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Chain conditions and continuous mappings on $C_p(X)$

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Chain Conditions and Continuous Mappings on $C_p(X)$.

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ABSTRACT - Let X, Y Tychonoff spaces and ∂: $C_p(X) \to C_p(Y)$ a one-to-one, continuous linear mapping. We prove that if Y satisfies a certain kind of chain conditions (caliber, c.c.c. e.t.c.) so does X. As a consequence of this, we prove $\{0, 1\}^\tau$ (τ regular) cannot be embedded into $C_p(X)$, if X has τ caliber. More generally, we prove that if X has τ caliber then $C_p(X)$ does not contain compact subspaces of weight τ. It follows, subject to GCH, that if B is a Banach space and (B, w) has ω_1 and ω_2 calibers then B is separable. Finally we prove that $C_p(X)$ with X dyadic of weight τ (of uncountable cofinality) does not admit a strictly positive measure.

All topological spaces are assumed to be infinite Tychonoff spaces. In the notations and terminology left unexplained below, we follow [4]. The symbols X, Y, Z always denote spaces and the symbols τ, λ denote infinite cardinals. The cofinality of a cardinal τ, denoted by $\mathrm{cf}\,\tau$, is the least ordinal β, such that τ is the cardinal sum of β many cardinals each smaller than τ. A cardinal τ is regular if $\tau = \mathrm{cf}\,\tau$. The symbol $\mathbb{N}$ stands for the set of all positive integers and the symbols k, l, m, n are used only to denote members of $\mathbb{N}$. Further d is the density, w is the weight and $|\cdot|$ is the cardinality. A space X satisfies τ.c.c. if there is no family $\gamma \subset \mathcal{T}^*(X)$ (the set of all non-empty, open subsets of X) of pairwise disjoint elements with $|\gamma| = \tau$. We set c.c.c. for ω_1.c.c.. A space X has (τ, λ) caliber (pre-caliber) if for every family $\gamma \subset \mathcal{T}^*(X)$ with $|\gamma| = \tau$ there is a subfamily $\gamma_1 \subset \gamma$ with $|\gamma_1| = \lambda$ and $\bigcap \gamma_1 \neq \emptyset$ (γ_1 is centered). We set τ caliber for (τ, τ) caliber. A space X satisfies property K_τ if for every family $\gamma \subset \mathcal{T}^*(X)$ with $|\gamma| = \tau$, there is a subfamily $\gamma_1 \subset \gamma$ with $|\gamma_1| = \tau$ with the 2-intersection property. It is well known that if X has

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τ caliber then X has also $\mathrm{cf}\,\tau$ caliber, and so $\mathrm{cf}\,\tau > \omega$. If $F = \{U_1, \ldots, U_n\}$ is a non-empty, finite subfamily of $\mathcal{T}^*(X)$, then $\mathrm{cal}(F)$ is the largest κ such that, there is $S \subset F$ with $|S| = \kappa$ and $\cap S \neq \emptyset$. If $J \subset \mathcal{T}^*(X)$ then $\kappa(J) = \inf\{\mathrm{cal}(F)/|F|: F \subset J, \text{ finite}\}$. A space X satisfies property (**) if $\mathcal{T}^*(X)$ can be written in the form, $\mathcal{T}^*(X) = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n$, with $\kappa(\mathcal{T}_n) > 0$, for all $n = 1, 2, \ldots$. A space X admits a strictly positive measure, if there is a non-negative Borel probability measure μ on X with $\mu(U) > 0$ for all non-empty open U. For a compact space, Kelley (see [4]), proved that property (**) is equivalent with the existence of a strictly positive measure on the space. It is well known that if X admits a strictly positive measure then it has (τ, ω) caliber for every cardinal τ with $\mathrm{cf}\,\tau > \omega$, [4].

If X is a space, $C_p(X)$ is the space of all real-valued continuous functions on X with the topology of pointwise convergence. For different points $x_1, \ldots, x_\kappa$ in X and $E_1, \ldots, E_\kappa$ non-empty, open intervals of $\mathbb{R}$, let

$$V(x_1, \ldots, x_\kappa: E_1, \ldots, E_\kappa) = \{f \in C_p(X): f(x_i) \in E_i, \text{ for } i = 1, \ldots, \kappa\}.$$

It is clear that $V(x_1, \ldots, x_\kappa: E_1, \ldots, E_\kappa)$ form a base of $C_p(X)$. It is well known that $C_p(X)$ is a dense subspace of $\mathbb{R}^{|X|}$ the set of all real-valued functions on X with the topology of pointwise convergence. It follows from well known properties of $\mathbb{R}^{|X|}$, that $C_p(X)$ has pre-caliber τ, for every cardinal τ with $\mathrm{cf}\,\tau > \omega$ and also satisfies property (**). In the case of a compact space X with $w(X) = \tau$ and $\mathrm{cf}\,\tau > \omega$, Arhangel'skii and Tkačuk in [3], proved that $C_p(X)$ does not have $\mathrm{cf}\,\tau$ caliber and also by a result of Tulcea [9], it follows that $C_w(X)$ (the space $C(X)$ with the weak topology) does not admit a strictly positive measure, although it satisfies property (**).

For $A \subset X$ and $f \in C_p(X)$ we set $f|_A$ for the restriction of f on A, and $\mathrm{supp}\, f = \{x \in X: f(x) \neq 0\}$ for the support of f.

THEOREM 1. Let $\partial: C_p(X) \to C_p(Y)$ be a 1-1, continuous mapping. Then we have the following:

(*a*) $d(X) \leqslant d(Y)$,

(*b*) Let τ, λ be cardinals with τ regular, $\tau \geqslant \lambda$ and $\mathrm{cf}\,\lambda > \omega$.

We suppose that Y has (τ, λ) caliber. Then X has (τ, λ) caliber.

PROOF. (*a*) We can suppose that $\partial(0) = 0$. Let D be a dense subset of Y. For every $y \in D$ and $n \in \mathbb{N}$, it follows from the continuity of ∂ at $0 \in C_p(X)$ that there exist $x_1^{y,n}, \ldots, x_{\kappa_{y,n}}^{y,n}$, pairwise different elements of

X, and $E_1^{y,n}, \ldots, E_{\kappa_{y,n}}^{y,n}$ open intervals in $\mathbb{R}$ containing $0 \in \mathbb{R}$, such that

$$\partial(V(x_1^{y,n}, \ldots, x_{\kappa_{y,n}}^{y,n}: E_1^{y,n}, \ldots, E_{\kappa_{y,n}}^{y,n})) \subset V\left(y: \left(-\frac{1}{n}, \frac{1}{n}\right)\right).$$

We claim that the set $A = \bigcup_{y \in D} \bigcup_{n \in \mathbb{N}} \{x_1^{y,n}, \ldots, x_{\kappa_{y,n}}^{y,n}\}$ is dense in X and since $|A| \leqslant |D|$, (a) follows. Indeed, let $x_0 \in X \setminus \overline{A}$, then there exists $f \in C_p(X)$ with $f(x_0) \neq 0$ and $f|_{\overline{A}} = 0$. But then $f|_{\{x_1^{y,n}, \ldots, x_{\kappa_{y,n}}^{y,n}\}} = 0$ for every $y \in D$ and $n \in \mathbb{N}$ and so $\partial(f)|_D = 0$ so $\partial(f) = 0$. This is contradiction since ∂ is one-to-one.

(b) Let $\{U_i\colon i < \tau\} \subset \mathcal{T}^*(X)$. For every $i < \tau$ we choose $f_i \in C_p(X)$ with $f_i \neq 0$ and $\operatorname{supp} f_i \subset U_i$ and set $V_i = \{y \in Y\colon \partial(f_i)(y) \neq 0\}$. From the regularity of τ, it follows that either exists τ V_i's equal elements or τ pairwise different. In both cases, since Y has (τ, λ) caliber, it follows that there exists $\Lambda \subset \tau$, $|\Lambda| = \lambda$ and $y_0 \in \bigcap \{V_i\colon i \in \Lambda\}$. Since $\operatorname{cf} \lambda > \omega$ and $\partial(f_i)(y_0) \neq 0$ for every $i \in \Lambda$ it follows that either there exists $\Lambda_1 \subset \Lambda$, $|\Lambda_1| = \lambda$ and $r_1 > 0$ such that $\partial(f_i)(y_0) \geqslant r_1$, for every $i \in \Lambda_1$, or $\Lambda_2 \subset \Lambda$, $|\Lambda_2| = \lambda$ and $r_2 < 0$ such that $\partial(f_i)(y_0) \leqslant r_2$ for every $i \in \Lambda_2$. We can suppose that we have the first. From the continuity of ∂ at $0 \in C_p(X)$ there exist $x_1, \ldots, x_\kappa$ pairwise different elements of X, and $E_1, \ldots, E_\kappa$ open intervals of $\mathbb{R}$ containing $0 \in \mathbb{R}$, such that

$$\partial(V(x_1, \ldots, x_\kappa\colon E_1, \ldots, E_\kappa)) \subset V(y_0\colon (-r_1, r_1)).$$

Then $f_i \notin V(x_1, \ldots, x_\kappa\colon E_1, \ldots, E_\kappa)$ for every $i \in \Lambda_1$ and so $\{x_1, \ldots, x_\kappa\} \cap U_i \neq \emptyset$ for every $i \in \Lambda_1$. Now (b) follows immediately.

REMARK. We note that the (a) of the above theorem follows also from well known results [7]. We also note that in the (b) of the above theorem the assumption that Y has (τ, λ) caliber cannot be relaxed to have pre-caliber. Indeed, since $C_p(X)$ is contained homeomorphically into $C_p(C_p(C_p(X)))$ and $C_p(C_p(X))$ has τ precaliber if $\operatorname{cf} \tau > \omega$, however X in general does not satisfy c.c.c..

COROLLARY 2. Let τ be an uncountable regular cardinal. Then X has τ caliber if and only if $C_p(C_p(X))$ has τ caliber. In particular the space $C_p(\mathbb{R}^\tau)$ has not τ caliber.

PROOF. From results in [8], follows that $C_p(C_p(X)) = \overline{\bigcup_{n \in \mathbb{N}} P_n}$, where P_n is a continuous image of the space $X^n \times \mathbb{R}^m$. If X has τ caliber, it follows that each P_n has τ caliber and so the space

$C_p(C_p(X))$. The «if» part follows from the fact that $C_p(X)$ is contained homeomorphically into $C_p(C_p(C_p(X)))$ and Th. 1(b).

COROLLARY 3. Let X be a space with τ caliber, τ regular. Let, also, $\delta_i \in \mathbb{R}^\tau$ with $\delta_i = (\delta_{ij})_{j<\tau}$, $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. Then there is no, one-to-one continuous mapping from the compact subspace $\{\delta_i\colon i < \tau\} \cup \{0\}$ of $\mathbb{R}^\tau$ into $C_p(X)$ and so there is no, one-to-one, continuous mapping from $\{0, 1\}^\tau$ into $C_p(X)$.

PROOF. On the discrete space τ, we consider the family $\{\{i\}\colon i < \tau\}$ and the set of continuous functions $\{\delta_i\colon i < \tau\}$ and repeat the argument of Th. 1(b).

REMARK. In the case that X is compact the above corollary follows also from well known arguments. Indeed in the case that X is compact and has τ caliber, if there exists $\partial\colon \{\delta_i\colon i < \tau\} \cup \{0\} \to C_p(X)$, a one-to-one continuous mapping then $\partial(\{\delta_i\colon i < \tau\} \cup \{0\})$ would be a compact subspace of $C_p(X)$, of weight τ, contradiction (see [3]). Also, if X is compact and $\{0, 1\}^\tau \subsetneq C_p(X)$ homeomorphically then $\{0, 1\}^\tau$ would be Eberlein compact and since it satisfies c.c.c., would be metrizable [6].

In connection with the above we prove the following stronger result.

THEOREM 4. Let X be a space and we suppose that there exists some $F \subset C_p(X)$ compact with $w(F) = \tau$ and $\operatorname{cf}\tau > \omega$. Then X has not $\operatorname{cf}\tau$ caliber.

PROOF. Let $\{\mu_j\colon j < \tau\}$ be a $\|\,\|$-dense subset of $C(F)$. We claim that for every $i < \tau$, there exist $f_i, g_i \in F$, $f_i \neq g_i$ and $\mu_j(f_i) = \mu_j(g_i)$ for all $j < i$. This follows easily from Stone-Weierstrass Theorem. Since $f_i \neq g_i$ there exist $r_i \in \mathbb{Q}$, $\delta_i > 0$ such that either

$$f_i^{-1}(-\infty, r_i) \cap g_i^{-1}(r_i + \delta_i, +\infty) \neq \emptyset,$$

or

$$g_i^{-1}(-\infty, r_i) \cap f_i^{-1}(r_i + \delta_i, +\infty) \neq \emptyset.$$

Since $\operatorname{cf}\tau > \omega$, without loss of generality, we can suppose that there exist $A \subset \tau$, $|A| = \tau$ and $r \in \mathbb{Q}$, $\delta > 0$ such that

$$U_i = f_i^{-1}(-\infty, r) \cap g_i^{-1}(r + \delta, +\infty) \neq \emptyset, \qquad \text{for every } i \in A.$$

Let $\{i_\eta\colon \eta < \operatorname{cf}\tau\} \subset A$ with $i_\eta < i_{\eta'}$, if $\eta < \eta' < \operatorname{cf}\tau$ and $\sum_{\eta < \operatorname{cf}\tau} i_\eta = \tau$. We suppose, if possible, that X has $\mathrm{cf}\,\tau$ caliber. Then there is a cofinal $B \subset \{i_\eta : \eta < \mathrm{cf}\,\tau\}$ with $|B| = \mathrm{cf}\,\tau$ and $\bigcap\{U_i : i \in B\} \neq \emptyset$. In case that there are no $\mathrm{cf}\,\tau$ pairwise different elements in $\{U_{i_\eta} : \eta < \mathrm{cf}\,\tau\}$, this follows immediately. Otherwise this follows from the assumption that X has $\mathrm{cf}\,\tau$ caliber. Let $x \in \bigcap\{U_i : i \in B\}$. Then $\delta_x \in C(F)$ and so there exists $i_0 < \tau$ such that

$$\|\mu_{i_0} - \delta_x\| < \frac{\delta}{4}.$$

If $i \in B$ with $i > i_0$ we have $\mu_{i_0}(f_i) = \mu_{i_0}(g_i)$ and so $|f_i(x) - g_i(x)| < \delta/2$ contradiction, since $f_i(x) \in (-\infty, r)$ and $g_i(x) \in (r + \delta, +\infty)$.

COROLLARY 5. (Arhangel'skii and Tkačuk, [3]). Let X be a compact space with $w(X) = \tau$ and $\mathrm{cf}\,\tau > \omega$. Then $C_p(X)$ does not have $\mathrm{cf}\,\tau$ caliber.

PROOF. It follows from the fact that X is contained homeomorphically into $C_p(C_p(X))$ and Th. 4.

COROLLARY 6. Suppose that $2^{\omega_1} = \omega_2$. Then we have the following:

(*a*) If X has ω_1 and ω_2 calibers, every compact $F \subset C_p(X)$ is metrizable.

(*b*) If X is compact and $C_p(X)$ has ω_1 and ω_2 calibers then X is metrizable.

PROOF. (*a*) We claim that F is separable. If not, there exists $\{f_i : i < \omega_1\} \subset F$ such that $f_j \notin \overline{\{f_i : i < j\}}$. Then $w(\overline{\{f_i : i < \omega_1\}}) \leq 2^{\omega_1} = \omega_2$ and so $w(\overline{\{f_i : i < \omega_1\}}) = \omega_1$ or ω_2 contradiction by Corol. 5. Therefore F is separable and so $w(F) \leq 2^{\omega} \leq 2^{\omega_1} = \omega_2$. It follows as before from Corol. 5 that F is metrizable.

(*b*) It follows from the fact $X \subsetneq C_p(C_p(X))$ homeomorphically and (*a*).

NOTE 1. I have recent information that Th. 4 follows also Corol. 5 and results in [8].

NOTE 2. The (*b*) of the above corollary has already been proved in [3].

NOTE 3. The above theorem is not valid if the assumption τ caliber is relaxed to pre-caliber. Indeed, in general $X \subsetneq C_p(C_p(X))$ and $C_p(X)$ has τ pre-caliber if $\mathrm{cf}\,\tau > \omega$, although X may have weight τ.

THEOREM 7 (GCH). If B is a Banach space, such that the space (B, w) has ω_1 and ω_2 calibers, then B is separable.

PROOF. It is well known that (S_{B^*}, w^*) the unit ball of B^* with the w^*-topology is contained homeomorphically into $C_p(B, w)$, and also that B is contained isometrically into $C(S_{B^*}, w^*)$. The result follows from Corol. 6.

For a set Γ we set $\Sigma(\mathbb{R}^\Gamma) = \{t \in \mathbb{R}^\Gamma : \{\gamma : t(\gamma) \neq 0\}$ is countable$\}$ with the relative topology in $\mathbb{R}^\Gamma$.

PROPOSITION 7. We assume that $C_p(X)$ has ω_1 caliber and there exists $\partial: C_p(X) \to \Sigma(\mathbb{R}^\Gamma)$, a 1-1, continuous mapping. Then X is separable.

PROOF. We claim that there exists a countable $A \subset \Gamma$ such that $\partial(f)(\gamma) = 0$ for every $\gamma \in \Gamma \setminus A$. Indeed, if not, there exists γ_ξ, $\xi < \omega_1$ in Γ and f_ξ, $\xi < \omega_1$ in $C_p(X)$ with $\partial(f_\xi)(\gamma_\xi) > 0$ for every $\xi < \omega_1$. We may suppose that $\partial(f_\xi) \geqslant r$ for some $r \in \mathbb{R}$. Since $C_p(X)$ has ω_1 caliber there exist $B \subset \omega_1$, $|B| = \omega_1$ and $f \in C_p(X)$ such that $\partial(f) \in \bigcap_{\xi \in B} V(\gamma_\xi : (r, +\infty))$ contradiction, because $\partial(f) \in \Sigma(\mathbb{R}^\Gamma)$.

It follows that the continuous mapping $f \to \partial(f)|_A$ remains 1-1, and the result follows from Th. 1(a).

In Corol. 2 we have that $C_p(\mathbb{R}^\tau)$ has not τ caliber, if τ is an uncountable regular cardinal. In connection with this we have the following stronger result.

PROPOSITION 8. Let τ be a cardinal with $\mathrm{cf}\,\tau > \omega$, then the space $C_p(\mathbb{R}^\tau)$, does not have (τ, ω) caliber (and so it does not admit a strictly positive measure).

PROOF. For every $i < \tau$, let $\delta_i \in \mathbb{R}^\tau$ with $\delta_i = (\delta_{ij})_{j<\tau}$, and $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. Then the family

$$V(\delta_i, \delta_{i+1} : (0, 1), (2, 3)), \qquad i < \tau$$

does not contain an infinite subfamily with non-empty intersection. Indeed, let $A \subset \tau$ and $f \in \bigcap_{i \in A} V(\delta_i, \delta_{i+1} : (0,1), (2,3))$. We set $0 = (0)_{i<\tau}$.

We may assume that $f(0) \notin (0,1)$. We consider $r > 0$ with $(f(0) - r, f(0) + r) \cap (0, 1) = 0$. From the continuity of f at 0 there exist $i_1, \ldots, i_\kappa$ in τ and $I_1, \ldots, I_\kappa$ open intervals of $\mathbb{R}$, containing 0 such that $f(\pi_{i_1}^{-1}(I_1) \cap \ldots \cap \pi_{i_\kappa}^{-1}(I_\kappa)) \subset (f(0) - r, f(0) + r)$. If $i \in A \setminus \{i_1, \ldots, i_\kappa\}$ then $\delta_i \in \pi_{i_1}^{-1}(I_1) \cap \ldots \cap \pi_{\kappa}^{-1}(I_\kappa)$ and so $f(\delta_i) \notin (0, 1)$, contradiction.

COROLLARY 9. Let X be a dyadic space with $w(X) = \tau$ and $\mathrm{cf}\,\tau > \omega$. Then $C_p(X)$ does not have (τ, ω) caliber, so it does not admit a strictly positive measure.

PROOF. By a result of Efimov, [5], it follows that $\{0, 1\}^\tau \subsetneq X$. If we repeat the proof of Prop. 8 we can prove that $C_p\{0, 1\}^\tau$ does not have (τ, ω) caliber. Now the mapping ∂: $C_p(X) \to C_p(\{0, 1\}^\tau)$ with $\partial(f) = f|_{\{0,1\}^\tau}$, is continuous, linear and onto. So if $C_p(X)$ had (τ, ω) caliber, then $C_p(\{0, 1\}^\tau)$ would have (τ, ω) caliber, contradiction.

The above theorem gives a partial answer to the following.

PROBLEM. Is there a non-metrizable compact Hausdorff space such that $C_p(X)$ has a strictly positive measure?

D. Fremlin in note of 10 Oct. 1989 proved that this problem is connected with the following

PROBLEM. (A. Bellow). Is there a probability space (Z, Σ, ν) with a $Y \subset \mathcal{L}^0(\Sigma)$ (= the space of real-valued measurable functions on Z) such that Y is compact and non-metrizable in the topology of pointwise convergence and any pair of distinct members of Y differ on a non-negligible set?

THEOREM 10. Let ∂: $C_p(X) \to C_p(Y)$ be a 1-1, continuous, linear mapping and τ be an uncountable regular cardinal. Then we have the following implications.

(*a*) If Y has (τ, ω) caliber, so does X.

(*b*) If Y admits a strictly positive measure then X has (τ, ω) caliber and satisfies property K_τ.

PROOF. (*a*) Let $\{U_i: i < \tau\}$ be a family of non-empty open sets in X. For every $i < \tau$ we find $f_i \in C_p(X)$ with $\mathrm{supp} f_i \subset U, f_i \neq 0$ and set $V_i = \{y \in Y: \partial(f_i)(y) \neq 0\}$. From the regularity of τ, it follows that either there exist τ V_i's equal elements, or τ pairwise different. In both cases there exists $\Lambda \subset \tau$, infinite and $y_0 \in \bigcap\{V_i: i \in \Lambda\}$. Then there exist $x_1, \ldots, x_\kappa$ pairwise different elements in X, and $E_1, \ldots, E_\kappa$ open intervals of $\mathbb{R}$ each containing 0 such that

$$\partial(V(x_1, \ldots, x_\kappa: E_1, \ldots, E_\kappa)) \subset V(y_0: (-1, 1)).$$

We claim that if $f|_{\{x_1, \ldots, x_\kappa\}} = 0$ then $\partial(f(y_0) = 0$. Indeed, if $\partial(f)(y_0) \neq 0$, there is a $\lambda \in \mathbb{R}$ such that $\partial(\lambda f)(y_0) = \lambda\partial(f)(y_0) \notin (-1, 1)$, by the linearity of ∂, but $\lambda f \in V(x_1, \ldots, x_\kappa: O_1, \ldots, O_\kappa)$, contradiction.

(*b*) If Y admits a strictly positive measure μ, then Y has (τ, ω) caliber and so X has (τ, ω) caliber by (*a*).

In the following we shall prove that X satisfies property K_τ. We suppose, if possible, that there exists a family $\{U_i\colon i < \tau\}$ of non-empty, open subsets of X which does not contain subfamily of the same cardinality with the 2-intersection property. Let f_i, V_i, $i < \tau$ as in (*a*). We can suppose that $\mu(V_i) \geqslant \delta$ for all $i < \tau$, for some $\delta > 0$.

For every $A \subset \tau$, $|A| = \tau$ we set

$$\mathcal{P}_A = \{B \subset A\colon \text{the family } \{U_i\colon i \in B\} \text{ has the 2-int-property}\}.$$

The set $\mathcal{P}_A$ is non-empty by (*a*), partially ordered by inclusion and satisfies the assumptions of Zorn's Lemma. Let B_A be maximal. Then $|B_A| < \tau$. For every $i \in A \setminus B_A$ there exists $j_i \in B_A$ with $U_i \cap U_{j_i} = 0$. Then

$$A = B_A U \bigcup_{j \in B_A} \{i \in A\colon j_i = j\}.$$

From the regularity of τ, there exists $j_1 \in B_A$, such that the set $A_1 = \{i \in A\colon j_i = j_1\}$ has cardinality τ. We repeat the same argument with A_1 in place of A.

Inductively we find $j_1, j_2, \dots, j_n, \dots$ pairwise different elements of τ, such that $U_{j_l} \cap U_{j_m} = \emptyset$, for every $l \neq m$, $l, m = 1, 2, \dots$. Now since $\mu(V_{i_l}) \geqslant \delta$, $l = 1, 2, \dots$ it follows that there exists a $B \subset \mathbb{N}$, infinite with $\bigcap_{l \in B} V_{j_l} \neq 0$. Now similar arguments as in (*a*) lead to contradiction.

COROLLARY 11. Let τ be an uncountable regular cardinal. If X admits a strictly positive measure, there is no, 1-1, linear continuous mapping from $\mathbb{R}^\tau$ into $C_p(X)$.

REMARK. Let X, Y be compact spaces and $\partial\colon C_p(X) \to C_p(Y)$ be a one-to-one, continuous linear mapping, then the mapping $\partial\colon (C(X), \|\ \|) \to (C(Y), \|\ \|)$ is also continuous (see Arhangel'skii [2]). However the existence of a 1-1, continuous linear mapping $\partial\colon (C(X), \|\ \|) \to (C(Y), \|\ \|)$ does not imply the existence of a 1-1, continuous $\phi\colon C_p(X) \to C_p(Y)$. By Dixmier's Theorem $L^\infty[0, 1] = C(\Omega)$ where Ω is a compact extremelly disconnected space. On the other hand $L^\infty[0, 1]$ is isomorphic to $C(\beta\mathbb{N})$. The space Ω is not separable, so from Th. 1(*a*) there exists no a one-to-one, continuous $\phi\colon C_p(\Omega) \to C_p(\beta\mathbb{N})$.

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