$. Then $\Sigma \simeq \mathcal{Q}(\bar{\Sigma}, G, \Xi)$ for a quadruple $\mathcal{Q}, \bar{\Sigma}, G, \Xi$, where $\mathcal{Q}$ is a n-dimensional affine space. Suppose that $\bar{\Sigma} \simeq \mathcal{B}(\tilde{\Sigma}, \tilde{G}, \tilde{\Xi})$ for some affine space $\mathcal{B}$ with $\dim \mathcal{B} \gtrless \dim \mathcal{Q}$. Since Σ can be regarded as the structure induced in the point-orbit Ξ of G over $\bar{\Sigma}$ by the line-plane structure of $\bar{\Sigma}$, then $\Sigma \simeq \mathcal{B}(\tilde{\Sigma}, G, \Xi)$. This contradicts the assumption that $a(\Sigma) = n$.

Certainly, our definition is not completely satisfactory. Indeed, if $\Sigma \simeq \mathcal{Q}(\bar{\Sigma}, G, \Xi)$, where $\dim \mathcal{Q} = a(\Sigma)$, then $\bar{\Sigma}$ induces in Σ the line-plane structure, but, in general, it does not induce in Σ the structure of the remaining subspaces. This makes clair when we determine the minimal affine dimension of p.t.structures of the form $\mathcal{Q}(\mathcal{Q}, G, \Xi)$, where the line-plane structure is just that induced by the corresponding line-plane structure of the affine space $\mathcal{Q}$. Indeed, we have the following proposition.

PROPOSITION 5.2. *If Σ is a p.t.structure of the form $\mathcal{C}(\mathcal{C}, G, \Xi)$ for some affine space $\mathcal{C}$, then $a(\Sigma) = 3$.*

PROOF.   It is enough to prove that for any n-dimensional vectorspace V over a field F, there exists a 3-dimensional vector space W over a overfield F_0 of F such that

(4) V is a F-invariant subgroup of W,

(5)   each 1-dimensional (2-dimensional) subspace of W (over F_0) meets V in a subspace of V which is at most 1-dimensional (2-dimensional) over F.

Let p_{ij} $i = 1, 2, 3$, $j = 1, \ldots, n$, be $3n$ pairwise distinct prime numbers, different from 1. Put $m_i = \prod_{j=1}^{n} p_{ij}$ and $m = m_1 m_2 m_3$. If $F = GF(q)$, let $F_0 = GF(q^m)$. Assume

$$W = F_0^3 = \{(\alpha, \beta, \gamma,): \alpha, \beta, \gamma \in F_0\}$$

and let

$$\mathfrak{B} = \{(\alpha_j, \beta_j, \gamma_j): \alpha_j \in GF(q^{p_{1j}}) - GF(q),$$

$$\beta_j \in GF(q^{p_{2j}}) - GF(q), \quad \gamma_j \in GF(q^{p_{3j}}) - GF(q), \quad j = 1, \ldots, n\}.$$

The vectors of $\mathfrak{B}$ are linearly independent over F. Indeed

$$\sum_{j=1}^{n} h_j(\alpha_j, \beta_j, \gamma_j) = 0 \rightarrow \sum_{j=1}^{n} h_j \alpha_j = 0, \qquad h_j \in F.$$

Nevertheless, $\sum_{j=1}^{n} h_j \alpha_j = 0$ for some $h_k \neq 0$ yields

$$\alpha_k \in \langle \alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n \rangle \text{ (over } F),$$

that is

$$\alpha_k \in GF(q^{p_{11} \cdots p_{1k-1} \cdot p_{1k+1} \cdots p_{1n}})$$

and this is impossible.

Denote by V the vector space generated by the elements on $\mathfrak{B}$ over F. Let W_1 be a 1-dimensional subspace of W which meets V in a vector $v = \sum_{j=1}^{n} h_j(\alpha_j, \beta_j, \gamma_j)$, $h_j \in F$, $v \neq 0$. So, $W_1 = F_0 v$. Suppose that

$$\delta \sum_{j=1}^{n} h_j(\alpha_j, \beta_j, \gamma_j) = \sum_{j=1}^{n} k_j(\alpha_j, \beta_j, \gamma_j)$$

for some $\delta \in F_0$, $k_j \in F$. In particular, this yields

$$\text{(6)} \qquad \delta \sum_{j=1}^{n} h_j \alpha_j = \sum_{j=1}^{n} k_j \alpha_j \,,$$

$$\text{(7)} \qquad \delta \sum_{j=1}^{n} h_j \beta_j = \sum_{j=1}^{n} k_j \beta_j \,.$$

As we have previously seen, $v \neq 0$ yields

$$\sum_{j=1}^{n} h_j \alpha_j \neq 0 \,, \qquad \sum_{j=1}^{n} h_j \beta_j \neq 0 \,,$$

So, $\delta \in GF(q^{m_1})$ by (6) and $\delta \in GF(q^{m_2})$ by (7). Therefore, $\delta \in GF(q^{m_1}) \cap GF(q^{m_2}) = F$ and hence $W_1 \cap V = Fv$.

Now, let W_2 be a 2-dimensional subspace of W which meets V in two vectors

$$u = \sum_{j=1}^{n} h_j (\alpha_j, \beta_j, \gamma_j), \quad v = \sum_{j=1}^{n} k_j (\alpha_j, \beta_j, \gamma_j), \qquad h_j, k_j \in F,$$

which are independent over F—and hence over F_0 for what we saw above. Suppose that $\delta u + \phi v = z$ for some $\delta, \phi \in F_0$, $z \in V$. If

$$z = \sum_{j=1}^{n} l_j (\alpha_j, \beta_j, \gamma_j), \qquad l_j \in F,$$

we have

$$\text{(8)} \qquad \delta \sum_{j=1}^{n} h_j \alpha_j + \phi \sum_{j=1}^{n} k_j \alpha_j = \sum_{j=1}^{n} l_j \alpha_j \,,$$

$$\text{(9)} \qquad \delta \sum_{j=1}^{n} h_j \beta_j + \phi \sum_{j=1}^{n} k_j \beta_j = \sum_{j=1}^{n} l_j \beta_j \,,$$

$$\text{(10)} \qquad \delta \sum_{j=1}^{n} h_j \gamma_j + \phi \sum_{j=1}^{n} k_j \gamma_j = \sum_{j=1}^{n} l_j \gamma_j \,.$$

Using (8) and (9) we obtain

$$\phi \left(\sum_{j=1}^{n} h_j \beta_j \sum_{j=1}^{n} k_j \alpha_j - \sum_{j=1}^{n} h_j \alpha_j \sum_{j=1}^{n} k_j \beta_j \right) = \sum_{j=1}^{n} h_j \beta_j \sum_{j=1}^{n} l_j \alpha_j - \sum_{j=1}^{n} h_j \alpha_j \sum_{j=1}^{n} l_j \beta_j \,.$$

This yields $\phi \in GF(q^{m_1 m_2})$ on condition that

(11)
$$\sum_{j=1}^{n} h_j \beta_j \sum_{j=1}^{n} k_j \alpha_j - \sum_{j=1}^{n} h_j \alpha_j \sum_{j=1}^{n} k_j \beta_j \neq 0 .$$

Likewise, we obtain $\phi \in GF(q^{m_1 m_3})$ and $\phi \in GF(q^{m_2 m_3})$. So,

$$\phi \in GF(q^{m_1 m_2}) \cap GF(q^{m_1 m_3}) \cap GF(q^{m_2 m_3}) = F .$$

This yields $\delta \in F$ by (8) and (9) and hence $W_2 \cap V = Fu \oplus Fv$. So, the
proof is complete if we prove that (11) must hold in any case. Suppose
that (11) does not hold, that is

(12)
$$\sum_{j=1}^{n} h_j \beta_j \sum_{j=1}^{n} k_j \alpha_j - \sum_{j=1}^{n} h_j \alpha_j \sum_{j=1}^{n} k_j \beta_j = 0 .$$

Then, (12) yields

(13)
$$\sum_{j=1}^{n} \beta_j \left(h_j \sum_{i=1}^{n} k_i \alpha_i - k_j \sum_{i=1}^{n} h_i \alpha_i \right) = 0 .$$

As $h_j \sum_{i=1}^{n} k_i \alpha_i - k_j \sum_{i=1}^{n} h_i \alpha_i \in GF(q^{m_1})$ for $j = 1, \ldots, n$ and the elements
β_j are independent over $GF(q^{m_1})$, relation (13) yields

(14)
$$h_j \sum_{i=1}^{n} k_i \alpha_i - k_j \sum_{i=1}^{n} h_i \alpha_i = 0$$

or also

(15)
$$\sum_{i=1}^{n} \alpha_i (h_j k_i - k_j h_i) = 0$$

for $j = 1, \ldots, n$. Again, as $h_j k_i - k_j h_i \in F$ and the elements α_i are inde-
pendent over F, (15) yields

(16)
$$h_j k_i - k_j h_i = 0$$

for $i, j = 1, \ldots, n$. By (16), $h_i = 0 \Leftrightarrow k_i = 0$. Let $h_m \neq 0$, $k_m \neq 0$, for some
$m \in \langle i, \ldots, n \rangle$. Then, $h_s / k_s = h_m / k_m$ for all $s \in \langle 1, \ldots, n \rangle$ such that
$h_s \neq 0$, $k_s \neq 0$, by (16). This yields $u = hv$ for some $h \in F$, contrary to our
assumptions.
We have also the following.

PROPOSITION 5.3. *P.t.structures* Σ *with* $a(\Sigma) = 4$ *do not exist.*

PROOF. Let $\Sigma \simeq \alpha(\bar{\Sigma}, T, \Xi)$ for some quadruple $\alpha, \bar{\Sigma}, T, \Xi$ with $\dim \alpha = 4$. It is enough to prove that $a(\bar{\Sigma}) = 3$. Let $\bar{\Sigma} = \bar{\Sigma}(\mathcal{O}(\alpha), \varepsilon(\alpha))$. Clearly, $\mathcal{O}(\alpha)$ can contain at most one subspace α_i with $\dim \alpha_i = 2$, because $\bar{\Sigma}$ must have more than one plane.

Suppose that $\dim \alpha_i = 1$, for each $\alpha_i \in \mathcal{O}(\alpha)$. In this case $\varepsilon(\alpha)$ can contain at most one subspace $\mathcal{E}_j$ with $\dim \mathcal{E}_j = 3$. If $\dim \mathcal{E}_j = 2$ for each $\mathcal{E}_j \in \varepsilon(\alpha)$ then $\bar{\Sigma} = \alpha$—with respect to the line-plane structure—and $a(\bar{\Sigma}) = 3$ by Prop. 5.2. Assume that $\dim \mathcal{E}_1 = 3$, $\mathcal{E}_1 \in \varepsilon(\alpha)$. Then, $\varepsilon(\alpha)$ consists of $\mathcal{E}_1$ and all 2-dimensional subspaces of α which meets $\mathcal{E}_1$ in a 1-dimensional subspace. Such a structure $\bar{\Sigma} = \bar{\Sigma}(\mathcal{O}(\alpha), \varepsilon(\alpha))$ is unique—up to an isomorphism.

Suppose that α is an affine space over the field $F = GF(q)$. As in Prop. 5.2, in order to prove that $a(\bar{\Sigma}) = 3$ we shall construct a vector space W—over a overfield F_0 of F—which contains a subgroup V satisfying the following conditions

— V is a 4-dimensional vector space over F,

— $\dim_F W_1 \cap V \leqslant 1$ for each 1-dimensional subspace of W,

— $\dim_F W_2 \cap V \leqslant 2$ for each 2-dimensional subspace of W with exactly one exception, that is there is a 2-dimensional subspace of W which meets V in a 3-dimensional subspace (over F).

Let p_{ij}, $i = 1, \ldots, 4$, $j = 1, \ldots, 4$, be 8 pairwise distinct prime numbers different from 1, and let $F_0 = GF(q^m)$ where $m = \prod_{i,j} p_{ij}$. Assume $W = F_0^3 = \{(\alpha, \beta, \gamma): \alpha, \beta, \gamma \in F_0\}$ and let

$$\mathcal{B}_1 = \{(\alpha_1, \beta_1, 1): \alpha_1 \in GF(q^{p_{11}}) - GF(q), \ \beta_1 \in GF(q^{p_{21}}) - GF(q)\},$$

$$\mathcal{B}_2 = \{(\alpha_j, \beta_j, 0): \alpha_j \in GF(q^{p_{1j}}) - GF(q), \ \beta_j \in GF(q^{p_{2j}}) - GF(q), \ j = 2, 3, 4\}.$$

Denote by V the vector space generated by the elements of $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ over F. As in Prop. 5.2, one can prove that

(17) $$\dim_F V = 4,$$

(18)   if W_1 is a 1-dimensional subspace of W, then $\dim_F W_1 \cap V \leqslant 1$.

Furthermore, the vectors in $\mathcal{B}_2$ generate a 2-dimensional subspace of W (over F_0) which meets V in a 3-dimensional subspace (over F). It is easily seen that no other 2-dimensional subspace of W can meet V in a subspace of $\dim > 2$ by (18).

Now, suppose that $\dim \alpha_1 = 2$ and $\dim \alpha_i = 1$ for each $\alpha_i \in \mathcal{O}(\alpha)$, $i \neq 1$. In this case $\varepsilon(\alpha)$ consists of the 3-dimensional subspaces of α containing α_1 and the 2-dimensional subspaces of α meeting α_1 in 0. Such a structure $\bar{\Sigma} = \bar{\Sigma}(\mathcal{O}(\alpha), \varepsilon(\alpha))$ is unique—up to an isomorphism. The proof

that $a(\overline{\Sigma}) = 3$ is similar to the previous one. Indeed, let p_{1j}, $j = 1, \ldots, 4$, p_{21}, p_{22} be 6 pairwise distinct prime numbers, different from 1, and let $F_0 = GF(q^m)$, where m is the product of these numbers. Put $W = F_0^3$ and let

$$\mathfrak{B}_1 = \{\{\alpha_1, \beta_1, i\}: \alpha_1 \in GF(q^{p_{11}}) - GF(q), \ \beta_1 \in GF(q^{p_{21}}) - GF(q)\},$$

$$\mathfrak{B}_2 = \{(\alpha_2, \beta_2, 0): \alpha_2 \in GF(q^{p_{12}}) - GF(q), \ \beta_2 \in GF(q^{p_{22}}) - GF(q),\},$$

$$\mathfrak{B}_3 = \{(\alpha_j, 0, 0): \alpha_j \in GF(q^{p_{3j}}) - GF(q), \ \ j = 3, 4\},$$

Put $V = \langle \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3 \rangle$. It is easily seen that $\mathfrak{B}_3$ generates the unique 1-dimensional subspace U of W meeting V in a 2-dimensional subspace, while all 2-dimensional subspaces of W containing U meet V in a 3-dimensional subspace. This completes the proof.

In Section 7, we shall see an example of a p.t.structure Σ with $a(\Sigma) = 5$.

6. P.t.structures induced by an affine space.

One may ask under what conditions a p.t.structure $\Sigma = \Sigma(\Pi(T), \varepsilon(T))$
can be represented in the form $\alpha(\alpha, G, \Xi)$ for a suitable affine space α, that is when Σ can be embedded in α so that the line-plane structure of Σ is induced by the line-plane structure of α. No intrinsic characterization of these structures is known to the authors. Nevertheless, the question seems to be difficult also in view of Prop. 5.2, because, in general, there is not a natural dimension for the embedding.

However, this question is strictly related to the more general question of the embedding of a planar linear space in a projective space. There are several results in this connection. In particular, the problem has been solved for the *locally projective semimodular lattices* (e.g. see [5]). The connections between the results on this last subject and the representation of a particular class of p.t.structures deserve some attention.

Suppose that $\mathcal{T} = \mathcal{T}(\Pi(T), \varepsilon(T))$ is a projective space. In this case, $(\Pi(T), \varepsilon(T))$ is called a *projective partition*. If we assume as *flats* of $\Sigma = \Sigma(\Pi(T), \varepsilon(T))$ the subspaces of Σ which meet $\mathcal{T}$ in a subspace of $\mathcal{T}$, then Σ satisfies the exchange property:

(E) *if x, y are points of Σ and U is a flat of Σ then*

$$y \in x \vee U \quad and \quad y \notin U \rightarrow x \in y \vee U,$$

where $x \vee U$ denotes the flat of Σ generated by x and U (see [8], Teorema 2.1).

From a lattice-theoretical point of view, Σ is what is usually called a *locally projective semimodular lattice* (e.g. see [5]). It is well known that if rank $\Sigma \geq 5$, then Σ can be embedded in a projective space. We recall as the embedding can be accomplished following [5].

Let $\Sigma = (\mathscr{P}, \mathscr{L}, \|)$. Two lines L and L' of Σ are *B-parallel*—we use the symbol $\|_B$—if they are coplanar and disjoint. Notice that $L\|L' \to \to L\|_B L'$. Let L, L' be two B-parallel lines of Σ. One can prove that any two distinct planes A, A' of Σ such that $L \subset A$, $L' \subset A'$ either are disjoint or meet in a line. Put

$[L, L'] = \{R: R \in \mathscr{L}, R = A \wedge A'$

$\qquad\qquad$ *for some planes* A, A' *of* Σ *with* $L \subset A$, $L' \subset A'\}$.

The set $[L, L']$ satisfy the following conditions:

(19) $[L, L']$ is a (set-theoretical) partition of the points of Σ,

(20) $[L, L']$ is the unique partition of the points of Σ containing both L and L' and consisting of pairwise B-parallel lines.

So, if $R, R' \in [L, L']$ then $[R, R'] = [L, L']$.

We introduce a new structure $\boldsymbol{P}$ whose point-set $\mathscr{P}^*$ consists of the points of Σ and of the sets $[L, L']$ as defined above.

For any point $x \in \mathscr{P}^*$ and any line $L \in \mathscr{L}$ we put

— $x \mathrel{\text{I}} L \Leftrightarrow x \in L$, for $x \in \mathscr{P}$,

— $x \mathrel{\text{I}} x \Leftrightarrow L \in x$, for $x \in \mathscr{P}^* - \mathscr{P}$.

For any flat S of Σ, we write $x \mathrel{\text{I}} S \Leftrightarrow x \mathrel{\text{I}} L$ for fome line $L \subseteq S$. For distinct planes π, π' of Σ such that rank $\pi \vee \pi' = 4$, define a line $[\pi, \pi']$ of $\boldsymbol{P}$ to be

$$[\pi, \pi'] = \{x: x \in \mathscr{P}^*, x \mathrel{\text{I}} \pi, x \mathrel{\text{I}} \pi'\}.$$

The line-set $\mathscr{L}^*$ of $\boldsymbol{P}$ consists of $\mathscr{L}$ and of these new lines.
The structure $\boldsymbol{P} = (\mathscr{P}^*, \mathscr{L}^*, \mathrel{\text{I}})$ is a projective space.

LEMMA 6.1. *The set* $\mathscr{P}_0^* = \{x: x = [L, L'], L\|L'\}$ *is a hyperplane of* $\boldsymbol{P}$.

PROOF. Notice that $[L, L']$ is a class of parallel lines of Σ by conditions (19) and (20). Let x, y be two points of $\mathscr{P}_0^*$. So, $x = [L, L']$ with $L\|L'$ and $y = [M, M']$ with $M\|M'$. We can suppose that L and M (L' and M') meet in a point and that $L' \not\subseteq L \vee M$. Let $\pi = L \vee M$ and $\pi' = L' \vee M'$. Then it is easily seen that $\pi\|\pi'$, $\pi \cap \pi' = \hbar$ and $[\pi, \pi']$ is the line through x and y.

Suppose that $z \perp [\pi, \pi']$ with $z = [N, N']$, $N \subseteq \pi$, $N' \subseteq \pi'$, and assume that $N\|_B N'$, but $N \not\Diamond N'$. Let $p \in N$, $R \in \mathcal{L}$ such that $p \in R$ and $R\|N'$. We have that $R \subseteq \pi$ since $R\|N' \subseteq \pi'$ and $\pi\|\pi'$. So, $R \vee N = \pi$. Nevertheless, $N \vee N' = R \vee N'$ because $R \subseteq N \vee N'$ as $R\|N'$. Thus, $\pi = R \vee N = N \vee N'$ and hence $N' \subseteq \pi$. This is a contradiction since π and π', are disjoint. Therefore, the line $[\pi, \pi']$ can contain only points of $\mathcal{P}_0^*$. Since every line of Σ lies in some class of parallel lines, then it meets $\mathcal{P}_0^*$ in a point.

LEMMA 6.2. *Any collineation of $\Sigma = \Sigma(\Pi(T), \varepsilon(T))$ preserving the planes of Σ extends to a collineation of $\mathbf{P}$ in a unique way. In particular, any translation of Σ extends to a collineation of P fixing the hyperplane $\mathcal{P}_0^*$ pointwise.*

PROOF. Let σ be any collineation of Σ preserving the planes and let L and L' be two B-parallel lines of Σ. The set $[L, L']$ is mapped by σ into a set of pairwise B-parallel lines as σ preserves the planes. So, $[L, L']\sigma = [L\sigma, L'\sigma]$ and hence σ may be well-defined on the points of $\mathcal{P}^*$. Likewise, $[\pi, \pi']\sigma = [\pi\sigma, \pi'\sigma]$. It is straightforward to show that σ preserves the incidence relation $\perp$ and hence σ extends to a collineation of $\mathbf{P}$. Using Lemma 6.1, we obtain the latter assertion.

From an affine point of view, the previous results may be reworded as follows.

PROPOSITION 6.3. *Let $\Sigma = \Sigma(\Pi(T), \varepsilon(T))$ be a p.t.structure such that $\mathcal{K}(\Pi(T), \varepsilon(T)) \cong PG(n-1, q)$ with $n \geq 4$. Then the group $\overline{T}$ is—up to isomorphism—a dilatation group of the affine space $\mathcal{A} = AG(n, q)$ and there is a regular point-orbit Ξ of $\overline{T}$ on $\mathcal{A}$ such that Σ is isomorphic to the p.t.structure $\mathcal{A}(\mathcal{A}, \overline{T}, \Xi)$ induced by $\mathcal{A}$ on the orbit Ξ of $\overline{T}$.*

Conversely, as we have seen in Section 4, if $\mathcal{A}$ is an affine space, G is a dilatation group of $\mathcal{A}$ and Ξ is a regular point-orbit of G, one can define the p.t.structure $\mathcal{A}(\mathcal{A}, G, \Xi)$ induced by $\mathcal{A}$ in the orbit Ξ of G. The following proposition shows under what conditions on G the geometry at infinity of $\mathcal{A}(\mathcal{A}, G, \Xi)$ coincides with that of $\mathcal{A}$.

PROPOSITION 6.4. *Let $\mathcal{A} = AG(n, q)$, $n \geq 3$, be an affine space. Let G be a dilatation group of $\mathcal{A}$ and let Ξ be a point-orbit of G. The p.t.structure $\mathcal{A}(\mathcal{A}, G, \Xi)$ has $PG(n-1, q)$ as the geometry at infinity if and only if one of the following holds*

(21) *G is a translation group and $|G| > q^{n-1}$*

(22) *G contains some proper dilatation and if K denotes the translation subgroup of G, then $|K| \geq q^{n-1}$.*

PROOF. Let x be any point in Ξ. The p.t.structure $\mathcal{A}(\mathcal{A}, G, \Xi)$ has $PG(n-1, q)$ as the geometry at infinity if and only if $G_L \neq \langle 1 \rangle$ for every line L of $\mathcal{A}$ through x. Suppose that G is a translation group and that $G_L \neq \langle 1 \rangle$ for every line L of $\mathcal{A}$ through x. Since there are $(q^n - 1)/(q - 1)$ lines of $\mathcal{A}$ through x we must have

$$|G| > (q^n - 1)/(q - 1) > q^{n-1}.$$

Conversely, if $|G| > q^{n-1}$ then $G_L \neq \langle 1 \rangle$ for each line L of $\mathcal{A}$ since G fixes the class $[L]$ consisting of the lines of $\mathcal{A}$ which are parallel to L and $|[L]| = q^{n-1}$.

Suppose that G is a Frobenius group with kernel K, where K is the translation subgroup of G. Assume that $G_L \neq \langle 1 \rangle$ for every line L of $\mathcal{A}$ through x. Then $|G| > q^{n-1}$. Thus $G_L \neq \langle 1 \rangle$ for each line L of $\mathcal{A}$. Let $\Gamma(G)$ be the exceptional point-orbit of G. For what we have seen above, if $|K| > q^{n-1}$ there exists a line L through a point of $\Gamma(G)$ such that $K_L = \langle 1 \rangle$. This yields $K_M = \langle 1 \rangle$ for each line M in $[L]$. Since $G_M \neq \langle 1 \rangle$ there exists a proper dilatation $\sigma \in G$ fixing M. The centre of σ lies in $\Gamma(G)$. So, M meets $\Gamma(G)$ in a point. Thus, each line of $[L]$ has a common point with $\Gamma(G)$. This yields $|\Gamma(G)| \geq q^{n-1}$, contrary to the assumption $|K| < q^{n-1}$ as K acts transitively on $\Gamma(G)$. Conversely, if $|K| \geq q^{n-1}$, then $|G| > q^{n-1}$ and hence $G_L \neq \langle 1 \rangle$ for each line L of A.

For what concerns the Sperner spaces, we have the following proposition which does not require that the projective dimension of the geometry at infinity is greater than 2.

PROPOSITION 6.5. *Let* $(\Pi(T), \varepsilon(T))$ *be a projective partition. If* $\Sigma(\Pi(T))$ *is a Sperner space then* $\Sigma(\Pi(T))$ *is an affine space and the elements of* $\Pi(T)$ $(\varepsilon(T))$ *are the 1-dimensional (2-dimensional) subspaces of* $\Sigma(\Pi(T))$.

PROOF. It is well known [11] that if $\Sigma(\Pi(T), \varepsilon(T))$ is a Sperner space, then T is a p-group and hence T is elementary abelian. Let $|T_i| = p^h$ for $T_i \in \Pi(T)$ and assume that $\mathcal{T}(\Pi(T), \varepsilon(T))$ has order q^t for q a prime. Pick any two distinct components $T_1, T_2 \in \Pi(T)$. If $|T_1 \vee T_2| = p^k$, then the following relation holds

(23) $$(p^k - 1)/(p^h - 1) = q^t + 1.$$

So, $p = q$ and relation (23) yields

(24) $$p^h(p^{k-h} - 1) = p^t(p^h - 1).$$

Therefore, $h = t$ and $k = 2h$. Thus $|T_1 \vee T_2| = p^{2h}$ and hence $T_1 \vee T_2 = T_1 \oplus T_2$ for each pair T_1, T_2 of components of $\Pi(T)$. So, $\Pi(T)$

is a geometric partition (see [4]) and $\Sigma(\Pi(T), \varepsilon(T))$ is an affine space over $GF(p^h)$ since the projective dimension of $\mathcal{T}(\Pi(T), \varepsilon(T))$ is greater than 1—recall that $|\varepsilon(T)| > 1$ by (1).

7. Example of linear partitions.

As we have seen, each linear partition is obtained starting from a linear partition of an abelian group. Nevertheless, it seem very difficult to carry out any systematic investigation on the linear partitions of elementary abelian groups. A very large class of p.t.structures over an abelian group is given by the p.t.structures $\mathcal{C}(\mathcal{C}, T, \Xi)$, where $\mathcal{C}$ is an affine space, T a translation group of $\mathcal{C}$ and Ξ a regular orbit of T on $\mathcal{C}$. Such p.t.structures may be or not associated to a projective partition. Certainly, there exist p.t.structures which are not of the form $\mathcal{C}(\mathcal{C}, T, \Xi)$. Below, we produce a class of such structures by extending a construction of Herzer (see [9], Theorem 2).

EXAMPLE 7.1.   Let V_1 be a linear partition of a vector space V over a field F and assume that every component of Π is a subspace of V—here, we assume that $\varepsilon(T)$ can also consist of a unique element. Let V_1 be a vector space over F such that $v \subset V_1$ and $\dim_F V_1/V = 1$. Put

$$\Pi_1 = \{W_i \colon W_i \in \Pi\} \cup \{wF \colon w \in V_1 - V\},$$

$$\varepsilon_1 = \{E_j \colon E_j \in \varepsilon\} \cup \{W_i \oplus wF \colon W_i \in \Pi, \; w \in V_1 - V\}.$$

It is very easy to see that (Π_1, ε_1) is a linear partition of V_1. Furthermore, starting from (Π_1, ε_1) we may repeat the same construction, thus obtaining (Π_2, ε_2) and so on.

Let start from a linear partition (Π, ε) such that $\Sigma = \Sigma(\Pi, \varepsilon)$ is a non-desarguesian translation plane (see [1]). In this case Σ is the unique plane in ε. The p.t.structure $\Sigma_n = \Sigma(\Pi_n, \varepsilon_n)$, $n \geq 1$, contains a plane which is isomorphic to Σ. Now, suppose that $\Sigma_n \simeq \mathcal{B}(\mathcal{B}, T, \Xi)$ for some affine space $\mathcal{B}$. Since rank $\mathcal{B} \geq 3$ because Σ_n has more than one plane, we realized an embedding of Σ, as an affine subplane, in a desarguesian affine plane. This is absurd.

The same argument shows that $a(\Sigma_n) > 3$. So, $a(\Sigma_n) \geq 5$ by Prop. 5.3. In particular, if $\Sigma = \Sigma(\Pi, \varepsilon)$ is a translation plane with $g(\Sigma) = 4$—that is Σ has dimension 2 over its kernel F in the usual terminology for translation planes—then $\Sigma_1 = \Sigma(\Pi_1, \varepsilon_1)$ gives an example of a basic p.t.structure with $a(\Sigma_1) = 5$.

As we have seen, when $(\Pi(T), \varepsilon(T))$ is a projective partition and the

projective dimension of $\mathfrak{I}(\Pi(T), \, \varepsilon(T))$ is greater than 2, then $\Sigma(\Pi(T), \, \varepsilon(T))$ is—up to an isomorphism—of the form $\mathfrak{Cl}(\mathfrak{Cl}, T, \Xi)$ for a suitable affine space $\mathfrak{Cl}$. When dim $\mathfrak{I}(\Pi(T), \, \varepsilon(T)) = 2$ this does not hold. A counterexample is given by the structure $\Sigma(\Pi_1, \, \varepsilon_1)$ in the Example 7.1, when $\Sigma(\Pi, \varepsilon)$ is a non-desarguesian affine plane. In this case, $\mathfrak{I}(\Pi_1, \, \varepsilon_1)$ is isomorphic to the projective extension of $\Sigma(\Pi, \varepsilon)$.

Similar examples can be constructed for p.t.structures over a Frobenius group using the results of Section 4. Nevertheless, they also appear in [9].

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