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Hölder regularity in non autonomous degenerate abstract parabolic equations

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Hölder Regularity in Non Autonomous Degenerate Abstract Parabolic Equations.

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ABSTRACT - We prove optimal Hölder regularity results for a class of non-autonomous degenerate parabolic equations, in general Banach space.

0. Introduction.

We consider a parabolic evolution equation in general Banach space X:

$$(0.1) \qquad \begin{cases} u'(t) = \varphi(t) A(t) u(t) + f(t), & 0 < t < T, \\ u(0) = u_0 . \end{cases}$$

Here «parabolic» means that for every $t \in [0, T]$, the operator $A(t)$: $D(A(t)) \subset X \rightarrow X$ generates an analytic semigroup in X. The domains $D(A(t))$ may possibly be not constant and not dense in X. We assume that the family $\{A(t)\colon 0 \leq t \leq T\}$ satisfies some conditions guaranteeing that there exists an evolution operator for problem

$$(0.2) \qquad \begin{cases} v'(t) = A(t) v(t), & 0 < t \leq T, \\ v(0) = v_0 . \end{cases}$$

The function $\varphi\colon [0, T] \rightarrow \mathbb{R}$ is continuous and nonnegative, and it is allowed to vanish at $t = 0$ and at $t = T$. Therefore (0.1) is an abstract degenerate parabolic initial value problem. We assume that

$$\varphi(t) = O(t^{\beta}) \quad \text{as } t \rightarrow 0,$$

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and

$$\varphi(t) = O(T - t)^{\beta_1} \quad \text{as } t \to T, \text{ with } \beta, \beta_1 \geq 0.$$

Then we can state precise regularity results (mainly, Hölder regularity results) for the solution of (0.1). The solution is a strong, classical, or strict one (see Def. 2.1, 3.1, 3.6) according to the regularity of the data. We get a representation formula for the solution by setting

$$\tau = \phi(t) = \int_0^t \varphi(\sigma)\, d\sigma \quad \text{and } u(t) = w(\phi(t)) \text{ in } (0.1),$$

so that (0.1) becomes

$$\begin{cases} w'(\tau) = A(\phi^{-1}(\tau))\, w(\tau) + f(\phi^{-1}(\tau))/\varphi(\phi^{-1}(\tau)), & 0 < \tau < \phi(T), \\ w(0) = u_0. \end{cases}$$

Now, w is given by the variation of constants formula

$$w(\tau) = G(\tau, 0)\, u_0 + \int_0^\tau G(\tau, \sigma) f(\phi^{-1}(\sigma))/\varphi(\phi^{-1}(\sigma))\, d\sigma,$$

so that

$$u(t) = G(\phi(t), 0)\, u_0 + \int_0^t G(\phi(t), \phi(s)) f(s)\, ds, \qquad 0 \leq t \leq T. \tag{0.3}$$

Here $G(\tau, \sigma)$ is the evolution operator associated to the family $\{A(\phi^{-1}(\tau),), \tau \in [0, \phi(T)]\}$. Our results are shown by a careful study of formula (0.3).

The literature on the subject is not very rich. In a previous paper ([5]) we studied the case where $A(t) = A$ is independent of time, and $\varphi(t) > 0$ for $t > 0$. Weak solutions to (0.1) are considered in [9] and [12], in the case where X is a Hilbert space, and $D(A(t)) = D$ is constant and dense in X. A certain class of degenerate equations could be studied also by means of the «sum of operators» method of [6]. However, such a method seems to be more fruitful in singular equations than in degenerate ones (see also [7], [8]). The paper is structured as follows. Section 1 is devoted to notation and preliminary estimates on the evolution operator relevant to problem (0.2). Section 2 deals with classical solvability of (0.1) and regularity properties of the classical solutions, whereas strict and strong solutions are studied in Section 3. Finally, in Section 4 we apply some of the abstract results to a class of initial boundary value problems for second order degenerate parabolic equations.

1. Notation and preliminaries.

Let $T > 0$ and let X be any Banach space whith norm $\|\cdot\|$. We shall use in the sequel the following functional spaces $B([0, T]; X)$ (the space of all bounded functions f: $[0, T] \to X$ endowed with the sup norm $\|\cdot\|_\infty$), Lip$([0, T)]; X)$ (the space of all Lipschitz continuous f: $[0, T] \to \to X$), $C([0, T]; X)$, $C^\alpha([0, T]; X)$, $C(]0, T]; X)$ with the usual meanings and norms. We consider a continuous function $\varphi: [0, T] \to \mathbb{R}$ such that $\varphi(t) > 0$ for $t \in]0, T[$. We shall see that the behaviour of the solution of (0.1) depends heavily on the behaviour of φ as $t \to 0$ and as $t \to T$.

Therefore we assume:

$$(1.1) \qquad \begin{cases} \varphi \in C([0, T]), \\ kt^\beta \leq \varphi(t) \leq Kt^\beta, & 0 \leq t \leq T/2 \\ k(T-t)^{\beta_1} \leq \varphi(t) \leq K(T-t)^{\beta_1}, & T/2 < t \leq T, \end{cases}$$

with $\beta \geq 0$, $\beta_1 \geq 0$ and $0 < k \leq K$.

It is convenient to introduce the notation

$$(1.2) \qquad \begin{cases} \psi(t, s) = \displaystyle\int_s^t \varphi(r)\, dr, \\ \phi(t) = \psi(t, 0), \qquad 0 \leq s \leq t \leq T. \end{cases}$$

Due to assumption (1.1), we get:

$$(1.3) \qquad \begin{cases} \dfrac{k}{\beta+1} t^{\beta+1} \leq \phi(t) \leq \dfrac{K}{\beta+1} t^{\beta+1}, \\ \dfrac{k}{\beta+1} t^\beta (t-s) \leq \psi(t, s) \leq Kt^\beta (t-s), \qquad 0 \leq s \leq t \leq T/2, \\ \dfrac{k}{\beta_1+1} (T-s)^{\beta_1} (t-s) \leq \psi(t, s) \leq K(T-s)^{\beta_1} (t-s), \\ \qquad\qquad T/2 < s \leq t \leq T. \end{cases}$$

Let

$$(1.4) \qquad g(t, s) = \begin{cases} t^\beta, & \text{if } 0 \leq s \leq t \leq T/2, \\ (T-s)^{\beta_1}, & \text{if } T/2 < s \leq t \leq T. \end{cases}$$

Then from (1.1) and (1.4) it follows:

$$(1.5)\qquad \begin{cases} kg(t, t) \leqslant \varphi(t) \leqslant Kg(t, t) \quad \text{for } 0 \leqslant t \leqslant T, \\ g(\tau, s) \leqslant g(t, s), \\ g(t, \tau) \leqslant g(t, s), \\ \text{both for } 0 \leqslant s \leqslant \tau \leqslant t \leqslant T/2 \text{ and for } T/2 < s \leqslant \tau \leqslant t \leqslant T; \end{cases}$$

and from (1.2), (1.3) and (1.4) it follows:

$$(1.6)\qquad \begin{cases} k_1 t^{\beta+1} \leqslant \phi(t) \leqslant K_1 t^{\beta+1} \quad \text{for } 0 \leqslant t \leqslant T/2, \\ k_1 g(t, s)(t-s) \leqslant \psi(t, s) \leqslant Kg(t, s)(t-s) \\ \qquad \text{both for } 0 \leqslant s \leqslant t \leqslant T/2 \text{ and for } T/2 < s \leqslant t \leqslant T, \\ \psi(t, \tau) \leqslant \psi(t, s) \\ \text{both for } 0 \leqslant s \leqslant \tau \leqslant t \leqslant T/2 \text{ and for } T/2 < s \leqslant \tau \leqslant t \leqslant T, \end{cases}$$

with $k_1 = \min\{k/(\beta+1),\ k/(\beta_1+1)\}$, $K_1 = K/(\beta+1)$.

Throughout the paper we shall assume that for each $t \in [0,]$, $A(t)$: $D(A(t)) \subset X \to X$ generates an analytic semigroup $e^{sA(t)}$, $s \geqslant 0$ in X. The domains $D(A(t))$ may change with t, however the resolvent sets $\rho(A(t))$ are assumed to contain a common sector

$$S_{\vartheta_0} = \{z \in \mathbb{C}\colon\ |\arg z| \leqslant \vartheta_0\} \cup \{0\} \qquad \forall t \in [0, T],$$

with $\vartheta_0 \in]\pi/2, \pi[$.

Moreover we shall assume, as in [2], [3]:

$$(1.7)\qquad \begin{cases} \text{There exist } C > 0,\ h \in \mathbb{N},\ \alpha_1, \dots, \alpha_h,\ \delta_1, \dots, \delta_h, \text{ with} \\ 0 \leqslant \delta_i < \alpha_i \leqslant 2, \text{ such that} \\ \|A(t) R(\lambda, A(t))(A(t)^{-1} - A(s)^{-1})\|_{L(X)} \leqslant C \sum\limits_{i,j=1}^{h} (t-s)^{\alpha_i} |\lambda|^{\delta_i - 1} \\ \forall \lambda \in S_{\vartheta_0} - \{0\},\ \forall 0 \leqslant s \leqslant t \leqslant T. \end{cases}$$

$D(A(t))$ is endowed with the graph norm and its closure in X is denoted by $\overline{D(A(t))}$. We refer to [13] and [3] for all properties of $e^{sA(t)}$ and of the interpolation spaces $D_{A(t)}(\vartheta, \infty)$, $D_{A(t)}(\vartheta + 1, \infty)$, $D_{A(t)}(\vartheta)$ and $D_{A(t)}(\vartheta + 1)$, $0 < \vartheta < 1$. We set $D_{A(t)}(0, \infty) = X$ and $D_{A(t)}(1, \infty) = D(A(t))$ $\forall t \in [0, T]$. Set

$$(1.8)\qquad B(\tau) = A(\phi^{-1}(\tau)), \qquad 0 \leqslant \tau \leqslant \phi(T).$$

Then the family $\{B(\tau)\}$ satisfies assumption (1.7). More precisely, we have (due to (1.1) and (1.3)):

$$(1.9)\quad \begin{cases} \text{There exist } \overline{C} > 0,\ h \in \mathbb{N},\ \alpha_1, \ldots, \alpha_h,\ \delta_1, \ldots \delta_h \text{ with} \\ 0 \leqslant \delta_i < \alpha_i \leqslant 2, \text{ such that } \forall \lambda \in S_{\vartheta_0} - \{0\} \\ \| B(\tau) R(\lambda, B(\tau)) [B(\tau)^{-1} - B(\sigma)^{-1}] \|_{L(X)} \leqslant \\ \qquad \leqslant \begin{cases} \overline{C}(\tau - \sigma)^{\alpha_i/(\beta+1)} |\lambda|^{\delta_i - 1} & \text{if } 0 \leqslant \sigma \leqslant \tau \leqslant \phi(T/2), \\ \overline{C}(\tau - \sigma)^{\alpha_i/(\beta_1+1)} |\lambda|^{\delta_i - 1} & \text{if } \phi(T/2) < \sigma \leqslant \tau \leqslant \phi(T). \end{cases} \end{cases}$$

We assume

$$(1.10)\quad \delta = \min\{\alpha_i/(\bar{\beta}+1) - \delta_i : 1 \leqslant i \leqslant h\} \in]0, 1[, \quad \text{with } \bar{\beta} = \max\{\beta, \beta_1\}.$$

By [2], [3] there exists an evolution operator $G(\tau, \sigma)$ relevant to the family $\{B(\tau) = A(\phi^{-1}(\tau)) : 0 \leqslant \tau \leqslant \phi(T)\}$. It can be represented as

$$(1.11)\quad G(\tau, \sigma) = e^{(\tau - \sigma) B(\sigma)} + \int_\sigma^\tau Z(r, \sigma)\, dr, \qquad 0 \leqslant \sigma \leqslant \tau \leqslant \phi(T)$$

(see [2]). Here $Z(r, \sigma) \in L(X)$ for $0 \leqslant \tau \leqslant r \leqslant \phi(T)$, and for $0 \leqslant s < \tau < t \leqslant \phi(T)$ we have:

$$(1.12)\quad \| Z(t, s) \|_{L(D_{B(s)}(\vartheta, \infty), X)} \leqslant c_1(\vartheta)(t - s)^{\vartheta + \delta - 1}, \qquad \vartheta \in [0, 1];$$

$$(1.13)\quad \begin{cases} \| Z(t, s) - Z(\tau, \sigma) \|_{L(D_{B(s)}(\vartheta, \infty), X)} \leqslant c_2(\vartheta, \eta)(t - \tau)^\eta (\tau - s)^{\vartheta + \delta - 1 - \eta} \\ \text{with } \vartheta \in [0, 1[,\ \ \eta \in \max\{\vartheta + \delta - 1, 0\}, \delta[\,; \end{cases}$$

$$(1.14)\quad \| Z(t, s) - Z(\tau, s) \|_{L(D(B(s)), X)} \leqslant c_3(\eta)(t - \tau)^\eta, \qquad \eta \in]0, \delta[\,;$$

where δ is defined in (1.10) (see [2], Lemma 2.2).

Moreover we have (recall that $A(\cdot) = B(\phi(\cdot))$):

$$(1.15)\quad \begin{cases} \text{if } \alpha, \vartheta \in [0, 1],\ \ 0 \leqslant t \leqslant T,\ \ 0 < \xi \leqslant \phi(T), \\ \| e^{\xi A(t)} \|_{L(D_{A(t)}(\vartheta, \infty), X)} \leqslant M_0(\vartheta); \\ \| A(t)^m e^{\xi A(t)} \|_{L(D_{A(t)}(\vartheta, \infty), D_{A(t)}(\alpha, \infty))} \leqslant M_1(\alpha, \vartheta, m)\, \xi^{-(m + \alpha - \vartheta)} \\ \text{with } m \geqslant 1; \end{cases}$$

$$
(1.16)\quad \begin{cases} \text{if } 0 \leqslant s \leqslant t \leqslant \phi(T), \ \ 0 < \xi \leqslant \phi(T), \ \ m \in \mathbb{N}, \\ \|B(t)^m e^{\xi B(t)} - B(s)^m e^{\xi B(s)}\|_{L(X)} \leqslant M_2(m) \sum_{i=1}^{h} \psi(t, s)^{\alpha_i/(\bar{\beta}+1)} \xi^{-(m+\delta_i)} \end{cases}
$$

(see [3, Lemmas 1.8 and 1.10] and our assumptions (1.9), (1.10)).

Let $0 \leqslant s < \tau < t \leqslant \phi(T)$, and let δ be defined in (1.10). We have:

$$
(1.17)\quad \begin{cases} \text{if } 0 \leqslant \vartheta \leqslant \alpha \leqslant 1, \\ \|G(t, s)\|_{L(D_{B(s)}(\vartheta, \infty), D_{B(t)}(\alpha, \infty))} \leqslant c_4(\alpha, \vartheta)(t-s)^{\vartheta - \alpha}; \end{cases}
$$

$$
(1.18)\quad \begin{cases} \text{if } \vartheta \in [0, 1], \ \ \alpha \in [0, \delta], \\ \|B(t)\, G(t, s)\|_{L(D_{B(s)}(\vartheta, \infty), D_{B(t)}(\alpha, \infty))} \leqslant c_5(\alpha, \vartheta)(t-s)^{\vartheta - \alpha - 1}; \end{cases}
$$

$$
(1.19)\quad \|G(t, s) - G(\tau, s)\|_{L(D_{B(s)}(\vartheta, \infty), X)} \leqslant c_6(\vartheta)(t-\tau)^{\vartheta}, \qquad \vartheta \in]0, 1];
$$

(see [10, Lemmas 4.1 and 4.2]).

$$
(1.20)\quad \begin{cases} \text{If } \vartheta \in]0, \delta], \ \ 0 \leqslant s \leqslant t \leqslant \phi(T), \\ \|B(t)\, G(t, s)\|_{L(D_{B(s)}(\vartheta+1, \infty), D_{B(t)}(\alpha, \infty))} \leqslant c_7; \end{cases}
$$

(see [3, Thm. 6.1]).

The following lemma will be useful in the sequel.

LEMMA 1.1. Let $y \in X$. Then $\int_0^t G(t, \sigma)\, y\, d\sigma \in D(B(t))$ $\forall t > 0$, and

$$
(1.21)\quad \left\| B(t) \int_a^t G(t, \sigma)\, y\, d\sigma \right\| \leqslant c\|y\| \qquad \forall t \in [0, \phi(T)];
$$

$$
(1.22)\quad \left\| B(t) \int_a^t G(t, \sigma)\, y\, d\sigma - B(\tau) \int_a^\tau G(\tau, \sigma)\, y\, d\sigma \right\| \leqslant c(\alpha) \frac{(t-\tau)^\alpha}{(\tau - a)^\alpha},
$$

for each $a < \tau < t \leqslant \phi(T)$, $0 < \alpha < 1$.

PROOF. It is an easy consequence of Propositions 2.1 (iv) and 2.6 (iii) (*d*) of [3] (with $\mu = 0$), together with the representation formula (1.22) of [3].

Let us state other estimates which will be used throughout the paper.

LEMMA 1.2. Let δ be defined in (1.10). Then, both for $0 \leqslant s < \tau < t \leqslant T/2$ and for $T/2 < s < \tau < t \leqslant T$ we have:

$$\|B(\phi(t))\, G(\phi(t), \phi(s))\|_{L(D_{A(s)}(\vartheta, \infty), D_{A(t)}(\alpha, \infty))} \leqslant \leqslant C_0(\alpha, \vartheta)\, g(t, s)^{\vartheta - \alpha - 1}(t - s)^{\vartheta - \alpha - 1}, \quad \vartheta \in [0, 1],\ \alpha \in [0, \delta]; \tag{1.23}$$

$$\|G(\phi(t), \phi(s)) - G(\phi(\tau), \phi(s))\|_{L(X)} \leqslant C_1(\nu)\frac{(t-\tau)^\nu}{(\tau - s)^\nu}, \quad \nu \in [0, 1]; \tag{1.24}$$

$$\|B(\phi(t))\, G(\phi(t), \phi(s)) - B(\phi(\tau))\, G(\phi(\tau), \phi(s))\|_{L(D_{A(s)}(\vartheta, \infty), X)} \leqslant \leqslant \frac{1}{g(\tau - s)^{1-\vartheta}}\left\{C_2(\vartheta)\left[\frac{1}{(\tau - s)^{1-\vartheta}} - \frac{1}{(t-s)^{1-\vartheta}}\right] + + C_3(\delta, \vartheta, \eta)\, g(t, \tau)^\eta g(\tau, s)^{\delta - \eta}\frac{(t-\tau)^\eta}{(\tau - s)^{1+\eta-\vartheta-\delta}}\right\}, \quad \vartheta \in [0, 1[,\ \eta \in [\max\{\vartheta + \delta - 1, 0\}, \delta]; \tag{1.25}$$

$$\|B(\phi(t))\, G(\phi(t), \phi(s)) - B(\phi(\tau))\, G(\phi(\tau), \phi(s))\|_{L(D(A(s)), X)} \leqslant \leqslant C_4(\nu)\frac{(t-\tau)^\nu}{(\tau - s)^\nu} + C_5(\delta, \eta)\, g(t, \tau)^\eta (t-\tau)^\eta, \quad \nu \in]0, 1],\ \eta \in]0, \delta[; \tag{1.26}$$

$$\|B(\phi(t))\, G(\phi(t), \phi(s)) - B(\phi(\tau))\, G(\phi(\tau), \phi(s))\|_{L(D_{A(s)}(\vartheta + 1, \infty), X)} \leqslant \leqslant C_6(\vartheta) g(t - s)^\vartheta (t - \tau)^\vartheta + C_7(\eta)\, g(t, \tau)^\eta (t-\tau)^\eta, \quad \vartheta \in [0, 1],\ \eta \in [0, \delta[. \tag{1.27}$$

PROOF. (1.23) is a simple consequence of (1.18), (1.2), (1.4) and (1.6). To show (1.24), ..., (1.27), we consider either $0 \leqslant s < \tau < t \leqslant T/2$ or $T/2 < s < \tau < t \leqslant T$. (1.24), for $\nu = 0$, holds with $C_1 = 2c_4$ thanks to (1.17). For $\nu > 0$, thanks to (1.23), (1.4) and (1.5), we have

$$\|G(\phi(t), \phi(s)) - G(\phi(\tau), \phi(s))\|_{L(X)} =$$

$$= \left\| \int_\tau^t \varphi(r)\, B(\phi(r))\, G(\phi(r), \phi(s))\, dr \right\|_{L(X)} \leqslant KC_0 \int_\tau^t \frac{g(r, r)\, dr}{g(r, s)(r - s)} \leqslant$$

$$\leqslant KC_0 \frac{1}{(\tau - s)^{1-\nu}} \int_\tau^t \frac{dr}{(r-s)^\nu} \leqslant \frac{KC_0}{\nu}\frac{(t-\tau)^\nu}{(\tau - s)^\nu}, \quad 0 < \nu \leqslant 1.$$

Concerning (1.25) and (1.26), thanks to [2, (2.10)] for $\vartheta \in [0,1]$ we have:

$$\|B(\phi(t))\,G(\phi(t),\,\phi(s)) - B(\phi(\tau))\,G(\phi(\tau),\,\phi(s))\|_{L(D_{A(s)}(\vartheta,\,\infty),\,X)} \leqslant$$

$$\leqslant \|B(\phi(s))[e^{\psi(t,\,s)B(\phi(s))} - e^{\psi(\tau,\,s)B(\phi(s))}]\|_{L(D_{A(s)}(\vartheta,\,\infty),\,X)} +$$

$$+ \|Z(\phi(t),\,\phi(s)) - Z(\phi(\tau),\,\phi(s))\|_{L(D_{A(s)}(\vartheta,\,\infty),\,X)}\,.$$

By (1.15), (1.4), (1.5) and (1.6) we have:

$$\|B(\phi(s))[e^{\psi(t,\,s)B(\phi(s))} - e^{\psi(\tau,\,s)B(\phi(s))}]\|_{L(D_{A(s)}(\vartheta,\,\infty),\,X)} =$$

$$= \left\| \int_\tau^t \varphi(r)\,B(\phi(s))^2\,e^{\psi(r,\,s),\,B(\phi(s))}\,dr \right\|_{L(D_{A(s)}(\vartheta,\,\infty),\,X)} \leqslant$$

$$\leqslant M_1(\vartheta) \int_\tau^t \frac{\varphi(r)\,dr}{\psi(r,\,s)^{2-\vartheta}} \leqslant \frac{KM_1(\vartheta)}{K_1^{2-\vartheta}\,g(\tau,\,s)^{1-\vartheta}} \int_\tau^t \frac{dr}{(r-s)^{2-\vartheta}}\,.$$

Now if $\vartheta \in [0,1[$

$$\int_\tau^t \frac{dr}{(r-s)^{2-\vartheta}} = \frac{1}{1-\vartheta}\left[\frac{1}{(\tau-s)^{1-\vartheta}} - \frac{1}{(t-s)^{1-\vartheta}}\right]$$

and from (1.13), (1.4), (1.6) it follows:

$$\|Z(\phi(t),\,\phi(s)) - Z(\phi(\tau),\,\phi(s))\|_{L(D_{A(s)}(\vartheta,\,\infty),\,X)} \leqslant$$

$$\leqslant c_2(\vartheta,\,\eta)\,\psi(t,\,\tau)^\eta\,\psi(\tau,\,s)^{\vartheta+\delta-\eta-1}, \qquad \eta \in [\max\{\vartheta+\delta-1,\,0\},\,\delta[$$

$$\leqslant c_2(\vartheta,\,\eta)\,K^{\vartheta+\delta-1}g(t,\,\tau)^\eta\,g(\tau,\,s)^{\vartheta+\delta-\eta-1}(t-\tau)^\eta(\tau-s)^{\vartheta+\delta-\eta-1}\,.$$

Hence (1.25) holds. If $\vartheta = 1$ and $\nu \in]0,1]$ we get:

$$\int_\tau^t \frac{dr}{r-s} \leqslant \frac{(t-\tau)^\nu}{\nu(\tau-s)^\nu}$$

and from (1.14), (1.4), (1.6) it follows, for $\eta \in [0,\delta[$:

$$\|Z(\phi(t),\,\phi(s)) - Z(\phi(\tau),\,\phi(s))\|_{L(D(A(s)),\,X)} \leqslant$$

$$\leqslant c_3(\eta)\,\psi(t,\,\tau)^\eta < c_3(\eta)\,K^\eta g(t,\,\tau)^\eta(t-\tau)^\eta;$$

hence (1.26) holds. Finally from [2, (2.10)], (1.15), (1.14), (1.4), (1.5) and (1.6) it follows:

$$\|B(\phi(t))\,G(\phi(t),\,\phi(s)) - B(\phi(\tau))\,G(\phi(\tau),\,\phi(s))\|_{L(D_{A(s)}(\vartheta+1,\,\infty),\,X)} \leqslant$$

$$\leqslant \|e^{\psi(t,\,s)B(\phi(s))} - e^{\psi(\tau,\,s)B(\phi(s))}\|_{L(D_{A(s)}(\vartheta,\,\infty),\,X)}\cdot$$

$$\cdot \sup_{0\leqslant s\leqslant T} \|B(\phi(s))\|_{L(D_{A(s)}(\vartheta+1,\,\infty),\,D_{A(s)}(\vartheta,\,\infty))} +$$

$$+ \|Z(\phi(t),\,\phi(s)) - Z(\phi(\tau),\,\phi(s))\|_{L(D(A(s)),\,X)} \leqslant$$

$$\leqslant \mathrm{const}\left\|\int_\tau^t \varphi(r)\,B(\phi(s))\,e^{\psi(r,\,s)B(\phi(s))}\,dr\right\|_{L(D_{A(s)}(\vartheta,\,\infty),\,X)} + c_3(\eta)\,\psi(t,\,\tau)^\eta \leqslant$$

$$\leqslant \mathrm{const}\,M_1(\vartheta)\int_\tau^t \frac{\varphi(r)\,dr}{\psi(r,\,s)^{1-\vartheta}} + Kc_3(\eta)\,g(t,\,\tau)^\eta(t-\tau)^\eta \leqslant$$

$$\leqslant \mathrm{const}\frac{KM_1(\vartheta)}{k_1^{1-\vartheta}}g(t,\,s)^\vartheta\int_\tau^t \frac{dr}{(r-s)^{1-\vartheta}} + Kc_3(\eta)\,g(t,\,\tau)^\eta(t-\tau)^\eta$$

and (1.27) holds. ■

We are now in position to state the main properties of the function

$$w(t) = G(\phi(t),\,0)\,x, \qquad 0\leqslant t\leqslant T,\ x\in X. \tag{1.28}$$

To simplify some statements, we introduce the following notation:

(1.29) $B([a,\,b];\,D_{A(\cdot)}(\vartheta,\,\infty)) =$

$$= \left\{u\colon [a,\,b]\to X\colon\ u(t)\in D_{A(t)}(\vartheta,\,\infty)\ \forall t\in[a,\,b];\ \sup_{a\leqslant t\leqslant b}\|u(t)\|_{D_{A(t)}(\vartheta,\,\infty)} < +\infty\right\},$$

for $a<b,\ 0<\vartheta<2,\ \vartheta\neq 1$.

PROPOSITION 1.3. Let δ be defined in (1.10). For each $x\in X$, we have:

$$w\in B([0,\,T];\,X); \tag{1.30}$$

$$w\in B([\varepsilon,\,T];\,D_{A(\cdot)}(\delta+1,\,\infty)), \qquad \forall\varepsilon\in]0,\,T[; \tag{1.31}$$

$$Aw\in C^\eta([\varepsilon,\,T];\,X), \qquad \forall\varepsilon\in]0,\,T[,\ \eta<\delta; \tag{1.32}$$

$$w \in C^1(]0, T]; X), \tag{1.33}$$

and

$$w'(t) = \varphi(t) A(t) w(t) = \varphi(t) B(\phi(t)) G(\phi(t), 0) x, \qquad \forall t \in]0, T]; \tag{1.34}$$

$$x \in \overline{D(A(0))} \Leftrightarrow w \in C([0, T]; X) \text{ and } w(0) = x; \tag{1.35}$$

$$\begin{cases} \text{if } \beta > 0, \text{ then:} \\ x \in D_{A(0)}(1/(\beta+1)) \Leftrightarrow w' \in C([0, T]; X) \text{ and } w'(0) = 0, \\ \text{if } \beta = 0, \text{ then: } x \in D(A(0)) \text{ and } A(0)x \in \overline{D(A(0))} \Leftrightarrow \\ \qquad \Leftrightarrow w' \in C([0, T]; X) \text{ and } w'(0) = \varphi(0) A(0) x. \end{cases} \tag{1.36}$$

$$\begin{cases} \text{if } \beta > 0, \ 0 < \vartheta \leqslant \min\{\beta/(\beta+1), \delta\}, \\ x \in D_{A(0)}(\vartheta + 1/(\beta+1), \infty) \Rightarrow w' \in B([0, T]; D_{A(\cdot)}(\vartheta, \infty)) \\ \text{if } \beta = 0, \ 0 < \vartheta \leqslant \delta, \\ x \in D_{A(0)}(\vartheta + 1, \infty) \Rightarrow w' \in B([0, T]; D_{A(\cdot)}(\vartheta, \infty)). \end{cases} \tag{1.37}$$

PROOF. (1.30), (1.34) and (1.35) follow from [2, Thm. 2.3 (i) and (v) and Thm. 4.1 (ii)]. By (1.23) and (1.4) for $0 < t \leqslant T/2$ we have:

$$\|B(\phi(t)) G(\phi(t), 0) x\|_{D_{A(t)}(\delta, \infty)} \leqslant C_0(\delta) \frac{\|x\|}{t^{(\beta+1)(\delta+1)}}.$$

In particular, setting

$$\bar{x} = w(T/2) = G(\phi(T/2), 0) x,$$

we have:

$$\bar{x} \in D_{A(T/2)}(\delta + 1, \infty). \tag{1.38}$$

Therefore by (1.38) and (1.20) we have, for $T/2 \leqslant t \leqslant T$:

$$\|B(\phi(t)) G(\phi(t), 0) x\|_{D_{A(t)}(\delta, \infty)} =$$

$$= \|B(\phi(t)) G(\phi(t), \phi(T/2)) \bar{x}\|_{D_{A(t)}(\delta, \infty)} \leqslant c_7 \|\bar{x}\|_{D_{A(T/2)}(\delta+1, \infty)}$$

and (1.31) holds. By (1.25) $Aw \in C^{\eta}([\varepsilon, T/2]; X)$; if $T/2 < \tau < t \leqslant T$, by

(1.38), (1.27) and (1.4) we have:

$$\|A(t)\,w(t) - A(\tau)\,w(\tau)\| =$$

$$= \|B(\phi(t))\,G(\phi(t),\,\phi(T/2))\,\bar{x} - B(\phi(\tau))\,G(\phi(\tau),\,\phi(T/2))\,\bar{x}\| \leqslant$$
$$\leqslant \{C_6(\delta)(T/2)^{\delta\beta_1}(t-\tau)^{\delta} + C_7(\eta)(T-\tau)^{\eta\beta_1}(t-\tau)^{\eta}\}\,\|\bar{x}\|_{D_{A(T/2)}(\vartheta+1,\,\infty)}, \qquad \eta < \delta.$$

Hence (1.32) holds. If $\beta_1 = 0$, (1.33) follows from [2, Thm. 2.3 (v)], (1.34) and (1.1). If $\beta_1 > 0$, $w \in C^1(]0, T[;\,X)$ and by (1.34), (1.38), (1.23) and (1.1) we get:

$$\lim_{t\to T^-} \|\varphi(t)\,B(\phi(t))\,G(\phi(t),\,0)\,x\| =$$

$$= \lim_{t\to T^-} \|\varphi(t)\,B(\phi(t))\,G(\phi(t),\,\phi(T/2))\,\bar{x}\| = C_0\,\|\bar{x}\|_{D(A(T/2))}\lim_{t\to T^-}\varphi(t) = 0\,.$$

Therefore $w \in C^1(]0, T];\,X)$. Let us show (1.36). The second equivalence follows from [3, thm. 6.1] and (1.33). Concerning the first equivalence, if $\beta > 0$, thanks to (1.23) we have:

$$x \in D(A(0)) \Rightarrow \lim_{t\to 0} \|\phi(t)^{1-1/(\beta+1)}B(\phi(t))\,G(\phi(t),\,0)\,x\| = 0$$

and thanks to [2, thm. 4.1 (iv)] we have:

$$x \in D_{A(0)}(1/(\beta+1)) \Rightarrow \exists \lim_{t\to 0} \|\phi(t)^{1-1/(\beta+1)}B(\phi(t))\,G(\phi(t),\,0)\,x\| = y\,,$$

and for each $t \in\,]0, T]$

$$\|\phi(t)^{1-1/(\beta+1)}B(\phi(t))\,G(\phi(t),\,0)\,x\| \leqslant \text{const}\,\|x_{D_{A(0)}(1/(\beta+1),\,\infty)}\,.$$

Since $D(A(0))$ is dense in $D_{A(0)}(1/(\beta+1))$, $y = 0$.
Hence fixed $\varepsilon > 0$, $\exists \bar{t} \leqslant T/2$: $\forall t < \bar{t}$

$$\|B(\phi(t))\,G(\phi(t),\,0)\,x\| \leqslant \varepsilon/\phi(t)^{\beta/(\beta+1)}\,;$$

so that

$$\|w'(t)\| = \|\phi(t)\,B(\phi(t))\,G(\phi(t),\,0)\,x\| \leqslant Kt^{\beta}\,\varepsilon/\phi(t)^{\beta/(\beta+1)} \leqslant K\varepsilon/k_1^{\beta/(\beta+1)}$$

and $\lim_{t\to 0} w'(t) = 0$. By (1.33) $w \in C^1([0, T];\,X)$. Conversely

$$w' \in C([0, T/2];\,X) \Rightarrow \exists \lim_{t\to 0} w'(t) = \lim_{t\to 0} \varphi(t)\,B(\phi(t))\,G(\phi(t),\,0)\,x = y\,.$$

Let us show that $y = 0$. Fix $\lambda \in S_{\vartheta_0}$; then we have:

$$[\lambda - B(0)]^{-1} y = \lim_{t \to 0} [\lambda - B(\phi(t))]^{-1} \varphi(t) B(\phi(t)) G(\phi(t), 0) x =$$

$$= \lim_{t \to 0} \lambda[\lambda - B(\phi(t))]^{-1} \varphi(t) G(\phi(t), 0) x - \lim_{t \to 0} \varphi(t) G(\phi(t), 0) x = 0.$$

Therefore $y = 0$. Fixed $\varepsilon > 0$, for t close to 0, we have: $\|\varphi(t) B(\phi(t)) G(\phi(t), 0) x\| < \varepsilon$, and consequently

$$\frac{\|G(\phi(t), 0) x - G(\phi(\tau), 0) x\|}{[\phi(t) - \phi(\tau)]^{1/(\beta+1)}} \leqslant \frac{\left\| \int_\tau^t \varphi(r) B(\phi(r)) G(\phi(r), 0) x \, dr \right\|}{k_1^{1/(\beta+1)} t^{1/(\beta+1)} (t - \tau)^{1/(\beta+1)}} \leqslant$$

$$\leqslant \left[\int_\tau^t \varepsilon \, dr \right] \Big/ [k_1^{1/(\beta+1)} (t - \tau)] = \varepsilon / k_1^{1/(\beta+1)}.$$

Hence, thanks to [2, Thm. 4.2 (iv)], $x \in D_{A(0)}(1/(\beta + 1))$.

Concerning (1.37), if $x \in D_{A(0)}(\vartheta + 1/(\beta + 1), \infty)$, then by (1.23), (1.1) and (1.4), for $\beta > 0$ and $0 \leqslant t \leqslant T/2$ we have:

$$\|\phi(t) B(\phi(t)) G(\phi(t), 0) x\|_{D_{A(t)}(\vartheta, \infty)} \leqslant K C_0(\vartheta) \|x\|_{D_{A(0)}(\vartheta + 1/(\beta+1), \infty)}$$

so that w' is bounded in $D_{A(t)}(\vartheta, \infty)$, thanks to (1.31). If $\beta = 0$, then w' is bounded in $D_{A(t)}(\vartheta, \infty)$, thanks to [3, Thm. 6.1] and (1.31). ■

In the case where φ is Hölder continuous, we can show other regularity properties of w'.

PROPOSITION 1.4. Let $\varphi \in C^\gamma([0, T])$, $0 < \gamma < 1$, $\gamma \leqslant \beta$ if $\beta > 0$, $\gamma \leqslant \beta_1$ if $\beta_1 > 0$ (see (1.1)), and let δ be defined in (1.10). Then

(1.39) $\quad x \in X \Rightarrow w' \in C^\eta([\varepsilon, T]; X), \qquad 0 < \varepsilon < T, \ \eta \leqslant \gamma, \ \eta < \delta;$

(1.40) $$\begin{cases} \text{if } \beta > 0, \ (1 + \beta - \gamma)/(\beta + 1) < \vartheta \leqslant 1 \\ x \in D_{A(0)}(\vartheta, \infty) \Rightarrow w' \in C^\eta([0, T]; X) \\ \text{with } \eta \leqslant \gamma - (\beta + 1)(1 - \vartheta), \ \ \eta < \delta; \end{cases}$$

(1.41) $$\begin{cases} \text{if } \beta = 0, \ \ 0 < \vartheta < 1 \\ x \in D_{A(0)}(\vartheta + 1, \infty) \Rightarrow w' \in C^\eta([0, T]; X) \\ \text{with } \eta \leqslant \min\{\gamma, \vartheta\}, \ \ \eta < \delta. \end{cases}$$

PROOF. From (1.25) we obtain that $B(\phi(t))\,w(t) = B(\phi(t))\,G(\phi(t), 0)\,x$ is η-Hölder continuos in $[\varepsilon, T]$ for $0 < \varepsilon < T$, $\eta < \delta$; then (1.36) is a consequence of the Hölder continuity of φ. Let us show that (1.40) holds. In the case $\vartheta < 1$, thanks to (1.23), (1.25), (1.1) and (1.4) for $0 \leqslant \tau < t \leqslant \leqslant T/2$ we have:

$$\|w'(t) - w'(\tau)\| \leqslant |\varphi(t) - \varphi(\tau)|\,\|B(\phi(t))\,G(\phi(t), 0)\,x\| +$$

$$+\varphi(\tau)\|B(\phi(t))\,G(\phi(\tau), 0)\,x - B(\phi(\tau))\,G(\phi(\tau), 0)\,x\| \leqslant$$

$$\leqslant \left\{C_0(\vartheta)[\varphi]_{C^\gamma}\frac{(t-\tau)^\gamma}{t^{(\beta+1)(1-\vartheta)}} + KC_2(\vartheta)\left|\frac{1}{\tau^{1-\vartheta(\beta+1)}} - \frac{1}{t^{1-\vartheta(\beta+1)}}\right| + \right.$$

$$\left. + C_3(\delta, \vartheta, \eta)\frac{\tau^\beta t^{\beta\eta}(t-\tau)^\eta}{\tau^{\beta+(\beta+1)(\eta-\delta-\vartheta)+1}}\right\}\|x\|_{D_{A(0)}(\vartheta, \infty)} \leqslant$$

$$\leqslant \{C_0(\vartheta)[\varphi]_{C^\gamma}(t-\tau)^\gamma t^{-(\beta+1)(1-\vartheta)} + KC_2(\vartheta)[t^{\vartheta(\beta+1)-1} - \tau^{\vartheta(\beta+1)-1}] +$$

$$+ KC_3(\delta, \vartheta, \eta)(t-\tau)^\eta t^{\beta\eta}\tau^{(\beta+1)(\delta+\vartheta-\eta)-1}\}\|x\|_{D_{A(0)}(\vartheta, \infty)}.$$

In the case $\vartheta = 1$, thanks to (1.23), (1.26), (1.1) and (1.4), for $0 \leqslant \tau < t \leqslant \leqslant T/2$ we have:

$$\|w'(t) - w'(\tau)\| \leqslant$$

$$\leqslant \{C_0[\varphi]_{C^\gamma}(t-\tau)^\gamma + KC_4(\eta)\,\tau^{\beta-\eta}(t-\tau)^\eta + KC_5(\delta, \eta)\,t^{\beta\eta}\tau^\beta(t-\tau)^\eta\}\|x\|_{D(A(0))}$$

and $w' \in C^\eta([0, T/2]; X)$, since $\eta \leqslant \gamma$. For $T/2 < \tau < t \leqslant T$, from (1.38), (1.23), (1.27), (1.1) and (1.4) we get:

$$\|w'(t) - w'(\tau)\| \leqslant |\varphi(t) - \varphi(\tau)|\,\|B(\phi(t))\,G(\phi(t), \phi(T/2))\,\bar{x}\| +$$

$$+\varphi(\tau)\|B(\phi(t))\,G(\phi(\tau), \phi(T/2))\,\bar{x} - B(\phi(\tau))\,G(\phi(\tau), \phi(T/2))\,\bar{x}\| \leqslant$$

$$\leqslant C_0(\vartheta)[\varphi]_{C^\gamma}(t-\tau)^\gamma\|\bar{x}\|_{D(A(T/2))} + K(T-\tau)^{\beta_1}\cdot$$

$$\cdot\{C_6(\vartheta)(T/2)^{\eta\beta_1}(t-\tau)^\eta + C_7(\eta)(T-\tau)^{\eta\beta_1}(t-\tau)^\eta\}\|\bar{x}\|_{D_{A(T/2)}(\vartheta+1, \infty)}$$

and (1.40) holds. In the same way, (1.41) follows from (1.23), (1.27), (1.1), and (1.4). ■

In the sequel we shall need also the following lemmas.

LEMMA 1.5. Let $\mu \geq 0$, $0 < \alpha < 1$, $\nu > 0$ and let $T/2 \leq a \leq b \leq t < T$. Then it holds:

$$\int_a^b (T-s)^{-\mu}(t-s)^{\alpha-1}ds \leq \frac{(b-a)^{\alpha-\mu}}{\alpha-\mu}, \qquad \text{if } \mu < \alpha; \tag{1.42}$$

$$\int_a^b (T-s)^{-\mu}(t-s)^{\alpha-1}ds \leq$$
$$\leq (T-t)^{\alpha-\mu}\int_0^{+\infty}(1+y)^{-\mu}y^{\alpha-1}dy, \qquad \text{if } \mu > \alpha; \tag{1.43}$$

$$\int_a^b (T-s)^{-\alpha}(t-s)^{\alpha-1}ds \leq$$
$$\leq (T/2)^{\nu}(T-t)^{-\nu}\int_0^{+\infty}(1+y)^{-(\alpha+\nu)}y^{\alpha-1}dy. \tag{1.44}$$

PROOF. Let $\int_a^b (T-s)^{-\mu}(t-s)^{\alpha-1}da =: I$. In the case $\mu < \alpha$ we get:

$$I \leq \int_a^b (t-s)^{\alpha-\mu-1}ds = \frac{(b-a)^{\alpha-\mu}}{\alpha-\mu},$$

and (1.42) holds.

In the case $\mu > \alpha$, (1.43) follows setting $(t-s) = (T-t)y$ in I.

In the case $\mu = \alpha$, for each $\nu > 0$, from (1.43) it follows:

$$I \leq (T/2)^{\nu}\int_a^b (T-s)^{-(\alpha+\nu)}(t-s)^{\alpha-1}ds \leq$$
$$\leq (T/2)^{\nu}(T-t)^{-\nu}\int_0^{+\infty}(1+y)^{-(\alpha+\nu)}y^{\alpha-1}dy,$$

and (1.44) holds. ■

LEMMA 1.6. Let $\varphi \in C([0,T])$ and let δ be defined in (1.10). Then it holds:

$$(1.45)\quad \begin{cases} \text{if } y \in X \text{ and } 0 \leqslant a < t \leqslant T, \text{ then} \\ \left\| B(\phi(t)) \int_a^t \varphi(s)\, G(\phi(t), \phi(s))\, y\, ds - [e^{\psi(t,a) B(\phi(t))} y - y] \right\| = \\ \qquad = O((t-a)^{\delta}); \end{cases}$$

$$(1.46)\quad \begin{cases} \text{if } \varphi \in C^{\gamma}([0,T]),\ 0<\gamma<1,\ \gamma \leqslant \beta \text{ if } \beta > 0,\ \gamma \leqslant \beta_1 \text{ if } \beta_1 > 0, \\ \text{(see (1.1)), then} \\ \left\| B(\phi(t)) \int_a^b G(\phi(t), \phi(s))\, ds \right\|_{L(X)} \leqslant \\ \leqslant \begin{cases} \dfrac{C_7(\gamma,\delta)}{\varphi(t)} \left[\dfrac{(b-a)^{\gamma}}{t^{\beta}} + t^{\beta\delta}(b-a)^{\delta} + 1 \right], \\ \text{for } 0 \leqslant a < b \leqslant t \leqslant T/2; \\ \qquad C_8(\gamma,\delta,T), \text{ for } T/2 < a \leqslant b \leqslant t \leqslant T,\ \beta_1 = 0; \\ \dfrac{C_9(\beta_1,\gamma,\delta,\nu,T)}{\varphi(t)} \left[\dfrac{1}{(T-t)^{\beta_1 - \gamma + \nu}} + (T-t)^{\delta\beta_1}(b-a)^{\delta} + 1 \right], \\ \text{for } T/2 < a \leqslant b \leqslant t < T,\ \beta_1 > 0; \\ \text{with } \nu = 0 \text{ if } \beta_1 > \gamma,\ \nu > 0 \text{ if } \beta_1 = \gamma; \\ C(\beta,\beta_1,\gamma,\delta,\nu,T)/[g(t,t)\varphi(t)], \\ \text{both for } 0 \leqslant a < b \leqslant t \leqslant T/2, \text{ and for } T/2 < a \leqslant b \leqslant t < T. \end{cases} \end{cases}$$

PROOF. From [2, (2.10)] it follows, for $0 \leqslant s < t \leqslant T$:

$$(1.47)\quad B(\phi(t))\, G(\phi(t), \phi(s)) = [B(\phi(s)) e^{\psi(t,s) B(\phi(s))} - B(\phi(t)) e^{\psi(t,s) B(\phi(t))}] + \\ + Z(\phi(t), \phi(s)) + B(\phi(t)) e^{\psi(t,s) B(\phi(t))} =: I_1 + I_2 + I_3 .$$

By (1.16), (1.12) and (1.10) we have:

$$(1.48)\quad \|I_1\|_{L(X)} + \|I_2\|_{L(X)} \leqslant M_2 \sum_{i=1}^{h} [\psi(t,s)^{\alpha_i/(\bar\beta+1)} / \psi(t,s)^{1+\delta_i}] + \\ + c_1 \psi(t-s)^{\delta - 1} \leqslant (hM_2 + c_1)\, \psi(t,s)^{\delta - 1}.$$

Hence by (1.48), (1.5) and (1.6) we get both for $0 \leqslant a \leqslant t \leqslant T/2$ and for $T/2 < a \leqslant t \leqslant T$:

$$(1.49)\quad \int_a^t \varphi(s) \| I_1 + I_2 |_{L(X)} ds \leqslant$$

$$\leqslant (hM_2 + c_1)(K/k_1^{1-\delta}) \int_a^t \frac{g(s, s)\, ds}{g(t, s)^{1-\delta}(t-s)^{1-\delta}} = O((t-a)^{\delta}).$$

Moreover from [13, (1.4)], for $y \in X$ it follows:

$$(1.50)\quad \int_a^t \varphi(s) I_3 y\, ds = e^{\psi(t, a) B(\phi(t))} y - y.$$

Therefore by (1.49) and (1.50), (1.45) holds. Let us show (1.46).

From [2, (2.10)] for $0 \leqslant s < t < T$ it follows:

$$(1.51)\quad B(\phi(t))\, G(\phi(t), \phi(s)) = [B(\phi(s)) e^{\psi(t, s) B(\phi(s))} - B(\phi(t)) e^{\psi(t, s) B(\phi(t))}] +$$

$$+ Z(\phi(t), \phi(s)) + \frac{1}{\varphi(t)} [\varphi(s) B(\phi(t)) e^{\psi(t, s) B(\phi(t))} +$$

$$+ (\varphi(t) - \varphi(s)) B(\phi(t)) e^{\psi(t, s) B(\phi(t))}] =: I_1 + I_2 + \frac{1}{\varphi(t)} [\varphi(t) I_3 + I_4].$$

Let $0 \leqslant a \leqslant b \leqslant t < T$. From (1.48) it follows:

$$(1.52)\quad \left\| \int_a^b (I_1 + I_2)\, ds \right\|_{L(X)} \leqslant (hM_2 + c_1) \int_a^b \psi(t, s)^{\delta - 1} ds;$$

from (1.15) it follows:

$$(1.53)\quad \left\| \int_a^b \varphi(s) I_3\, ds \right\|_{L(X)} = \| e^{\psi(t, a) B(\phi(t))} - e^{\phi(t, b) B(\phi(t))} \|_{L(X)} \leqslant 2M_0,$$

and

$$(1.54)\quad \left\| \int_a^b I_4\, ds \right\|_{L(X)} \leqslant M_1 [\varphi]_{C^{\gamma}} \int_a^b (t-s)^{\gamma} \psi(t, s)^{-1} ds.$$

Therefore by (1.51), ..., (1.54) we get:

$$(1.55)\quad \left\| B(\phi(t)) \int_a^b G(\phi(t), \phi(s))\, ds \right\|_{L(X)} \leqslant$$

$$\leqslant (hM_2 + c_1) \int_a^b \psi(t, s)^{\delta - 1} ds + \frac{2M_0}{\varphi(t)} + M_1 [\varphi]_{C^\gamma} \frac{1}{\varphi(t)} \int_a^b (t-s)^\gamma \psi(t, s)^{1-} ds.$$

From (1.6) and (1.4) it follows:

$$(1.56)\quad \int_a^b \psi(t, s)^{\delta - 1} ds \leqslant [1/(k_1^{1-\delta} g(t, t)^{1-\delta})] \int_a^b (t-s)^{\delta - 1} ds =$$

$$= \begin{cases} (b-a)^\delta / [\delta k_1^{1-\delta} t^{\beta(1-\delta)}] & \text{if } 0 \leqslant a < b \leqslant t \leqslant T/2, \\ (b-a)^\delta / [\delta k_1^{1-\delta} (T-t)^{\beta_1(1-\delta)}] & \text{if } T/2 < a \leqslant b \leqslant t < T; \end{cases}$$

By (1.6) we have, both for $0 \leqslant a < b \leqslant t \leqslant T/2$ and for $T/2 < a \leqslant b \leqslant$ $\leqslant t < T$:

$$(1.57)\quad \int_a^b (t-s)^\gamma \psi(t, s)^{-1} ds \leqslant k_1^{-1} \int_a^b g(t, s)^{-1} (t-s)^{\gamma - 1} ds =: J.$$

Moreover by (1.4) we have:

$$(1.58)\quad J = k_1^{-1} t^{-\beta} \int_a^b (t-s)^{\gamma - 1} ds = \frac{(b-a)^\gamma}{\gamma k_1 t^\beta} \qquad \text{if } 0 \leqslant a < b \leqslant t \leqslant T/2;$$

$$(1.59)\quad J = k_1^{-1} \int_a^b (T-s)^{\beta_1} (t-s)^{\gamma - 1} ds \qquad \text{if } T/2 < a \leqslant b \leqslant t < T.$$

Hence if $0 \leqslant a < b \leqslant t < T/2$, by (1.55), ..., (1.58) we have:

$$\left\| B(\phi(t)) \int_a^b G(\phi(t), \phi(s))\, ds \right\|_{L(X)} \leqslant$$

$$\leqslant \frac{K(hM_2 + c_1)(b-a)^\delta t^{\beta\delta}}{\delta k_1^{1-\delta} \varphi(t)} + \frac{M_1 [\varphi]_{C^\gamma} (b-a)^\gamma}{\gamma k_1 \varphi(t) t^\beta} + \frac{2M_0}{\varphi(t)},$$

and the first inequality of (1.46) follows with

$$C_7(\gamma, \delta) = \max\{K(hM_2 + c_1)/[\delta k_1^{1-\delta}],\ M_1 [\varphi]_{C^\gamma}/[\gamma k_1],\ 2M_0\}.$$

If $T/2 < a \leqslant b \leqslant t \leqslant T$ and $\beta_1 = 0$, by (1.55), (1.56), (1.57), (1.59) and (1.42) we have:

$$\left\| B(\phi(t)) \int_a^b G(\phi(t), \phi(s))\, ds \right\|_{L(X)} \leqslant$$

$$\leqslant [C_7(\gamma, \delta)/\varphi(t)][(b-a)^\gamma + (b-a)^\delta + 1] \leqslant [C_7(\gamma, \delta)/k][T^\gamma + T^\delta + 1],$$

and the second inequality of (1.46) follows.

Finally, if $T/2 < a \leqslant b \leqslant t < T$ and $\beta_1 > 0$, by (1.55), (1.56), (1.57), (1.59), (1.43) and (1.44) we have:

$$\left\| B(\phi(t)) \int_a^b G(\phi(t), \phi(s))\, ds \right\|_{L(X)} \leqslant \frac{K(hM_2 + c_1)(b-a)^\delta (T-t)^{\beta\delta_1}}{\delta k_1^{1-\delta} \varphi(t)} +$$

$$+ \frac{M_1 [\varphi]_{C^\gamma} (T/2)^\nu}{k_1} \int_0^{+\infty} (1+y)^{-\beta_1 - \nu} y^{\gamma - 1} dy \cdot \frac{(T-t)^{\gamma - \beta_1 - \nu}}{\varphi(t)} + \frac{2M_0}{\varphi(t)},$$

with $\nu = 0$ in the case $\beta_1 > \gamma$ and $\nu > 0$ in the case $\beta_1 = \gamma$.

Hence the third inequality of (1.46) follows with

$$C_9(\beta_1, \gamma, \delta, \nu, T) =$$

$$= \max \left\{ \frac{K(hM_2 + c_1)}{\delta k^{1-\delta}}, \ \frac{M_1 [\varphi]_{C^\gamma} (T/2)^\nu}{k_1} \int_0^{+\infty} (1+y)^{-\beta_1 - \nu} y^{\gamma - 1} dy, \ 2M_0 \right\}.$$

The last inequality is a trivial consequence of the previous inequalities and of (1.4). We introduced it to simplify some statements in the sequel. ■

2. The classical solution.

DEFINITION 2.1. Let $f \in C([0,T]; X)$. A function $u \in C([0,T]; X)$ is said to be a classical solution of (0.1) in the interval $[0,T]$ if $u \in C^1(]0,T[; X)$, $t \to A(t)u(t)$ belongs to $C(]0,T[; X)$ and (0.1) holds. ■

Argiung exactly as in [3, Prop. 3.7 (ii)] we get that $x \in \overline{D(A(0))}$ is a necessary condition in order that problem (0.1) has a classical solution u such that $\|A(t)u(t)\| \leqslant \text{const}\, t^{-\mu}$, $\mu \in [0, 1+\delta[$.

In Section 1 we showed (see (1.33), (1.34) and (1.35)) that if $x \in \overline{D(A(0))}$ then the function w defined in (1.28) is a classical solution of (0.1) for $f \equiv 0$. In the general case, arguing exactly as in [2, Thm. 5.2], we get that if problem (0.1) has a classical solution u, then u is given by the representation formula (0.3). Consequently, the classical solution of (0.1) is unique. We shall show that if f is either Hölder continuous with values in X, or bounded with values in some interpolation space, then the function u given by (0.3) is in fact a classical solution of (0.1). We begin by studying the function

$$(2.1)\quad \begin{cases} v(t) = \int_a^t G(\phi(t), \phi(s)) f(s)\, ds, \\ \text{with either } a=0 \text{ and } 0 \leq t \leq T/2, \text{ or } a = T/2 \text{ and } T/2 < t \leq T. \end{cases}$$

We recall that, by (0.3) we have:

$$u(t) = G(\phi(t), 0)\, x + \int_0^t G(\phi(t), \phi(s)) f(s)\, ds \qquad \text{if } 0 \leq t \leq T/2,$$

and

$$u(t) = G(\phi(t), \phi(T/2))\, \overline{x} + \int_{T/2}^t G(\phi(t), \phi(s)) f(s)\, ds \qquad \text{if } T/2 < t \leq T$$

with $\overline{x} = u(T/2)$.

PROPOSITION 2.2. For every continuous $f\colon [0, T] \to X$, v enjoys the following properties: for $0 < \alpha < 1$ we have:

$$(2.2)\qquad v \in C^{\alpha}([0, T]; X);$$

$$(2.3)\qquad v \in B([\varepsilon, T - \varepsilon]; D_{A(\cdot)}(\alpha, \infty)) \qquad \varepsilon \in]0, T/2[;$$

$$(2.4)\quad \begin{cases} \text{if } \beta > 0, & v \in B([0, T/2]; D_{A(\cdot)}(1/(\beta + 1)), \infty), \\ \text{if } \beta = 0, & v \in B([0, T/2]; D_{A(\cdot)}(\alpha, \infty)), \\ \text{if } 0 < \alpha < 1/(\beta_1 + 1), & v \in B([T/2, T]; D_{A(\cdot)}(\alpha, \infty)). \end{cases}$$

PROOF. (2.2) is a simple consequence of estimates (1.17) and (1.24). By (1.17), (1.6) and (1.5) we get, for $0 < t < T$:

$$\|v(t)\|_{D_{A(t)}(\alpha,\infty)} = \left\| \int_a^t G(\phi(t), \phi(s)) f(s)\, ds \right\|_{D_{A(t)}(\alpha,\infty)} \leqslant$$

$$\leqslant c_4(\alpha) \int_a^t \psi(t,s)^{-\alpha} ds \|f\|_\infty \leqslant$$

$$\leqslant \frac{c_4(\alpha)}{k_1^\alpha g(t,t)^\alpha} \int_a^t \frac{ds}{(t-s)^\alpha} \|f\|_\infty \leqslant \frac{c_4(\alpha)(t-a)^{1-\alpha}}{(1-\alpha) k_1^\alpha g(t,t)^\alpha} \|f\|_\infty ,$$

so that (2.3) holds. Concerning (2.4), by (1.17), (1.6) and (1.4) we have for $0 \leqslant t \leqslant T/2$:

$$\|v(t)\|_{D_{A(t)}(\alpha,\infty)} \leqslant \frac{c_4(\alpha)}{(1-\alpha) k_1^\alpha} t^{1-\alpha(\beta+1)} \|f\|_\infty ;$$

for $T/2 < t < T$, if $\alpha < 1/(\beta_1 + 1)$:

$$\|v(t)\|_{D_{A(t)}(\alpha,\infty)} \leqslant [c_4(\alpha)/k_1^\alpha] \int_{T/2}^t (T-s)^{-\alpha\beta_1} (t-s)^{-\alpha} ds \|f\|_\infty \leqslant$$

$$\leqslant [c_4(a)/[k_1^\alpha(1-\alpha(\beta_1+1))]](t-T/2)^{1-\alpha(\beta_1+1)} \|f\|_\infty ,$$

thanks to (1.42). Hence v is bounded with values in $D_{A(t)}(\alpha, \infty)$ as stated in (2.4). ■

Now we state two existence and uniqueness results for the classical solution of (0.1). The first one details with the case where f has values in same interpolation space.

THEOREM 2.3. Let $f \in C([0,T]; X) \cap B([0,T]; D_{A(\cdot)}(\alpha, \infty))$, $0 < \alpha < 1$ and let $x \in \overline{D(A(0))}$. Then the function u defined in (0.3) is the unique classical solution of (0.1). Moreover for $\varepsilon \in]0, T/2[$, $\beta \geqslant 0$ and $\eta \leqslant \alpha$, $\eta < \delta$, we have:

(2.5) $Au \in C^\eta([\varepsilon, T-\varepsilon]; X)$ if $\beta_1 > 0$, $\quad Au \in C^\eta([\varepsilon, T]; X)$ if $\beta_1 = 0$;

(2.6) $u', Au \in B([\varepsilon, T-\varepsilon]; D_{A(\cdot)}(\vartheta, \infty))$, $\quad \vartheta = \min\{\alpha, \delta\}$, $\varepsilon \in]0, T/2[$.

PROOF. Since $u = w + v$, with w given by (1.28), it is sufficient to show that v is the classical solution of (0.1) with $x = 0$. From (2.2) and

(2.1) it follows that $v \in C([0, T]; X)$ and $v(0) = 0$. From (1.23) and (1.5) we get:

$$(2.7) \qquad \|A(t)\,v(t)\| = \left\| B(\phi(t)) \int_a^t G(\phi(t), \phi(s)) f(s)\, ds \right\| \leqslant$$

$$\leqslant \frac{C_0(\alpha)}{g(t, t)^{1-\alpha}} \int_a^t (t-s)^{\alpha - 1} ds \sup_{0 \leqslant s \leqslant T} \|f(s)\|_{D_{A(s)}(\alpha, \infty)} \leqslant$$

$$\leqslant \frac{C_0(\alpha)}{\alpha g(t, t)^{1-\alpha}} (t-a)^{\alpha} \sup_{0 \leqslant s \leqslant T} \|f(s)\|_{D_{A(s)}(\alpha, \infty)}$$

and $v(t) \in D(A(t))$ for each $0 < t < T$. By (2.1), (1.23) and (1.25), both for $0 \leqslant a < \tau < t \leqslant T/2$ and for $T/2 \leqslant a < \tau < t < T$ we have:

$$(2.8) \qquad \|A(t)\,v(t) - A(\tau)\,v(\tau)\| \leqslant \left\| \int_\tau^t B(\phi(t))\, G(\phi(t), \phi(s)) f(s)\, ds \right\| +$$

$$+ \left\| \int_a^\tau [B(\phi(t))\, G(\phi(t), \phi(s)) - B(\phi(\tau))\, G(\phi(\tau), \phi(s))] f(s)\, ds \right\| \leqslant$$

$$\leqslant \left\{ \frac{C_0(\alpha)}{g(t, t)^{1-\alpha}} \int_\tau^t \frac{ds}{(t-s)^{1-\alpha}} + \frac{C_2(\alpha)}{g(\tau, \tau)^{1-\alpha}} \int_a^\tau \left[\frac{1}{(\tau - s)^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}} \right] ds + \right.$$

$$\left. + \frac{C_3(\alpha, \delta, \eta)\, g(t, \tau)^{\eta} g(\tau, a)^{\delta - \eta}}{g(\tau, \tau)^{1-\alpha}} (t - \tau)^{\eta} \int_a^\tau \frac{ds}{(\tau - s)^{1 + \eta - \alpha - \delta}} \right\} \cdot$$

$$\cdot \sup_{0 \leqslant s \leqslant T} \|f(s)\|_{D_{A(s)}(\alpha, \infty)} \leqslant \left\{ \frac{C_0(\alpha)(t-\tau)^{\alpha}}{\alpha g(t, t)^{1-\alpha}} + \frac{2C_2(\alpha)(t-\tau)^{\alpha}}{\alpha g(\tau, \tau)^{1-\alpha}} + \right.$$

$$\left. + \frac{C_3(\alpha, \delta, \eta)\, g(t, \tau)^{\eta} g(\tau, a)^{\delta - \eta}}{(\alpha + \delta - \eta)\, g(\tau, \tau)^{1-\alpha}} (\tau - a)^{\alpha + \delta - \eta} (t - \tau)^{\eta} \right\} \sup_{0 \leqslant s \leqslant T} \|f(s)\|_{D_{A(s)}(\alpha, \infty)},$$

and taking into account (1.4) we get, for $\varepsilon \in]0, T/2[$:

$$(2.9) \quad Av \in C^{\eta}([\varepsilon, T - \varepsilon]; X) \quad \text{if } \beta_1 > 0, \qquad Av \in C^{\eta}([\varepsilon, T]; X) \quad \text{if } \beta_1 = 0.$$

Hence (2.5) follows from (1.32).

Let us show that v is differentiable for $0 < t < T$. For $0 < \varepsilon < 1$ set:

$$v_\varepsilon(t) = \int_a^{t\varepsilon} G(\phi(t), \phi(s)) f(s)\, ds. \tag{2.10}$$

As ε goes to 1, v_ε converges uniformly to v on each compact subset of $]0, T[$; moreover v_ε is differentiable in $]0, T[$ with

$$v_\varepsilon'(t) = \varepsilon G(\phi(t), \phi(t\varepsilon)) f(t\varepsilon) + \varphi(t) B(\phi(t)) \int_a^{t\varepsilon} G(\phi(t), \phi(s)) f(s)\, ds.$$

Therefore, since $f \in C([0, T]; \overline{D(A(t))})$, $v_\varepsilon' \to \varphi(\cdot) A(\cdot) v(\cdot) + f(\cdot)$ as $\varepsilon \to 1$.

Hence v is differentiable in $]0, T[$ with $v'(t) = \varphi(t) A(t) v(t) + f(t)$. Since $\varphi \in C([0, T])$, by (2.5) it follows $v' \in C(]0, T[; X)$. Summing up, we find that $u = w + v$ is a classical solution of (0.1). Concerning (2.6), by (1.31) $w \in B([\varepsilon, T]; D_{A(\cdot)}(\delta + 1, \infty))$. Moreover setting, for $0 < s < t < T$ and $0 < \sigma < \tau < \phi(T)$, $\phi(s) = \sigma$ and $\phi(t) = \tau$ in (2.1), we have:

$$A(t) v(t) = A(\phi^{-1}(\tau)) \int_{\phi(a)}^{\tau} G(\tau, \sigma) \frac{f(\phi^{-1}(\sigma))}{\varphi(\phi^{-1}(\sigma))} d\sigma = \tag{2.11}$$

$$= B(\tau) \int_{\phi(a)}^{\tau} G(\tau, \sigma) \bar{f}(\sigma)\, d\sigma = B(\tau) \bar{v}(\tau).$$

By (1.1) and (1.3) for $0 \leqslant t \leqslant T/2$ and $0 \leqslant \sigma \leqslant \phi(T/2)$ it follows:

$$\varphi(\phi^{-1}(\sigma)) \geqslant k(\phi^{-1}(\sigma))^\beta \geqslant (k/k_1^{\beta/(\beta+1)})\, \sigma^{\beta/(\beta+1)};$$

hence

$$\sigma \to \sigma^{\beta/(\beta+1)} \bar{f}(\sigma) \in B([0, \phi(T/2)]; X) \cap B([0, \phi(T/2)]; D_{B(\cdot)}(\alpha, \infty)). \tag{2.12}$$

Plugging (2.12) in (2.11) and applying Prop. 3.1 (vi) of [3] with $x = 0$, $\mu = \beta/(\beta + 1) < 1$, we get, if $\vartheta = \min\{\alpha, \delta\}$:

$$\tau \to \tau^{\vartheta + \beta/(\beta+1)} B(\tau) \bar{v}(\tau) \in B([0, T/2]; D_{B(\cdot)}(\vartheta, \infty)), \tag{2.13}$$

so that by (2.11), $Av \in B([\varepsilon, T/2]; D_{A(\cdot)}(\vartheta, \infty))$, $0 < \varepsilon < T/2$.

If $T/2 < t \leqslant T$ and $\phi(T/2) < \sigma \leqslant \phi(T)$, by (1.1) and (1.3) it follows:

$$\varphi(\phi^{-1}(\sigma)) \geqslant k[T - \phi^{-1}(\sigma)]^{\beta_1} \geqslant (k/k_1^{\beta_1/(\beta_1+1)})[\phi(T) - \sigma]^{\beta_1/(\beta_1+1)};$$

hence

$$\sigma \to [\phi(T) - \sigma]^{\beta_1/(\beta_1+1)} \tilde{f}(\sigma) \in \tag{2.14}$$

$$\in B([\phi(T/2), \phi(T)]; X) \cap B([\phi(T/2), \phi(T)]; D_{B(\cdot)}(\alpha, \infty)).$$

So that, plugging (2.14) in (2.11), we obtain that

$$Av \in B([T/2, T - \varepsilon]; D_{A(\cdot)}(\vartheta, \infty)), \qquad 0 < \varepsilon < T/2.$$

Since $u = w + v$, then (2.6) holds. ■

Now we consider the case where both f and φ are Hölder continuous functions.

THEOREM 2.4. Let $\varphi \in C^{\gamma}([0, T])$, $0 < \gamma < 1$, $\gamma \leq \beta$ if $\beta > 0$, $\gamma \leq \beta_1$ if $\beta_1 > 0$ (see (1.1)), and let $f \in C^{\alpha}([0, T]; X)$, $0 < \alpha < 1$, $x \in \overline{D(A(0))}$. Then the function u defined in (0.3) is the unique classical solution of (0.1) and

$$u'(t) = w'(t) + \varphi(t) A(t) \int_0^t G(\phi(t), \phi(s))(f(s) - f(t))\, ds + \tag{2.15}$$

$$+ \varphi(t) A(t) \int_0^t G(\phi(t), \phi(s)) f(t)\, ds + f(t) = \varphi(t) A(t) u(t) + f(t).$$

Moreover for each $\varepsilon \in]0, T/2[$, $\eta \leq \min\{\alpha, \gamma\}$ and $\eta < \delta$ we have:

$$u' \in C^{\eta}([\varepsilon, T - \varepsilon]; X) \text{ if } \beta_1 > 0, \qquad u' \in C^{\eta}([\varepsilon, T]; X) \text{ if } \beta_1 = 0; \tag{2.16}$$

$$u' \in B([\varepsilon, T - \varepsilon]; D_{A(\cdot)}(\eta, \infty)). \tag{2.17}$$

PROOF. As before, it is sufficient to show that v is the classical solution of (0.1) with $x = 0$. From (2.2) and (2.1) it follows $v \in C([0, T]; X)$ and $v(0) = 0$. Let us estimate $A(t) v(t)$. By (1.23) and (1.46), for $0 < t < T$ we have (with either $a = 0$ and $0 \leq t \leq T/2$, or a $a = T/2$ and

$T/2 \leqslant t \leqslant T$):

$$\|A(t)v(t)\| = \left\| B(\phi(t)) \int_a^t G(\phi(t), \phi(s))(f(s) - f(t))\, ds \right\| + $$

$$+ \left\| B(\phi(t)) \int_a^t G(\phi(t), \phi(s)) f(t)\, ds \right\| \leqslant$$

$$\leqslant \frac{C_0 [f]_{C^\alpha}}{g(t, t)} \int_a^t \frac{ds}{(t-s)^{1-\alpha}} + \frac{C(\beta, \beta_1, \gamma, \delta, \nu, T)}{g(t, t)\varphi(t)} \|f\|_\infty$$

and $v(t) \in D(A(t))$ for $0 < t < T$. To show (2.16) we set, for $0 < t < < T$:

$$\begin{cases} A(t)v(t) = h_1(t) + h_2(t)/\varphi(t), \\ h_1(t) = A(t) \int_a^t G(\phi(t), \phi(s))(f(s) - f(t))\, ds, \\ h_2(t) = \varphi(t) A(t) \int_a^t G(\phi(t), \phi(s)) f(t)\, ds. \end{cases} \tag{2.18}$$

Both for $0 \leqslant a < \tau < t < T/2$ and for $T/2 \leqslant a < \tau < t < T$ we have:

$$\|h_1(t) - h_1(\tau)\| \leqslant \left\| B(\phi(t)) \int_\tau^t G(\phi(t), \phi(s))(f(s) - f(t))\, ds \right\| + \tag{2.19}$$

$$+ \left\| \int_a^\tau [B(\phi(t)) G(\phi(t), \phi(s)) - B(\phi(\tau)) G(\phi(\tau), \phi(s))](f(s) - f(t))\, ds \right\| +$$

$$+ \left\| B(\phi(t)) \int_a^\tau G(\phi(t), \phi(s))(f(\tau) - f(t))\, ds \right\| = I_1 + I_2 + I_3 .$$

By (1.23) we get:

$$I_1 \leqslant C_0 \frac{(t-\tau)^\alpha}{\alpha g(t, t)} [f]_{C^\alpha};$$

by (1.25) for $\eta < \delta$ we get:

$$I_2 \leqslant \left\{ C_2 \frac{t-\tau}{g(\tau,\tau)} \int_a^\tau (t-s)^{-1}(\tau-s)^{\alpha-1} ds + \right.$$

$$\left. + C_3(\delta,\eta) \frac{g(t,\tau)^\eta g(\tau,a)^{\delta-\eta}}{g(\tau,\tau)} (t-\tau)^\eta \int_a^\tau (\tau-s)^{\alpha+\delta-\eta-1} ds \right\} [f]_{C^\alpha} \leqslant$$

$$\leqslant \left\{ C_2 \frac{(t-\tau)^\alpha}{g(\tau,\tau)} \int_0^{+\infty} (1+y)^{-1} y^{\alpha-1} dy + \right.$$

$$\left. + C_3(\delta,\eta) \frac{g(t,\tau)^\eta g(\tau,a)^{\delta-\eta}}{(\alpha+\delta-\eta)\, g(\tau,\tau)} (t-\tau)^\eta (\tau-a)^{\alpha+\delta-\eta-1} \right\} [f]_{C^\alpha};$$

by (1.46) we have:

$$I_3 \leqslant \frac{C(\beta,\beta_1,\gamma,\delta,\nu,T)}{g(t,t)\,\varphi(t)} (t-\tau)^\alpha [f]_{C^\alpha};$$

and taking into account (1.4) and (1.1), for $0 < \varepsilon < T/2$, we get:

(2.20) $\quad h_1 \in C^\eta([\varepsilon, T-\varepsilon]; X)$ if $\beta_1 > 0$, $\qquad h_1 \in C^\eta([\varepsilon, T]; X)$ if $\beta_1 = 0$.

On the other hand we have:

(2.21)
$$\|h_2(t) - h_2(\tau)\| \leqslant \left\| B(\phi(t)) \int_\tau^t (\varphi(t)-\varphi(s))\, G(\phi(t),\phi(s)) f(t)\, ds \right\| +$$

$$+ \left\| B(\phi(t)) \int_a^\tau (\varphi(t)-\varphi(\tau))\, G(\phi(t),\phi(s)) f(t)\, ds \right\| +$$

$$+ \left\| \int_a^\tau (\varphi(\tau)-\varphi(s)) [B(\phi(t))\, G(\phi(t),\phi(s)) f(t) - B(\phi(\tau))\, G(\phi(\tau),\phi(s)) f(\tau)]\, ds \right\| +$$

$$+ \left\| \int_a^t \varphi(s) B(\phi(t))\, G(\phi(t),\phi(s)) f(t)\, ds - \int_a^\tau \varphi(s) B(\phi(\tau)) G(\phi(\tau),\phi(s)) f(\tau)\, ds \right\| =:$$

$$=: I_1 + I_2 + I_3 + I_4.$$

By (1.23) we have:

$$I_1 \leqslant C_0 \frac{(t-\tau)^\gamma}{\gamma g(t,t)} \|f(t)\| [\varphi]_{C^\gamma};$$

by (1.46) we have:

$$I_2 \leqslant \frac{C(\beta, \beta_1, \gamma, \delta, \nu, T)}{g(t,t)\,\varphi(t)} (t-\tau)^\gamma \|f(t)\| [\varphi]_{C^\gamma};$$

by (1.25) and (1.23) for $\eta < \delta$ we have:

$$I_3 \leqslant \left\{ \int_a^\tau (\tau-s)^\gamma \left[\frac{C_2(t-\tau)}{g(\tau,\tau)(\tau-s)(t-s)} + C_3(\delta,\eta) \frac{g(t,\tau)^\eta g(\tau,s)^{\delta-\eta}(t-\tau)^\eta}{g(\tau,\tau)(\tau-s)^{1-\delta+\eta}} \right] ds \cdot \right.$$

$$\left. \cdot \|f(\tau)\| + \int_a^\tau (\tau-s)^\gamma \|B(\phi(t))\, G(\phi(t), \phi(s))(f(t) - f(\tau))\| ds \right\} [\varphi]_{C^\gamma} \leqslant$$

$$\leqslant \left\{ \left[C_2 \frac{(t-\tau)^\gamma}{g(\tau,\tau)} \int_0^{+\infty} (1+y)^{-1} y^{\gamma-1} dy + \right. \right.$$

$$\left. C_3(\delta,\eta) \frac{g(t,\tau)^\eta g(\tau,a)^{\delta-\eta}(\tau-a)^{\gamma+\delta-\eta}}{(\gamma+\delta-\eta)\, g(\tau,\tau)} (t-\tau)^\eta \right] \cdot$$

$$\left. \cdot \|f(\tau)\| + C_0 \frac{(t-\tau)^\alpha}{g(t,\tau)} \int_a^\tau (\tau-s)^{\gamma-1} ds [f]_{C^\alpha} \right\} [\varphi]_{C^\gamma} \leqslant$$

$$\leqslant \left\{ \left[C(\gamma)\, C_2 \frac{(t-\tau)^\gamma}{g(\tau,\tau)} + C_3(\delta,\eta) \frac{g(t,\tau)^\eta g(\tau,a)^{\delta-\eta}(\tau-a)^{\gamma+\delta-\eta}}{(\gamma+\delta-\eta)\, g(\tau,\tau)} (t-\tau)^\eta \right] \cdot \right.$$

$$\left. \cdot \|f(\tau)\| + C_0 \frac{(t-\tau)^\alpha (\tau-a)^\gamma}{\gamma g(t,\tau)} [f]_{C^\alpha} \right\} [\varphi]_{C^\gamma}.$$

Concerning I_4, set $\phi(s) = \sigma$. Then we obtain:

$$I_4 = \left\| B(\phi(t)) \int_{\phi(a)}^{\phi(t)} G(\phi(t), \sigma) f(t)\, d\sigma - B(\phi(\tau)) \int_{\phi(a)}^{\phi(\tau)} G(\phi(\tau), \sigma) f(\tau)\, d\sigma \right\| =$$

$$= \left\| B(\phi(t)) \int_{\phi(a)}^{\phi(t)} G(\phi(t), \sigma)(f(t) - f(\tau))\, d\sigma \right\| +$$

$$+ \left\| B(\phi(t)) \int_{\phi(a)}^{\phi(t)} G(\phi(t), \sigma) f(\tau)\, d\sigma - B(\phi(\tau)) \int_{\phi(a)}^{\phi(\tau)} G(\phi(\tau), \sigma) f(\tau)\, d\sigma \right\|.$$

Thanks to (1.21), (1.22), (1.2), (1.6) and (1.4) we get:

$$I_4 \leqslant c(t-\tau)^\alpha [f]_{C^\alpha} + c(\alpha) \frac{[\phi(t) - \phi(\tau)]^\alpha}{[\phi(\tau) - \phi(a)]^\alpha} \|f(\tau)\| \leqslant$$

$$\leqslant c(t-\tau)^\alpha [f]_{C^\alpha} + C(\alpha, K, k_1) \frac{g(t, \tau)^\alpha (t-\tau)^\alpha}{g(\tau, a)^\alpha (\tau - a)^\alpha} \|f(\tau)\| \leqslant$$

$$\leqslant \left[c + C(\alpha, K, k_1, \beta, \beta_1, T) \frac{1}{\tau^{\alpha\beta} (\tau - a)^\alpha} \right] (t-\tau)^\alpha \|f\|_{C^\alpha}.$$

Hence by (1.1) and (1.4), for $0 < \varepsilon < T/2$, we get:

$$(2.22) \qquad Av \in C^\eta([\varepsilon, T-\varepsilon]; X) \ \text{ if } \beta_1 > 0, \qquad Av \in C^\eta([\varepsilon, T]; X) \ \text{ if } \beta_1 = 0.$$

Let us show that v is differentiable for $0 < t < T$. For $0 < \varepsilon < 1$. let v_ε be defined by (2.10): then v_ε is differentiable in $]0, T[$ with

$$v_\varepsilon'(t) = \varepsilon G(\phi(t), \phi(t\varepsilon)) f(t\varepsilon) + \varphi(t) B(\phi(t)) \int_a^{t\varepsilon} G(\phi(t), \phi(s))(f(s) - f(t))\, ds +$$

$$+ B(\phi(t)) \int_a^{t\varepsilon} (\varphi(t) - \varphi(s))\, G(\phi(t), \phi(s)) f(t)\, ds +$$

$$+ B(\phi(t)) \int_a^{t} \varphi(s)\, G(\phi(t), \phi(s)) f(t)\, ds - B(\phi(t)) \int_{t\varepsilon}^{t} \varphi(s)\, G(\phi(t), \phi(s)) f(t)\, ds.$$

By (1.45) we get:

$$\lim_{\varepsilon\to 1}\left\| -B(\phi(t))\int_{t\varepsilon}^{t}\varphi(s)\,G(\phi(t),\,\phi(s))f(t)\,ds + e^{\psi(t,\,t\varepsilon)B(\phi(t))}f(t) - f(t)\right\| = 0.$$

By (1.11), (1.12), (1.15) and (1.16), thanks to continuity of f, we have:

$$\lim_{\varepsilon\to 1}\|\varepsilon G(\phi(t),\,\phi(t\varepsilon))f(t\varepsilon) - e^{\psi(t,\,t\varepsilon)B(\phi(t))}f(t)\| =$$

$$= \lim_{\varepsilon\to 1}\left\| e^{\psi(t,\,t\varepsilon)B(\phi(t\varepsilon))}\,\varepsilon f(t\varepsilon) + \int_{\phi(t\varepsilon)}^{\phi(t)} Z(r,\,\phi(t\varepsilon))\,\varepsilon f(t\varepsilon)\,dr -\right.$$

$$\left. - e^{\psi(t,\,t\varepsilon)B(\phi(t\varepsilon)}f(t) - e^{\psi(t,\,t\varepsilon)B(\phi(t))}f(t) + e^{\psi(t,\,t\varepsilon)B(\phi(t\varepsilon))}f(t)\right\| =$$

$$= \lim_{\varepsilon\to 1}\|e^{\psi(t,\,t\varepsilon)B(\phi(t\varepsilon))}(\varepsilon f(t\varepsilon) - f(t)) + [e^{\psi(t,\,t\varepsilon)B(\phi(t\varepsilon))} - e^{\psi(t,\,t\varepsilon)B(\phi(t))}]f(t)\| \leqslant$$

$$\leqslant \lim_{\varepsilon\to 1} hM_2\,\psi(t,\,t\varepsilon)^{\delta}\|f(t)\| = 0.$$

Hence as $\varepsilon \to 1$, $v_\varepsilon'(t)$ converges uniformly to the function

$$z(t) = \varphi(t)\,B(\phi(t))\int_a^t G(\phi(t),\,\phi(s))(f(s) - f(t))\,ds +$$

$$+\varphi(t)\,B(\phi(t))\int_a^t G(\phi(t),\,\phi(s))f(t)\,ds + f(t), \qquad 0 < t < T,$$

on each compact subset of $]0, T[$. Therefore v is differentiable in $]0, T[$ with $v'(t) = z(t)$ for $0 < t < T$ and (2.15) follows. Since $\varphi \in C^{\gamma}([0, T])$ and $u'(t) = w'(t) + \varphi(t)A(t)v(t) + f(t)$, (2.16) follows from (1.39) and (2.22). Summing up, we find that $u = w + v$ is a classical solution of (0.1). Finally from (1.32), (2.22) and (2.16) it follows that

$$u \in C^{1,\,\eta}([\varepsilon,\,T-\varepsilon];\,X) \quad\text{and}\quad t \to A(t)\,u(t) \in C^{\eta}([\varepsilon,\,T-\varepsilon];\,X),$$

so that (2.17) follows by interpolation (it is sufficient to argue as in [11, Lemma 1.1], with obvious modifications). ∎

3. The strict solution.

DEFINITION 3.1. If $f: [0, T] \to X$ is continuous, any function $u \in C^1([0, T]; X)$ is said to be a strict solution of (0.1) in $[0, T]$, if $t \to A(t)u(t)$ belongs to $C(]0, T[; X)$ and (0.1) holds. ■

Since any strict solution is also a classical one, then the strict solution of (0.1) is unique, and it is given by the representation formula (0.3). If u is the strict solution of (0.1) then $\varphi(\cdot)A(\cdot)u(\cdot)$ belongs to $C([0. T]; X)$. We showed in Section 1 that in the case $f \equiv 0$, the function w defined in (1.28) is the strict solution of (0.1) if and only if either $x \in D_{A(0)}(1/(\beta + 1))$ (in the case $\beta > 0$), or $x \in D(A(0))$, $A(0)x \in \overline{D(A(0))}$ (in the case $\beta = 0$) (see (1.32) and (1.36)).

Let us give some sufficient conditions for the existence and uniqueness of the strict solution to problem (0.1). As for the classical solution, we begin with the case where f has values in an interpolation space.

PROPOSITION 3.2. Let $f \in C([0, T]; X) \cap B([0, T]; D_{A(\cdot)}(\alpha, \infty))$, $0 < \alpha < 1$ and assume $x \in D_{A(0)}(1/(\beta + 1))$ (in the case $\beta > 0$), $x \in D(A(0))$, $A(0)x \in \overline{D(A(0))}$ (in the case $\beta = 0$). Then the function u defined in (0.3) is the unique strict solution of (0.1) in $[0, T]$.

PROOF. By Theorem 2.3 u is the classical solution of (0.1). Moreover from (1.36) w belongs to $C^1([0, T]; X)$. Therefore it is sufficient to show that $t \to v'(t)$ is continuous at $t = 0$ and $t = T$.

Since $v'(t) = \varphi(t)A(t)v(t) + f(t)$, $0 < t < T$; by (2.1), (2.7) and (1.4) we get $\lim_{t \to 0^+} v'(t) = f(0)$ and, in the case $\beta_1 > 0$, $\lim_{t \to T^-} v'(t) = f(T)$.

In the case $\beta_1 = 0$, $t \to v'(t)$ is continuous at $t = T$ thanks to (2.9), so that the statement follows. ■

Now we consider the case where both f and φ are Hölder continuous.

THEOREM 3.3. Let $\varphi \in C^\gamma([0, T])$, $0 < \gamma < 1$, $\gamma \leq \beta$ if $\beta > 0$, $\gamma \leq \beta_1$ if $\beta_1 > 0$ (see (1.1)), $\max\{\beta - \gamma, \beta_1 - \gamma\} < 1$ and let $f \in C^\alpha([0, T]; X)$ with $\max\{0, \beta - \gamma, \beta_1 - \gamma\} < \alpha < 1$. Let moreover:

a) $x \in D_{A(0)}(1/(\beta + 1))$, $f(0) \in D_{A(0)}(\vartheta, \infty)$, $\vartheta > (\beta - \gamma)/(\beta + 1)$ if $\beta > 0$;

b) $x \in D(A(0))$, $\varphi(0)A(0)x + f(0) \in \overline{D(A(0))}$ if $\beta = 0$; $f(T) = 0$ if $\beta_1 > 0$.

Then the function u given by (0.3) is the unique strict solution of (0.1) in $[0,T]$.

PROOF. From Theorem 2.4 we know that u is the classical solution of (0.1). Therefore we have only to show that $t \to u'(t) = w'(t) + v'(t)$ is continuous at $t=0$ and $t=T$. First we consider the behaviour near $t=T$. By (1.36) $w' \in C([0,T];X)$. Moreover in the case $\beta_1 = 0$, $t \to u'(t)$ is continuous at $t=T$ thanks to (2.16). We consider now the case $\beta_1 > 0$. By (2.15) using notation (2.18) for $T/2 < t < T$ we have:

$$u'(t) = w'(t) + \varphi(t) h_1(t) + h_2(t) + f(t).$$

From (2.18), (1.23), (1.4) and (1.1) it follows:

$$\|\varphi(t) h_1(t)\| \leq K C_0 (T-t)^{\beta_1} \int_{T/2}^{t} (T-s)^{-\beta_1} (t-s)^{\alpha-1} ds [f]_{C^\alpha}.$$

Hence from (1.42), (1.43) and (1.44) it follows:

$$\lim_{t \to T^-} \|\varphi(t) h_1(t)\| = 0.$$

Moreover by (2.18) and (1.46) it follows:

$$\lim_{t \to T^-} \|h_2(t)\| \leq$$

$$\leq C_9(\beta_1, \gamma, \delta, \nu, T) \lim_{t \to T^-} [1/(T-t)^{\beta_1 - \gamma + \nu} + (T-t)^{\delta\beta_1}(t - T/2)^{\delta} + 1] \cdot$$

$$\cdot (T-t)^{\alpha} [f]_{C^\alpha} = 0 \qquad (\text{choosing } 0 < \nu < \alpha - \beta_1 + \gamma).$$

Concerning the behaviour of u' as $t \to 0$, we set for $0 < t \leq T/2$:

$$\text{(3.1)} \qquad \begin{cases} \overline{h}_2(t) = \varphi(t) A(t) \int_0^t G(\phi(t), \phi(s))(f(t) - f(0))\, ds, \\ h_3(t) = \varphi(t) A(t) \int_0^t G(\phi(t), \phi(s)) f(0)\, ds. \end{cases}$$

By (2.15), using notation (2.18) and (3.1) for $0 < t \leq T/2$ we have:

$$u'(t) = w'(t) + \varphi(t) h_1(t) + \overline{h}_2(t) + h_3(t) + f(t).$$

From (2.18), (1.23), (1.4) and (1.1) it follows, for $\beta \geqslant 0$:

$$\lim_{t \to 0^+} \|\varphi(t) h_1(t)\| \leqslant KC_0 \lim_{t \to 0^+} \int_0^t (t-s)^{\alpha-1} ds [f]_{C^\alpha} = 0 .$$

From (3.1) and (1.46) it follows, for $\beta \geqslant 0$:

$$\lim_{t \to 0^+} \|\bar{h}_2(t)\| \leqslant C_7(\gamma, \delta) \lim_{t \to 0^+} [t^{\gamma-\beta} + t^{\delta(\beta+1)} + 1] t^\alpha [f]_{C^\alpha} = 0 .$$

By (3.1) we get:

$$h_3(t) = \int_0^t \varphi(s) B(\phi(t)) G(\phi(t), \phi(s)) f(0)\, ds + \tag{3.2}$$

$$+ B(\phi(t)) \int_0^t (\varphi(t) - \varphi(s)) G(\phi(t), \phi(s)) f(0)\, ds =: I_1 + I_2 ,$$

and by (1.45) we get:

$$\|I_1 - e^{\phi(t) B(\phi(t))} f(0) + f(0)\| = O(t^\delta) . \tag{3.3}$$

We distinguish now two cases: $\beta > 0$ and $\beta = 0$.

In the case $\beta > 0$, from [2, Thm. 4.1 (iii)], (1.23) and (1.4) we get:

$$\|I_2\| \leqslant \left\| B(\phi(t)) \int_0^t (\varphi(t) - \varphi(s)) G(\phi(t), \phi(s)) [1 - G(\phi(s), 0)] f(0)\, ds \right\| + \tag{3.4}$$

$$+ \left\| B(\phi(t)) \int_0^t (\varphi(t) - \varphi(s)) G(\phi(t), \phi(s)) G(\phi(s), 0) f(0)\, ds \right\| \leqslant$$

$$\leqslant \left[C(\vartheta) t^{-\beta} \int_0^t (t-s)^{\gamma-1} \phi(s)^\vartheta ds + C_0(\vartheta) t^{-(\beta+1)(1-\vartheta)} \int_0^t (t-s)^\gamma ds \right] \cdot$$

$$\cdot [\varphi]_{C^\gamma} \|f(0)\|_{D_{A(0)}(\vartheta, \infty)} \leqslant \left[C(K, \vartheta) t^{-\beta} \int_0^t s^{(\beta+1)\vartheta} (t-s)^{\gamma-1} ds + C(\gamma, \vartheta) t^{\gamma-\beta+(\beta+1)\vartheta} \right] \cdot$$

$$\cdot [\varphi]_{C^\gamma} \|f(0)\|_{D_{A(0)}(\vartheta, \infty)} \leqslant C(K, \gamma, \vartheta) t^{\gamma-\beta+(\beta+1)\vartheta} [\varphi]_{C^\gamma} \|f(0)\|_{D_{A(0)}(\vartheta, \infty)} .$$

Therefore by (3.2), (3.3) and (3.4), if $\beta > 0$, $\lim_{t \to 0^+} h_3(t) = 0$, since $f(0) \in$

$\in \overline{D(A(0))}$ so that $e^{\phi(t)B(\phi(t))}f(0) \to f(0)$ as $t \to 0^+$. In the case $\beta = 0$ by (1.23) and (1.4) we have:

$$\lim_{t \to 0^+} \|I_2\| \leq \lim_{t \to 0^+} C_0 \int_0^t (t-s)^{\gamma - 1} ds [\varphi]_{C^\gamma} \|f\|_\infty = 0. \tag{3.5}$$

Then, from (3.2), (3.3), (3.5), (1.34), [2, (2.10)], (1.12) and (1.16) it follows:

$$\lim_{t \to 0^+} [w'(t) + h_3(t) + f(t)] = \lim_{t \to 0^+} [w'(t) + e^{\phi(t)B(\phi(t))}f(0) - f(0) + f(t)] =$$

$$= \lim_{t \to 0^+} \{(\varphi(t) - \varphi(0))B(\phi(t))G(\phi(t), 0)x + \varphi(0)[B(0)e^{\phi(t)B(0)}x + Z(\phi(t), 0)x] +$$

$$+ [e^{\phi(t)B(\phi(t))} - e^{\phi(t)B(0)}]f(0) + e^{\phi(t)B(0)}f(0)\} =$$

$$= \lim_{t \to 0^+} [\varphi(0)B(0)e^{\phi(t)B(0)}x + e^{\phi(t)B(0)}f(0)] =$$

$$= \lim_{t \to 0^+} e^{\phi(t)A(0)}[\varphi(0)A(0)x + f(0)] = \varphi(0)A(0)x + f(0),$$

since $\varphi(0)A(0)x + f(0) \in \overline{D(A(0))}$. Therefore both for $\beta > 0$ and for $\beta = 0$ there exist $\lim_{t \to 0^+} u'(t)$.

Summarizing, under the previous assumptions u' is continuous up to $t = 0$ and $t = T$, with $u'(0) = f(0)$ if $\beta > 0$, $u'(0) = \varphi(0)A(0)x + f(0)$ if $\beta = 0$, and $u'(T) = 0$ if $\beta_1 > 0$, $u'(T) = \varphi(T)A(T)u(T) + f(T)$ if $\beta_1 = 0$. ■

In Theorem 3.3 we assumed $\beta - \gamma < 1$ and $\beta_1 - \gamma < 1$ for simplicity.

In fact, we could study the existence of a strict solution to (0.1) for any value of $\beta - \gamma$ and $\beta_1 - \gamma$; obviously, we should made much more regularity assumptions on f.

Now we show some further regularity properties of the strict solution of problem (0.1): roughly speaking, the regularity of u up to $t = 0$ increases as the regularity of the initial value x increases.

PROPOSITION 3.4. Let $f \in C([0, T]; X) \cap B([0, T]; D_{A(\cdot)}(\alpha, \infty))$, $0 < \alpha \leq \delta$, and let $x \in D_{A(0)}(\alpha + 1, \infty)$ if $\beta = 0$, $x \in D_{A(0)}(\alpha + 1/(\beta + 1), \infty)$,

$0 < \alpha < \beta/(\beta+1)$ if $\beta > 0$. Let u be the strict solution of (0.1). Then

$$\text{(3.6)}\quad \begin{cases} \text{if } \beta = \beta_1 = 0 & u' \in B([0,T]; D_{A(\cdot)}(\alpha, \infty)), \\ \text{if } \beta = 0,\ \beta_1 > 0,\ 0 < \varepsilon < T/2 & u' \in B([0, T-\varepsilon]; D_{A(\cdot)}(\alpha, \infty)); \end{cases}$$

$$\text{(3.7)}\quad \begin{cases} \text{if } \beta > 0 \text{ and } \beta_1 = 0 & t^{\alpha(\beta+1)} u' \in B([0,T]; D_{A(\cdot)}(\alpha, \infty)), \\ \text{if } \beta > 0,\ \beta_1 > 0,\ 0 < \varepsilon < T/2 & t^{\alpha(\beta+1)} u' \in B([0, T-\varepsilon]; D_{A(\cdot)}(\alpha, \infty)); \end{cases}$$

$$\text{(3.8)}\qquad u' \in B([0,T]; D_{A(\cdot)}(\alpha, \infty)) \quad \text{for every } \vartheta \in]0, \alpha[.$$

PROOF. Since $u'(t) = w'(t) + v'(t)$, $0 \leq t \leq T$, and by (1.37) $w' \in B[0,T]; D_{A(\cdot)}(\alpha, \infty)$, it is sufficient to show that $v'(t) = \varphi(t) A(t) v(t) + f(t)$ satisfies (3.6), (3.7) and (3.8). Arguing as in Theorem 2.3, in the case $\beta = 0$, thanks to (2.11), (2.12) and (2.14), (3.6) follows from Prop. 3.1 (iii) of [3]. In the case $\beta > 0$, by (1.3) we get:

$$\text{(3.9)}\qquad \begin{cases} \tau = \phi(t) \geq k_1 t^{\beta+1}, \quad 0 \leq t \leq T/2 \\ \text{and} \\ \tau^{\alpha+\beta/(\beta+1)} \geq k_1^{\alpha+\beta/(\beta+1)} t^{\beta+\alpha(\beta+1)}. \end{cases}$$

Therefore from (2.11), (2.13) and (3.9) it follows that $t^{\alpha(\beta+1)} v'(t)$ is bounded with values in $D_{A(t)}(\alpha, \infty)$ for $0 \leq t \leq T/2$, so that (3.7) holds thanks to (3.6). Moreover for each β and β_1, by (1.23) for $0 < \vartheta < \alpha$, $0 \leq t \leq T$ we have:

$$\varphi(t) \| A(t) v(t) \|_{D_{A(t)}(\vartheta, \infty)} \leq$$

$$\leq K C_0(\alpha, \vartheta) g(t,t)^{\alpha-\vartheta} \int_a^t (t-s)^{\alpha-\vartheta-1} ds \sup_{0 \leq s \leq T} \|f(s)\|_{D_{A(s)}(\alpha, \infty)} \leq$$

$$\leq \frac{K C_0(\vartheta, \infty)}{\alpha - \vartheta} g(t,t)^{\alpha-\vartheta} (t-a)^{\alpha-\vartheta} \sup_{0 \leq s \leq T} \|f(s)\|_{D_{A(s)}(\alpha, \infty)} < +\infty.$$

Hence $v'(t)$ belongs to $B([0,T]; D_{A(\cdot)}(\vartheta, \infty))$ and (3.8) holds. ∎

PROPOSITION 3.5. Let $\varphi \in C^\gamma([0,T])$, $0 < \gamma < 1$, $\gamma \leq \beta$ if $\beta > 0$, $\gamma \leq \beta_1$, if $\beta_1 > 0$, and let $f \in C([0,T]; X) \cap B([0,T]; D_{A(\cdot)}(\alpha, \infty))$, $\max\{0, (\beta-\gamma)/(\beta+1), \beta_1/(\beta_1+1)\} < \alpha < 1$, $x \in D_{A(0)}(\alpha + 1/(\beta+1), \infty)$.

Let u be the strict solution of (0.1). Then

$$\varphi A u \in C^{\eta}([0, T]; X) \tag{3.10}$$

with $\eta < \delta$, $\eta \leqslant \min\{\alpha, \gamma, \alpha(\beta+1) - (\beta - \gamma)\}$.

PROOF. By Prop. 3.2, $\varphi(t)A(t)u(t) = w'(t) + \varphi(t)A(t)v(t)$, $0 \leqslant t \leqslant T$. By (1.40) and (1.41) $w' \in C^{\eta}([0, T]; X)$. Moreover from (2.1), (1.23), (1.4), (2.7) and (2.8) it follows for $0 \leqslant \tau < t \leqslant T/2$:

$$\|\varphi(t)A(t)v(t) - \varphi(\tau)A(\tau)v(\tau) \leqslant$$

$$\leqslant |\varphi(t) - \varphi(\tau)|\,\|A(t)v(t)\| + \varphi(\tau)\|A(t)v(t) - A(\tau)v(\tau)\| \leqslant$$

$$\leqslant \left\{\frac{C_0(\alpha)}{\alpha} t^{\alpha(\beta+1)-\beta}(t-\tau)^{\gamma}[\varphi]_{C^{\gamma}} + \frac{KC_0(\alpha)}{\alpha} t^{\alpha\beta}\cdot(t-\tau)^{\alpha} + \frac{2KC_2(\alpha)}{\alpha}\tau^{\alpha\beta}\cdot\right.$$

$$\left.\cdot(t-\tau)^{\alpha} + \frac{KC_3(\alpha,\delta,\eta)}{\alpha+\delta-\eta} t^{\beta\eta}\tau^{(\alpha+\delta-\eta)(\beta+1)}(t-\tau)^{\eta}\right\} \sup_{0 \leqslant s \leqslant T/2} \|f(s)\|_{D_{A(s)}(\alpha,\infty)};$$

and for $T/2 < \tau < t \leqslant T$:

$$\|\varphi(t)A(t)v(t) - \varphi(\tau)A(\tau)v(\tau) \leqslant$$

$$\leqslant |\varphi(t) - \varphi(\tau)|\,\|A(\tau)v(\tau)\| + \varphi(t)\|A(t)v(t) - A(\tau)v(\tau)\| \leqslant$$

$$\leqslant \left\{C_0(\alpha)(t-\tau)^{\gamma}\int_{T/2}^{\tau}(T-s)^{\beta_1(\alpha+1)}(\tau-s)^{\alpha-1}ds[\varphi]_{C^{\gamma}} + \right.$$

$$+ \frac{KC_0(\alpha)}{\alpha}(T-t)^{\alpha\beta_1}(t-\tau)^{\alpha} + \frac{2KC_2(\alpha)}{\alpha}\cdot$$

$$\cdot(T-\tau)^{\alpha\beta_1}(t-\tau)^{\alpha} + \frac{KC_3(\alpha,\delta,\eta)}{\alpha+\delta-\eta}(T-\tau)^{(\alpha-\eta)\beta_1}\cdot$$

$$\left.\cdot(T-T/2)^{(\delta-\eta)\beta_1}(\tau-T/2)^{\alpha+\delta-\eta}(t-\tau)^{\eta}\right\} \sup_{T/2 < s \leqslant T} \|f(s)\|_{D_{A(s)}(\alpha,\infty)}.$$

Hence (3.7) follows, thanks to (1.42). ■

DEFINITION 3.6. Let

$$f \in C([0, T]; X), \quad x \in X.$$

A function $u \in C([0, T]; X)$ is said to be a strong solution of (0.1) in the interval $[0, T]$ if there is a sequence $u_n \in C^1([0, T]; X)$ with $\varphi(\cdot)A(\cdot)u_n(\cdot) \in$

$\in C([0, T]; X)$ such that

$$u_n \to u \quad \text{uniformly in } [0, T] \text{ as } n \to \infty,$$

$$u_n' - \varphi(\cdot)A(\cdot)u_n(\cdot) \to u \quad \text{uniformly in } [0, T] \text{ as } n \to \infty. \quad ■$$

PROPOSITION 3.7. Let $f \in C([0, T]; X)$, $x \in \overline{D(A(0))}$. Moreover, in the case $\beta > 0$, we assume also $f(0) \in \overline{D(A(0))}$. Then the function u defined in (0.3) is the unique strong solution of (0.1).

PROOF. Let us consider first the case $\beta > 0$. Let $\varphi_n \in C^\gamma([0, T]; X)$, $\gamma > 0$, be such that $\varphi_n \to \varphi$ in $C([0, T])$ as $n \to \infty$, and $\varphi_n(t) > 0$ for every t. Let $f_n \in C^\alpha([0, T]; X)$, $0 < \alpha < 1$, be such that $f_n \to f$ in $C([0, T]; X)$ as $n \to \infty$, and $f_n(0) = f(0)$. Let finally $x_n \in D(A^2(0))$ be such that $x_n \to x$ in X. We can apply Theorem 3.3 to problem

$$\text{(3.11)} \qquad \begin{cases} u_n'(t) = \varphi_n(t)A(t)u_n(t) + f_n(t), & 0 < t \leq T, \\ u_n(0) = x_n. \end{cases}$$

By Theorem 3.3*b*), the function

$$\text{(3.12)} \qquad u_n(t) = G(\phi_n(t), 0)x_n + \int_0^t G(\phi_n(t), \phi_n(s))f_n(s)\,ds,$$

where $\phi_n(t) = \int_0^t \varphi_n(s)\,ds$, is the unique strict solution to (3.11). Recalling that $x \in \overline{D(A(0))}$ and letting $n \to \infty$ in (3.12), we obtain that $u_n \to u$ uniformly in $[0, T]$, where u is defined in (0.3). Let us consider now the case $\beta = 0$. Let φ_n, f_n be as before, and choose $\lambda \in \rho(A(0))$.

Let $z_n \in D(A^2(0))$ be such that

$$\lim_{n\to\infty} z_n = x - (\lambda - A(0))^{-1}f(0)/\varphi(0)$$

and set

$$x_n = z_n + (\lambda - A(0))^{-1}f(0)/\varphi(0).$$

Then x_n belongs to $D(A(0))$ and $\lim_{n\to\infty} x_n = x$; moreover

$$A(0)\varphi(0)x_n + f_n(0) = A(0)\varphi(0)z_n + \lambda(\lambda - A(0))^{-1}f(0) \in D(A(0)).$$

By Theorem 3.3*b*), the function u_n defined in (3.12) is the strict solution of (3.11) and the conclusion is the same as in the case $\beta > 0$. ■

4. An application.

We apply here some results of the previous sections to a degenerate parabolic initial boundary value problem:

$$
(4.1)\quad \begin{cases} u_t(t,x) = \varphi(t) \sum_{i,j=1}^{n} a_{ij}(t,x)\, u_{x_i x_j}(t,x) + f(t,x), & 0<t<T,\ x \in \overline{\Omega}, \\ u(0,x) = u_0(x), & x \in \overline{\Omega}, \\ \sum_{i=1}^{n} b_i(t,x)\, u_{x_i}(t,x) + c(t,x)\, u(t,x) = 0, & 0<t<T,\ x \in \partial\Omega, \end{cases}
$$

where Ω is a bounded open set in $\mathbb{R}^n$ with regular boundary $\partial\Omega$:

$$
(4.2)\quad \begin{cases} a_{ij} \in C([0,T] \times \overline{\Omega}), \qquad i,j = 1, \dots, n \ \text{and} \\ \operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(t,x)\, \xi_i \xi_j \geq \nu |\xi|^2, \\ \qquad \nu > 0,\ \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,\ \forall t \in [0,T],\ \forall x \in \overline{\Omega}, \\ \sum_{i=1}^{n} b_i(x)\, \nu_i(x) \neq 0, \qquad \forall x \in \partial\Omega,\ \nu(t) = (\nu_1(x), \dots, \nu_n(x)) = \text{unit} \\ \qquad \text{exterior normal vector to } \partial\Omega \text{ at } x, \\ t \to b_i(t,\cdot), \quad c(t,\cdot) \in C^{\mu_1}([0,T]; C^1(\partial\Omega)), \qquad \mu_1 > 1/2, \\ t \to a_{ij}(t,\cdot) \in C^{\mu}([0,T]; C(\overline{\Omega})), \qquad 0 < \mu < 1, \\ \delta = \min\{\mu - \varepsilon, \mu_1 - 1/2\}, \qquad \varepsilon > 0. \end{cases}
$$

Setting $u(t,\cdot) = u(t)$, $f(t,\cdot) = f(t)$ we can write problem (4.1) as an abstract Cauchy problem of the type (0.1) choosing:

$$
(4.3)\quad \begin{cases} X = C(\overline{\Omega}), \\ D(A(t)) = \Big\{ g \in \bigcap_{p>1} W^{2,p}(\Omega) \colon A(t) g \in C(\overline{\Omega}),\ C(t) g \equiv \\ \qquad \equiv \sum_{i=1}^{n} b_i(t,x)\, g_{x_i} + c(t,x)\, g \big|_{\partial\Omega} = 0 \Big\}, \\ A(t) g = \sum_{i,i=1}^{n} a_{ij}(t,\cdot)\, g_{x_i x_j}, \qquad \forall g \in D(A(t)). \end{cases}
$$

Then $\overline{D(A(t))} = X$ and $A(t)$ generates, for $t \in [0,T]$, an analytic semigroup in X thanks to [14].

The interpolation spaces $D_{A(t)}(\vartheta, \infty)$ are given by (see [1], [4]):

$$\begin{cases} D_{A(t)}(\vartheta, \infty) = \{g \in C^{2\vartheta}(\overline{\Omega}) \colon C(t) g|_{\partial\Omega} = 0\} \quad \text{if } \vartheta > 1/2, \\ D_{A(t)}(\vartheta, \infty) = C^{2\vartheta}(\overline{\Omega}) \quad \text{if } \vartheta < 1/2, \\ D_{A(t)}(1/2, \infty) = \Big\{ g \in C^0(\overline{\Omega}) \colon \exists \overline{K} \in \mathbb{R} \text{ such that:} \\ \left| g(x) + g(y) - 2g\left(\dfrac{x+y}{2}\right)\right| \leq \overline{K}|x-y|, \ \forall x, y \in \overline{\Omega} \text{ with } \dfrac{x+y}{2} \in \overline{\Omega}, \\ |g(x - \sigma b(x)) - g(x)| \leq \overline{K}\sigma \ \forall x \in \partial\Omega, \ \forall \sigma > 0 \text{ with } x - \sigma b(x) \in \overline{\Omega} \Big\}. \end{cases}$$

If in addition $a_{ij}(t, \cdot) \in C^{2\vartheta}(\overline{\Omega})$, $b_i(t, \cdot)$, $c(t, \cdot) \in C^{2\vartheta+1}(\partial\Omega)$, then $D_{A(t)}(\vartheta + 1, \infty)$ is given by

$$D_{A(t)}(\vartheta + 1, \infty) = \{g \in C^{2\vartheta+2}(\overline{\Omega}) \colon C(t) g|_{\partial\Omega} = C(t) A(t) g|_{\partial\Omega} = 0\} \qquad \text{if } \vartheta > 1/2,$$

$$D_{A(t)}(\vartheta + 1, \infty) = \{g \in C^{2\vartheta+2}(\overline{\Omega}) \colon C(t) g|_{\partial\Omega} = 0\} \quad \text{if } \vartheta < 1/2.$$

We state now two existence theorems for the classical and strict solution to (4.1) in the case where f is Hölder continuous either with respect to x or with respect to time.

PROPOSITION 4.1. Let φ satisfy (1.1), let $u_0 \in C(\overline{\Omega})$ and let $f \in C([0, T] \times \overline{\Omega})$ be such that

$$f(t, \cdot) \in C^{2\alpha}(\overline{\Omega}) \quad \forall t \in [0, T], \qquad \sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{C^{2\alpha}(\overline{\Omega})} < +\infty \tag{4.4}$$

with $0 < 2\alpha < 1$. Then there is a unique function $u \in C([0, T] \times \overline{\Omega})$ such that there exist u_t, $A(\cdot) u \in C(]0, T[\times \overline{\Omega})$ and u satisfies (4.1).

Moreover, let δ be defined by (4.2). The following statements hold true:

1) if either $\beta > 0$ and $u_0 \in D_{A(0)}(1/(\beta + 1))$, or $\beta = 0$ and $u_0 \in D(A(0))$, then

$$u_t, \ A(\cdot) u \in C([0, T] \times \overline{\Omega}); \tag{4.5}$$

2) if either $\beta > 0$ and $u_0 \in D_{A(0)}(\alpha + 1/(\beta + 1), \infty)$, $0 < \alpha < \beta/(\beta +$

$+ 1)$, $\alpha \leqslant \delta$, or $\beta = 0$ and $u_0 \in D_{A(0)}(\alpha + 1, \infty)$, then, for $0 < \vartheta < 2\alpha$,

$$(4.6) \quad \begin{cases} u_t(t, \cdot), \ \ \varphi(t) A(t) u(t, \cdot) \in C^{\vartheta}(\overline{\Omega}) \text{ and} \\ \sup\limits_{0 \leqslant t \leqslant T} \|u_t(t, \cdot)\|_{C^{\vartheta}(\overline{\Omega})} < +\infty, \quad \sup\limits_{0 \leqslant t \leqslant T} \|\varphi(t) A(t) u(t, \cdot)\|_{C^{\vartheta}(\overline{\Omega})} < +\infty; \end{cases}$$

3) if $\beta = \beta_1 = 0$, then

$$(4.7) \qquad u_t(t, \cdot), \qquad \varphi(t) A(t) u(t, \cdot) \in C^{2\alpha}(\overline{\Omega}),$$

and u_t, $\varphi A u$ enjoy the same regularity properties of f.

PROOF. Under our assumptions, $\underline{t \to f(t, \cdot)}$ belongs to $C([0, T]; X) \cap B([0, T]; D_{A(\cdot)}(\alpha, \infty))$, and $u_0 \in \overline{D(A(0))}$. Then Theorem 2.3 is applicable and yields the first part of the statement. (4.5) and (4.6) follow from Propositions 3.2 and 3.4 respectively; (4.7) follows from Proposition 3.4. ■

PROPOSITION 4.2. Let $\varphi \in C^{\gamma}([0, T])$, $0 < \gamma < 1$ ($\gamma \leqslant \beta$ if $\beta > 0$, $\gamma \leqslant \beta_1$ if $\beta_1 > 0$), satisfy (1.1); let $u_0 \in C(\overline{\Omega})$ and let $f \in C([0, T] \times \overline{\Omega})$ be such that

$$(4.8) \quad f(\cdot, x) \in C^{\alpha}([0, T]) \quad \forall x \in (\overline{\Omega}); \qquad \sup_{x \in \overline{\Omega}} \|f(\cdot, x)\|_{C^{\alpha}([0, T])} < +\infty.$$

Then there is a unique function $u \in C([0, T] \times \overline{\Omega})$ such that there exist u_t, $A(\cdot)u \in C(]0, T[\times \overline{\Omega})$ and u satisfies (4.1).

Moreover for each $\varepsilon \in]0, T/2[$,

$$(4.9) \qquad u_t(\cdot, x), \ A(\cdot) u(\cdot, x) \text{ belong to } C^{\eta}([\varepsilon, T - \varepsilon]),$$

uniformly with respect to $x \in \overline{\Omega}$,

$$(4.10) \quad u_t(t, \cdot) \in C^{2\eta}(\overline{\Omega}) \ \forall t \in [\varepsilon, T - \varepsilon] \ \text{ and } \ \sup_{\varepsilon \leqslant t \leqslant T - \varepsilon} \|u_t(t, \cdot)\|_{C^{2\eta}(\overline{\Omega})} < +\infty,$$

with $\eta \leqslant \min\{\alpha, \gamma\}$, $\eta < \delta$.

In addition if $\max\{\beta - \gamma, \beta_1 - \gamma\} < 1$, $\max\{0, \beta - \gamma, \beta_1 - \gamma\} < \alpha < 1$ and

$$\begin{aligned} &u_0 \in D_{A(0)}(1/(\beta + 1)), \ f(0) \in D_{A(0)}(\vartheta, \infty), \ \vartheta > (\beta - \gamma)/(\beta + 1) && \text{if } \beta > 0, \\ &u_0 \in D(A(0)) && \text{if } \beta = 0, \\ &f(T) = 0 && \text{if } \beta_1 > 0, \end{aligned}$$

then

$$(4.11) \qquad u_t, \quad A(\cdot) u \in C([0, T] \times \overline{\Omega}).$$

PROOF. Assumption (4.8) implies that $t \to f(t, \cdot)$ belongs to $C^{\alpha}([0, T]; X)$. The first part of the statement, (4.9) and (4.10) follow applying Theorem 2.4. Finally (4.11) follows from Theorem 3.3. ∎

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