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Direct and Inverse Problems in the Theory of Materials with Memory.

A. LORENZI - E. PAPARONI (*)

SUMMARY - We prove some existence, uniqueness and stability results for a direct and an inverse problem related to the linear integrodifferential equations arising in the theory of materials with memory having a non-smooth memory function.

0. Introduction.

In this paper we deal with two different, but related, problems arising in the theory of materials with memory. The former is a direct problem, while the latter is an identification problem. Both are characterized by the fact that the function k describing the history of the material is assumed to be non smooth.

Our direct problem is the following: *determine a function* $u \in W^{1+\sigma,p}((0,T);X) \cap W^{\sigma,p}((0,T);Y)$ *such that*

$$u'(t) - Au(t) - \int_0^t k(t-s)[Bu(s) + b(s)]\,ds = f(t), \tag{0.1}$$

for a.e. $t \in (0, T)$,

$$u(0) = u_0. \tag{0.2}$$

We assume that X is a Banach space, $b, f \in W^{\sigma,p}((0,T);X)$ $(1 < p < +\infty)$, $k \in L^q(0,T)$ for some $q \in (1, +\infty]$, $u_0 \in X$ and $A\colon D_A = Y \subset X \to X$, $B\colon D_B \subset X \to X$ are two linear closed operators such that:

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(0.3) there exist two positive constants M and $\phi \in (\pi/2, \pi)$ such that
i) $(z - A)^{-1} \in \mathcal{L}(X)$ $\forall z \in \Sigma_\phi$;
ii) $\|z(z-A)^{-1}\|_{\mathcal{L}(X)} \leq M$ $\forall z \in \Sigma_\phi$, where $\Sigma_\phi = \{z \in \mathbb{C}: |\arg z| \leq \phi\}$;

$$D_A \subset D_B \subset \overline{D_A} = X. \tag{0.4}$$

We recall that $\mathcal{L}(X)$ denotes the Banach space of all bounded linear operators from X into itself endowed with the Sup-norm. Moreover D_A and D_B are equipped with the graph-norm.

Under assumption (0.3) A defines an analytic semigroup e^{tA} of bounded linear operators satisfying the following estimates for some positive constant M_0 and M_1:

$$\text{(0.5)} \qquad \begin{aligned} &\text{iii)}\ \|e^{tA}\|_{\mathcal{L}(X)} \leq M_0, && \forall t \in (0, +\infty), \\ &\text{iv)}\ \|Ae^{tA}\|_{\mathcal{L}(X)} \leq M_1 t^{-1}, && \forall t \in (0, +\infty). \end{aligned}$$

We assume also that A admits an inverse $A^{-1} \in \mathcal{L}(X)$. This is not a restriction: in fact, it suffices to replace our unknown (u, k) and datum (b, f) by the pairs $(\bar{u}, \bar{k})$, and $(\bar{b}, \bar{f})$ respectively, where $\bar{u}(t) = e^{-\lambda t}u(t)$, $\bar{k}(t) = e^{-\lambda t}k(t)$, $\bar{b}(t) = e^{-\lambda t}b(t)$, $\bar{f}(t) = e^{-\lambda t}f(t)$ and λ is any (fixed) positive constant. By such a procedure we get the equation $(\bar{A}, \bar{B}) = (A - \lambda, B)$.

According to the closed graph theorem we can assume that $BA^{-1} \in \mathcal{L}(X)$. Hence there exists a positive constant M_2 such that

$$\|BA^{-1}\|_{\mathcal{L}(X)} \leq M_2. \tag{0.6}$$

REMARK 0.1. The presence of a known function b inside the integral allows a unified treatment of our direct and inverse problems.

Assume now that k is unknown too. We consider the following inverse problem: *determine a pair of functions* $(u, k) \in [W^{2+\sigma, p}((0, T); X) \cap W^{1+\sigma, p}((0, T); Y)]\, W^{\sigma, p}(0, T)$ *solution to problem* (0.1), (0.2) *and satisfying the additional information*

$$\Phi[u(t)] = g(t) \qquad t \in [0, T], \tag{0.7}$$

Φ and g being, respectively, a prescribed functional in X^* (the dual to X) and a function in $W^{2+\sigma, p}(0, T)$. As far as b, f and u_0 are concerned, we assume that $b, f \in W^{1+\sigma, p}((0, T); X)$ and $u_0 \in D_A$.

Also in this case we can assume that A admits an inverse $A^{-1} \in \mathcal{L}(X)$. It suffices, in fact, to perform the same transformation as before and to set $\bar{g}(t) = e^{-\lambda t}g(t)$.

Several authors studied direct linear problems similar to ours. They proved existence, uniqueness and regularity theorems. However, we limit ourselves to mentioning only papers [1] and [3], since they deal with non-smooth memory operators.

We note that in [3] the author studies a more general equation, but under more restrictive assumptions than ours. In fact, in our case she should require that $k \in L^{p'}(0, T)$ $(1/p + 1/p' = 1)$, while it suffices to require $k \in L^q(0, T)$ for some $q \in (1, +\infty]$ (cfr. Section 1).

As far as our inverse problem is concerned, we recall that similar problems (with $b=0$) were studied in [4], [5] and solved in the class $\mathcal{K}_{\alpha,\beta}$ of functions k such that $t^{1-\alpha}k \in C^\beta([0, T])$, where $\alpha \in (0,1)$ and $\beta \in (0, \min(\alpha, 1-\alpha))$. In the quoted papers results of existence, uniqueness and continuous dependence with respect to data were obtained in the class of *smooth* data $f \in C^{1+\beta}([0, T]; X)$ and $g'' \in \mathcal{K}_{\alpha,\beta}$.

On the contrary, in this paper we consider the case of *admissible* data with *nonsmooth* derivatives. This means that we are allowed to deal with data $f \in W^{1+\sigma,p}((0, T); X)$ and $g \in W^{2+\sigma,p}(0, T)$, possibly with σ and p near 0 and 1 respectively. This choice of spaces allows the function g'' to have (weak) singularities spread over points of the interval $[0, T]$ other than $t=0$.

Moreover we observe that the fractional Sobolev space $W^{\sigma,p}(0, T)$ contains class $\mathcal{K}_{\alpha,\beta}$ iff $\alpha \in (1/p', 1)$ and $\sigma \in (0, \min(\alpha - 1/p', 1-\alpha))$.

We conclude this section by noting that an application of our abstract result to a concrete case is given in Section 5.

1. Equivalent problems and main results.

In this section we prove that our direct and inverse problems are equivalent to two integral equations, both being particular cases of a more general integral one.

We begin by considering first the direct problem (0.1), (0.2). To this purpose we introduce the intermediate spaces $D_A(\beta, p)$ between D_A and X defined by (see e.g. [1, Section 1]):

$$D_A(\beta, p) = \left\{ x \in X:\ |x|_{\beta,p} = \left(\int_0^{+\infty} t^{(1-\beta)p-1} \|Ae^{tA}x\|^p dt \right)^{1/p} < +\infty \right\},$$

$$\beta \in (0, 1),\quad p \in (1, +\infty)$$

and

$$D_A(\beta, p) = \{x \in D_{A^n} : A^n x \in D_A(\beta - n, p)\} \quad \beta \in (n, n+1),\quad p \in (1, +\infty).$$

Moreover $D_A(\beta, p)$ is a Banach space, when endowed with the norm

$$\|x\|_{\beta,p} = \sum_{j=0}^{n} \|A^j x\| + |Ax|_{\beta-n,p} \qquad \text{if } \beta \in (n, n+1), \; n \in \mathbb{N}.$$

Under the previous assumptions on data k, b, f and the membership of u_0 in $D_A(\sigma + 1/p', p)$, we can show (reasoning as in [2]) that problem (0.1), (0.2) is equivalent to determining a solution $u \in W^{1+\sigma,p}((0,T); X) \cap W^{\sigma,p}((0,T); Y)$ to the following operator integral equation:

$$(1.1) \qquad u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} f(s)\, ds + \\ + \int_0^t e^{(t-\sigma)A} d\sigma \int_0^t k(\sigma - s)[Bu(s) + b(s)]\, ds, \qquad \text{for a.e. } t \in (0, T).$$

Introduce then the auxiliary function

$$(1.2) \qquad w(t) = Au(t) \qquad t \in (0, T)$$

and apply operator A to both members in (1.1). Using again the arguments in [2] it easy to check that equation (1.1) is equivalent to determining a solution $w \in W^{\sigma,p}((0,T); X)$ to the following integral equation, where $*$ denotes convolution:

$$(1.3) \qquad w(t) = Ae^{tA} u_0 + \int_0^t Ae^{(t-s)A}[f(s) - f(t)]\, ds + (e^{tA} - I) f(t) + \\ + \int_0^t Ae^{(t-s)A}[k * (BA^{-1}w + b)(s) + k * (BA^{-1}w + b)(t)]\, ds + \\ + (e^{tA} - I) k * (BA^{-1}w + b)(t) := N_1(u_0)(t) + N_2(f)(t) + \\ + N_2[k * (BA^{-1}w + b)](t), \qquad \text{for a.e. } t \in (0, T).$$

To derive (1.3) we have used the following formula (see e.g. [1, Section 7, Lemma 2]):

$$(1.4) \qquad A \int_0^t e^{(t-s)A} f(s)\, ds = \int_0^t Ae^{(t-s)A}[f(s) - f(t)]\, ds + (e^{tA} - I) f(t), \\ \text{for a.e. } t \in (0, T).$$

Now we can state our existence uniqueness and continuous dependence theorems for the direct problem, whose proofs we postpone to Section 2.

THEOREM 1.1. *Let* A: $Y = D_A \subset X \to X$ *and* B: $D_B \subset X \to X$ *satisfy properties* (0.3), (0.4) *and let* $(b, f, u_0) \in W^{\sigma, p}((0, T); X) \times \times W^{\sigma, p}((0, T); X) \times D_A(\sigma + 1/p', p)$ $(\sigma \in (0, 1) \setminus \{1/p\})$ *be a triplet such that*

i) $Au_0 + f(0) \in D_A(\sigma - 1/p, p)$ *when* $\sigma \in (1/p, 1)$.

Assume further that:

ii) $k \in L^q(0, T)$ *for some* $q \in (1, +\infty]$ *when* $\sigma \in (0, 1/p)$;

iii) $k \in L^p((0, T); t^{p(1-\sigma)}dt)$ *when* $\sigma \in (1/p, 1)$.

Then problem (0.1), (0.2) *admits a unique solution*

$$u \in W^{1+\sigma, p}((0, T); X) \cap W^{\sigma, p}((0, T); Y).$$

REMARK 1.1. The singular case $\sigma = 1/p$ is not covered by the methods developed in this paper.

REMARK 1.2. If k satisfies property iii), then $k \in L^q(0, T)$ for any $q \in [1, p(1 + (1 - \sigma)p)^{-1}]$.

Now we can state our stability result, where $C(\sigma, p, M_0, M_1, M_2, T)$ denotes (here and throughout the paper) a positive constant depending continuously and increasingly on M_0, M_1, M_2.

THEOREM 1.2. *Let* A: $Y = D_A \subset X \to X$ *and* B: $D_B \subset X \to X$ *satisfy properties* (0.3), (0.4). *Assume further that* $(k_j, b_j, f_j, u_{0,j}) \in \in W^{\sigma, p}(0, T) \times W^{\sigma, p}((0, T); X) \times W^{\sigma, p}((0, T); X) \times X$ *are two quadruplets fulfilling assumptions listed in the statement of Theorem* 1.1 *and the following bounds*:

$$\max_{1 \leqslant j \leqslant 2} \|k_j\|_{L^q(0, T)} \leqslant M_3, \qquad \text{if } \sigma \in (0, 1/p), \tag{1.5}$$

$$\max_{1 \leqslant j \leqslant 2} \|t^{1-\sigma} k_j\|_{L^p(0, T)} \leqslant M_3, \qquad \text{if } \sigma \in (1/p, 1), \tag{1.6}$$

$$\max_{1 \leqslant j \leqslant 2} (\|u_{0,j}\|_{\sigma + 1/p', p} + \|b_j\|_{W^{\sigma, p}((0, T); X)} + \|f_j\|_{W^{\sigma, p}((0, T); X)}) \leqslant M_4, \qquad \text{if } \sigma \in (0, 1/p), \tag{1.7}$$

$$\max_{1 \leqslant j \leqslant 2} (\|Au_{0,j}\| + \|Au_{0,j} + f_j(0)\|_{\sigma - 1/p, p} + \|b_j\|_{W^{\sigma, p}((0, T); X)} + \|f_j\|_{W^{\sigma, p}((0, T); X)}) \leqslant M_4, \quad \text{if } \sigma \in (1/p, 1). \tag{1.8}$$

Then the solution u_j $(j=1,2)$ *to problems* (0.1), (0.2) *corresponding to data* $(k_j, b_j, f_j, u_{0,j})$ $(j=1,2)$ *satisfy the estimate*

(1.9) $\|u_2 - u_1\|_{W^{1+\sigma,p}((0,t);X)\cap W^{\sigma,p}((0,t);Y)} \leq C_1(\sigma, p, M_0, M_1, M_2, M_3, T)\cdot$

$\cdot(1 + M_4)(\|k_1 - k_2\|_{L^1(0,t)} + \|b_2 - b_1\|_{W^{\sigma,p}((0,t);X)} +$

$+ \|f_2 - f_1\|_{W^{\sigma,p}((0,t);X)} + \|u_{0,2} - u_{0,1}\|_{\sigma+1/p'p}), \qquad \forall t \in (0, T], \ \sigma \in (0, 1/p),$

(1.10) $\|u_2 - u_1\|_{W^{1+\sigma,p}((0,t);X)\cap W^{\sigma,p}((0,t);Y)} \leq C_2(\sigma, p, M_0, M_1, M_2, M_3, T)\cdot$

$$\cdot(1 + M_4)(\|t^{1-\sigma}(k_2 - k_1)\|_{L^p(0,t)} + \|b_2 - b_1\|_{W^{\sigma,p}((0,t);X)} +$$
$$+ \|f_2 - f_1\|_{W^{\sigma,p}((0,t);X)} + \|Au_{0,2} - Au_{0,1} + f_2(0) - f_1(0)\|_{\sigma-1/p,p} +$$
$$+ \|Au_{0,2} - Au_{0,1}\|), \qquad \forall t \in (0, T], \ \sigma \in (1/p, 1).$$

Assume now that kernek k is itself unknown. Then our identification problem is the following: *determine a pair of functions* $(u, k) \in \in [W^{2+\sigma,p}((0, T); X) \cap W^{1+\sigma,p}((0, T); Y)] \times W^{\sigma,p}(0, T)$ *such that*

(1.11) $$u'(t) - Au(t) - \int_0^t k(t-s)[Bu(s) + b(s)]\,ds = f(t),$$

for a.e. $t \in (0, T)$,

(1.12) $$u(0) = u_0,$$

(1.13) $$\Phi[u(t)] = g(t), \qquad t \in [0, T].$$

We assume that $b, f \in W^{1+\sigma,p}((0, T); X)$, $u_0 \in D_A$, $Au_0 + f(0) \in D_A(\sigma + + 1/p', p)$, $g \in W^{2+\sigma,p}(0, T)$ and $\Phi \in X^*$.

Introduce the function

(1.14) $$v(t) = u'(t), \qquad t \in [0, T].$$

It is easy to realize that the pair (v, k) belongs to $[W^{1+\sigma,p}((0, T); X) \cap \cap W^{\sigma,p}((0, T); Y)] \times W^{\sigma,p}(0, T)$ and solves the problem

(1.15) $$v'(t) - Av(t) - \int_0^t k(t-s)[Bv(s) + b'(s)]\,ds =$$

$$= f'(t) + k(t)[Bu_0 + b(0)], \qquad \text{for a.e. } t \in (0, T),$$

(1.16) $$v(0) = Au_0 + f(0),$$

$$(1.17)\quad g''(t) - \Phi\left\{Av(t) + \int_0^t k(t-s)[Bv(s) + b'(s)]\,ds\right\} =$$

$$= \Phi[f'(t)] + k(t)\Phi[Bu_0 + b(0)], \qquad \text{for a.e. } t \in (0, T).$$

Introduce then the auxiliary function w defined by

$$(1.18)\qquad w(t) = Av(t), \qquad \text{for a.e. } t \in (0, T).$$

Consequently we derive the following equations for v and k:

$$(1.19)\quad v(t) = e^{tA}[Au_0 + f(0)] + \int_0^t e^{(t-s)A}\{f'(s) + k(s)[Bu_0 + b(0)]\}\,ds +$$

$$+ \int_0^t e^{(t-s)A} k * (BA^{-1}w + b')(s)\,ds, \qquad \text{for a.e. } t \in (0, T).$$

$$(1.20)\quad k(t) = \chi\{g''(t) - \Phi[f'(t)]\} - \chi\Phi[w(t) + k * (BA^{-1}w + b')(t)],$$

$$\text{for a.e. } t \in (0, T),$$

where we have assumed that

$$(1.21)\qquad \chi^{-1} := \Phi[Bu_0 + b(0)] \neq 0.$$

Finally apply the linear operator A to both members of equation (1.19). You obtain the equivalent inverse problem for the pair (w, k):

$$(1.22)\quad w(t) = Ae^{tA}[Au_0 + f(0)] + \int_0^t Ae^{(t-s)A}[f'(s) - f'(t)]\,ds +$$

$$+ (e^{tA} - I)f'(t) - \int_0^t Ae^{(t-s)A}k(s)[Bu_0 + b(0)]\,ds +$$

$$+ \int_0^t Ae^{(t-s)A}[k * (BA^{-1}w + b')(s) - k * (BA^{-1}w + b')(t)]\,ds +$$

$$+ (e^{tA} - I)\,k * (BA^{-1}w + b')(t) := N_1[Au_0 + f(0)](t) + N_2(f')(t) +$$

$$+ N_3[k(Bu_0 + b(0))](t) + N_2[k * (BA^{-1}w + b')](t), \qquad \text{for a.e. } t \in (0, T),$$

$$(1.23)\quad k(t) = \chi\{g''(t) - \Phi[f'(t)]\} - \chi\Phi[w(t) + k * (BA^{-1}w + b')(t)],$$

$$\text{for a.e. } t \in (0, T).$$

REMARK 1.3. We observe that equations (1.3) and (1.22) are special cases of the following integral equation

(1.24) $$w = N_2[k*(BA^{-1}w + \tilde{b})] + N_1(\tilde{u}_0) + N_2(\tilde{f}) + \lambda N_3(kx),$$

where $\tilde{u}_0 \in D_A(\sigma + 1/p', p)$, $\tilde{b}, \tilde{f} \in W^{\sigma, p}((0, T); X)$, $\lambda \in \mathbb{R}$ and $x \in \in D_A(\theta, p)$ for a suitable $\theta \in (0,1)$ (see e.g. Lemma 4.1).

We note also that integral equations (1.24) corresponding to different values of λ differ only in their free members.

REMARK 1.4. From equations (1.11)-(1.13) it is easy to deduce the following consistency conditions:

(1.25) $$\Phi(u_0) = g(0),$$

(1.26) $$\Phi[Au_0 + f(0)] = g'(0).$$

Recall now the embedding $W^{\sigma, p}((0, T); X) \hookrightarrow C([0, T]; X)$ when $\sigma \in (1/p, 1)$ and use the membership $Au_0 + f(0) \in D_A$. Then from equation (1.17) we derive the further consistency condition

(1.27) $$g''(0) = \Phi[A(Au_0 + f(0)) + f'(0)] + k(0)\Phi[Bu_0 + b(0)],$$

if $\sigma \in (1/p, 1)$.

Owing to (1.20) we get

(1.28) $$k(0) = \chi\{g''(0) - \Phi[A(Au_0 + f(0)) + f'(0)]\}, \qquad \text{if } \sigma \in (1/p, 1).$$

Now we can state our existence, uniqueness and continuous dependence results related to inverse problem (1.11)-(1.13).

THEOREM 1.3. *Let A: $Y = D_A \subset X \to X$ and B: $D_B \subset X \to X$ satisfy properties* (0.3), (0.4). *Assume further that* $(b, f, u_0) \in \in W^{1+\sigma, p}((0, T_0); X) \times W^{1+\sigma, p}((0, T_0); X) \times W^{2+\sigma, p}((0, T_0); X) \times D_A$ $(\sigma \in (0, 1)\setminus\{1/p\})$ *is a quadruplet enjoying the following properties*:

i) $Au_0 + f(0) \in D_A(\sigma + 1/p', p)$ *when* $\sigma \in (0,1)\setminus\{1/p\}$;

ii) $Bu_0 + b(0) \in D_A(\theta, p)$ *for some* $\theta \in (0,1)$ *when* $\sigma \in (0, 1/p)$;

iii) $Bu_0 + b(0) \in D_A(\sigma - 1/p, p)$ *when* $\sigma \in (1/p, 1)$;

iv) $A[Au_0 + f(0)] + f'(0) + \chi\{g''(0) - \Phi[A(Au_0 + f(0)) + f'(0)]\} \cdot \cdot [Bu_0 + b(0)] := E(b, f, g, u_0) \in D_A(\sigma - 1/p, p)$ *when* $\sigma \in (1/p, 1)$;

Then, if conditions (1.21) *and* (1.25)-(1.28) *are fulfilled, problem* (1.11)-(1.13) *admits a unique solution* $(u, k) \in [W^{2+\sigma, p}((0, T); X) \cap \cap W^{1+\sigma, p}((0, T); Y] \cap W^{\sigma, p}(0, T)$ *for some* $T \in (0, T_0]$.

THEOREM 1.4. *Let* A: $Y = D_A \subset X \to X$ *and* B: $D_B \subset X \to X$ *satisfy properties* (0.3), (0.4). *Assume further that* $(b_j, f_j, g_j, u_{0,j}) \in$ $\in W^{1+\sigma,p}((0, T_0); X) \times W^{1+\sigma,p}((0, T_0); X) \times W^{2+\sigma,p}(0, T_0) \times D_A$ *are two quadruplets enjoying properties* i)-iv) *listed in the statement of Theorem* 1.3 *and fulfilling conditions* (1.21), (1.25)-(1.28) *and the following estimates for some positive constants* M_5 *and* $\theta \in (0,1)$:

$$(1.29) \qquad \|b\|_{W^{1+\sigma,p}((0,T_0);X)} + \|f\|_{W^{1+\sigma,p}((0,T_0);X)} + \|g\|_{W^{2+\sigma,p}(0,T_0)} +$$

$$+ \|Au_0 + f(0)\|_{\sigma+1/p',p} + \|Bu_0 + b(0)\|_{\theta,p} \le M_5, \qquad \text{if } \sigma \in (0, 1/p),$$

$$(1.30) \qquad \|b\|_{W^{1+\sigma,p}((0,T_0);X)} + \|f\|_{W^{1+\sigma,p}((0,T_0);X)} + \|g\|_{W^{2+\sigma,p}(0,T_0)} + \|Au_0\| +$$

$$+ \|Bu_0 + b(0)\|_{\sigma-1/p,p} + \|E(b, f, g, u_0)\|_{\sigma-1/p,,p} \le M_5, \qquad \text{if } \sigma \in (1/p, 1),$$

Then the solutions (u_j, k_j) $(j=1,2)$ *to problems* (1.11)-(1.13) *corresponding to data* $(b_j, f_j, g_j u_{0,j})$ $(j=1,2)$ *satisfy the following estimates for some* $T \in (0, T_0]$, *where* $\chi_j^{-1} = \Phi[Bu_{0,j} + b_j(0)]$ $(j=1,2)$:

$$(1.31) \qquad \|u_2 - u_1\|_{W^{2+\sigma,p}((0,t);X)\cap W^{1+\sigma,p}((0,t);Y)} + \|k_2 - k_1\|_{W^{\sigma,p}(0,t)} \le$$

$$\le C_3((\sigma, p, M_0, M_1, M_2, M_5, \max_{1\le j\le 2} |\chi_j|, \|\Phi\|_{X^*}, T)\cdot$$

$$\cdot[\|b_2 - b_1\|_{W^{1+\sigma,p}((0,t);X)} + \|f_2 - f_1\|_{W^{1+\sigma,p}((0,t);X)} + \|g_2 - g_1\|_{W^{2+\sigma,p}(0,t)} +$$

$$+ \|u_{0,2} - u_{0,1}\|_{\sigma+1/p',p} + \|Au_{0,2} - Au_{0,1} + f_2(0) - f_1(0)\|_{\sigma+1/p',p} +$$

$$+ \|Bu_{0,2} - Bu_{0,1} + b_2(0) - b_1(0)\|_{\theta,p}], \qquad \forall t \in (0, T], \ \sigma \in (0, 1/p),$$

$$(1.32) \qquad \|u_2 - u_1\|_{W^{2+\sigma,p}((0,t);X)\cap W^{1+\sigma,p}((0,t);Y)} + \|k_2 - k_1\|_{W^{\sigma,p}(0,t)} \le$$

$$\le C_4(\sigma, p, M_0, M_1, M_2, M_5, \max_{1\le j\le 2} |\chi_j|, \|\Phi\|_{X^*}, T)\cdot$$

$$\cdot[\|b_2 - b_1\|_{W^{1+\sigma,p}((0,t);X)} + \|f_2 - f_1\|_{W^{1+\sigma,p}((0,t);X)} + \|g_2 - g_1\|_{W^{2+\sigma,p}(0,t)} +$$

$$+ \|Au_{0,2} - Au_{0,1}\| + \|E(b_2, f_2, g_2, u_{0,2}) - E(b_1, f_1, g_1, u_{0,1})\|_{\sigma-1/p,p} +$$

$$+ \|Bu_{0,2} - Bu_{0,1} + b_2(0) - b_1(0)\|_{\sigma-1/p,p}], \qquad \forall t \in (0, T], \ \textit{if } \sigma \in (1/p, 1).$$

2. Proof of Theorem 1.1.

First we state some preliminary lemmas.

LEMMA 2.1. *Let* $f \in W^{\sigma,p}((0,T);X)$, $\sigma \in (0,1)\setminus\{1/p\}$. *Then the function* F *defined by*

$$(2.1)\qquad F(t) = \int_0^t A e^{(t-s)A}[f(s)-f(t)]\,ds, \quad \textit{for a.e. } t \in (0,T),$$

belongs to $W^{\sigma,p}((0,T);X)$ *and satisfies the estimates*

$$(2.2)\qquad \|F\|_{L^p((0,t);X)} \leq c_1(\sigma,p,M_0,M_1)\,t^\sigma |f|_{W^{\sigma,p}((0,t);X)}, \qquad \forall t \in (0,T],$$

$$(2.3)\qquad |F|_{W^{\sigma,p}((0,t);X)} \leq c_2(\sigma,p,M_0,M_1)|f|_{W^{\sigma,p}((0,t);X)}, \qquad \forall t \in (0,T],$$

where M_0 *and* M_1 *are the positive constants appearing in* (0.5) *and*

$$|f|_{W^{\sigma,p}((0,t);X)} = \left(\int_0^T dt_2 \int_0^T (t_2-t_1)^{-1-\sigma p}\|f(t_2)-f(t_1)\|^p dt_1\right)^{1/p}.$$

PROOF. See the proof of Lemma 2 and Theorem 24 in [1, Section 7] and recall estimates

$$(2.4)\qquad \|t^{-\sigma}f\|_{L^p((0,T);X)} \leq c_3(\sigma,p)|f|_{W^{\sigma,p}((0,T);X)}, \qquad \forall f \in W^{\sigma,p}((0,T);X),\ \forall \sigma \in (0,1/p),$$

$$(2.5)\qquad \|t^{-\sigma}[f-f(0)]\|_{L^p((0,T);X)} \leq c_4(\sigma,p)|f|_{W^{\sigma,p}((0,T);X)}, \qquad \forall f \in W^{\sigma,p}((0,T);X),\ \forall \sigma \in (1/p,1),$$

We note that the embedding constants c_j $(j=3,4)$ are independent of T. In fact, (2.4), (2.5) hold with $T=1$ (see e.g. Lemmas 7 and 8 in [1, appendix]). To derive the general case, associate with any $f \in W^{\sigma,p}((0,T);X)$ the function $f_T(t) = f(tT)$ $(t \in (0,1))$ belonging to $W^{\sigma,p}((0,1);X)$. Apply then the previous result to f_T. ■

LEMMA 2.2. *Let* $f \in W^{\sigma,p}((0,T);X)$ $(\sigma \in (0,1)\setminus\{1/p\})$ *and let* $x \in D_A(\sigma + 1/p', p)$. *Then the following estimates hold:*

$$(2.6)\qquad \|Ae^{tA}x\|_{W^{\sigma,p}((0,T);X)} \leq c_5(\sigma,p,M_0,M_1)\|x\|_{\sigma+1/p',p},$$

$$(2.7)\qquad \|e^{tA}f\|_{W^{\sigma,p}((0,T);X)} \leq c_6(\sigma,p,M_0,M_1)\|f\|_{W^{\sigma,p}((0,T);X)}, \text{ if } \sigma \in (0,1/p),$$

$$(2.8)\qquad \|e^{tA}[f-f(0)]\|_{W^{\sigma,p}((0,T);X)} \leq c_7(\sigma,p,M_0,M_1)\|f\|_{W^{\sigma,p}((0,T);X)}, \quad \text{if } \sigma \in (1/p,1).$$

PROOF. Estimates (2.6) (with $\sigma \in (0,1/p)$) follows from Theorem 4 and 10 in [1]. On the contrary estimate (2.6) with $\sigma \in (1/p,1)$ is implied by Theorem 8 in [1] and equation $Ae^{tA}x = e^{tA}Ax$, since $Ax \in D_A(\sigma - 1/p, p)$.

Finally, estimates (2.7), (2.8) can be proved reasoning as in Lemma 2 and Theorem 24 in [1] and using estimates (2.4), (2.5). ■

LEMMA 2.3. *Let* $\sigma \in (0,1)$ *and* $t^{1-\sigma}k \in L^p(0, T)$. *Then the function* $\tilde{k}(t) = \int_0^t k(s)\,ds$ *belongs to* $W^{\sigma,p}((0, T); X)$ *and satisfies the estimate*

$$(2.9)\quad |\tilde{k}|_{W^{\sigma,p}((0,t);X)} \leqslant 2^{1/p}[p(1-\sigma)]^{-1/p}\sigma^{-1}\|\tau^{1-\sigma}k\|_{L^p(0,t)}, \qquad \forall t \in (0, T].$$

PROOF. For any triplet (t_1, t, t_2) such that $0 \leqslant t_1 < t \leqslant t_2 \leqslant T$ consider the identity

$$\tilde{k}(t_1, t_2) := \tilde{k}(t_2) - \tilde{k}(t_1) = (t_2 - t_1)\int_0^1 k[t_1 + s(t_2 - t_1)]\,ds.$$

From it (performing simple computations) we get the following chain of inequalities which proves the assertion:

$$(2.10)\quad \left(\int_0^t dt_1 \int_0^t |t_2 - t_1|^{-1-\sigma p}\|\tilde{k}_1(t_1, t_2)\|^p\,dt_2\right)^{1/p} =$$

$$= 2^{1/p}\left(\int_0^t dt_1 \int_{t_1}^t (t_1 - t_2)^{-1-\sigma p}\|\tilde{k}_1(t_1, t_2)\|^p\,dt_2\right)^{1/p} =$$

$$= 2^{1/p}\int_0^1\left(\int_0^t dt_1 \int_{t_1}^t (t_2 - t_1)^{p(1-\sigma)-1}|k[t_1 + s(t_2 - t_1)]|^p\,dt_2\right)^{1/p} ds =$$

$$= 2^{1/p}\int_0^1 s^{-1+\sigma}\left(\int_0^t dt_1 \int_{t_1}^{t_1+s(t-t_1)} (t_2 - t_1)^{p(1-\sigma)-1}|k(t_2)|^p\,dt_2\right)^{1/p} ds \leqslant$$

$$\leqslant 2^{1/p}\int_0^1 s^{-1+\sigma}\left(\int_0^t dt_1 \int_{t_1}^t (t_2 - t_1)^{p(1-\sigma)-1}|k(t_2)|^p\,dt_2\right)^{1/p} ds =$$

$$= 2^{1/p}\sigma^{-1}\left(\int_0^t |k(t_2)|^p dt_2 \int_0^{t_2}(t_2 - t_1)^{p(1-\sigma)-1} dt_1\right)^{1/p} =$$

$$= 2^{1/p}[p(1-\sigma)]^{-1/p}\sigma^{-1}\|t^{1-\sigma}k\|_{L^p(0,t)}, \qquad \forall t \in (0, T]. \qquad \blacksquare$$

LEMMA 2.4. *Let k and w be two functions belonging respectively to $L^1(0, T)$ and $W^{\sigma,p}((0, T); X)$, ($\sigma \in (0, 1)\setminus\{1/p\}$, $p \in (1, +\infty)$) respectively. When $\sigma \in (1/p, 1)$ assume moreover that $k \in L^p((0, T); t^{(1-\sigma)p} dt)$. Then the following estimates hold for any $t \in (0, T]$:*

$$(2.11) \qquad \|k * w\|_{L^p((0,t);X)} \leqslant \|k\|_{L^1(0,t)}^{1/p'}\left(\int_0^t |k(s)|\,\|w\|_{L^p((0,t-s);X)}^p ds\right)^{1/p} \leqslant$$

$$\leqslant \|k\|_{L^1(0,t)}\|w\|_{L^p((0,t-s);X)},$$

$$(2.12) \qquad |k * w|_{W^{\sigma,p}(0,t);X)} \leqslant$$

$$\leqslant c_8(\sigma, p)\|k\|_{L^1(0,t)}^{1/p'}\left(\int_0^t |k(s)|\,|w|_{W^{\sigma,p}((0,t-s);X)}^p ds\right)^{1/p} \leqslant$$

$$\leqslant c_8(\sigma, p)\|k\|_{L^1(0,t)}\,|w|_{W^{\sigma,p}((0,t-s);X)} \qquad \text{if } \sigma \in (0, 1/p),$$

$$(2.13) \qquad |k * w|_{W^{\sigma,p}(0,t);X)} \leqslant c_9(\sigma, p)\,\cdot$$

$$\cdot\left\{\|k\|_{L^1(0,t)}^{1/p'}\left(\int_0^t |k(s)|\,|w|_{W^{\sigma,p}((0,t-s);X)}^p ds\right)^{1/p} + \|t^{1-\sigma}k\|_{L^p(0,t)}\|w(0)\|\right\} \leqslant$$

$$\leqslant c_9(\sigma, p)\{\|k\|_{L^1(0,t)}\,|w|_{W^{\sigma,p}((0,t);X)} + \|t^{1-\sigma}k\|_{L^p(0,t)}\|w(0)\|\}, \qquad \text{if } \sigma \in (1/p, 1).$$

PROOF. From the inequalities

$$\|k * w\| \leqslant \left(\int_0^t |k(s)|^{1/p'}\,|k(s)|^{1/p}\|w(t-s)\|\,ds\right)^{1/p} \leqslant$$

$$\leqslant \|k\|_{L^1(0,t)}^{1/p'}\left(\int_0^t |k(s)|\,\|w(t-s)\|^p ds\right)^{1/p}, \qquad \text{for a.e. } t \in (0, T),$$

we easily get the following estimates for any $t \in (0,T]$ and any $\theta \in [0,1)$:

$$(2.14)\qquad \|\tau^{-\theta}(k*w)\|_{L^p((0,t);X)} \leqslant$$

$$\leqslant \|k\|_{L^1(0,t)}^{1/p'}\left(\int_0^t \tau^{-\theta p}\,d\tau\int_0^\tau |k(s)|\,\|w(\tau-s)\|^p ds\right)^{1/p} \leqslant$$

$$\leqslant \|k\|_{L^1(0,t)}^{1/p'}\left(\int_0^t |k(s)|\,ds\int_s^t (\tau-s)^{-\theta p}\|w(\tau-s)\|^p d\tau\right)^{1/p} =$$

$$= \|k\|_{L^1(0,t)}^{1/p'}\left(\int_0^t |k(s)|\,\|\tau^{-\theta p}w\|_{L^p((0,t-s);X)}^p ds\right)^{1/p} \leqslant \|k\|_{L^1(0,t)}\|\tau^{-\theta p}w\|_{L^p((0,t);X)}\,.$$

We observe that (2.11) is an immediate consequence of (2.14) with $\theta=0$. To prove (2.12) we consider the identity

$$(2.15)\qquad (k*w)(t_2)-(k*w)(t_1) = \int_{t_1}^{t_2} k(s)\,w(t_2-s)\,ds +$$

$$+\int_0^{t_1} k(s)[w(t_2-s)-w(t_1-s)]\,ds = F_1(t_1,t_2)+F_2(t_1,t_2),\qquad 0\leqslant t_1<t_2\leqslant T.$$

First we observe that

$$\left(\int_{t_1}^t (t_2-t_1)^{-1-\sigma p}\|F_2(t_1,t_2)\|^p dt_2\right)^{1/p} \leqslant$$

$$\leqslant \int_0^{t_1} |k(s)|\left(\int_{t_1}^t (t_2-t_1)^{-1-\sigma p}\|w(t_2-s)-w(t_1-s)\|^p dt_2\right)^{1/p} ds,$$

$$0\leqslant t_1<t\leqslant T.$$

Hence we derive the following chain of inequalities

$$(2.16)\qquad \left(\int_0^t dt_1 \int_{t_1}^t (t_2-t_1)^{-1-\sigma p}\|F_2(t_1,t_2)\|^p dt_2\right)^{1/p} \leqslant$$

$$\leqslant \left\{\int_0^t \left[\int_0^{t_1} |k(s)| \left(\int_{t_1}^t (t_2-t_1)^{-1-\sigma p}\|w(t_2-s)-w(t_1-s)\|^p dt_2\right)^{1/p} ds\right]^p dt_1\right\}^{1/p} \leqslant$$

$$\leqslant \left\{\int_0^t \left[\left(\int_0^{t_1} |k(s)|\,ds\right)^{p/p'} \int_0^{t_1} |k(s)|\,ds \cdot \right.\right.$$

$$\left.\left. \cdot \int_{t_1}^t (t_2-t_1)^{-1-\sigma p}\|w(t_2-s)-w(t_1-s)\|^p dt_2\right] dt_1\right\}^{1/p} \leqslant$$

$$\leqslant \|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t dt_1 \int_0^t |k(s)|\,ds \int_{t_1}^t (t_2-t_1)^{-1-\sigma p}\|w(t_2-s)-w(t_1-s)\|^p dt_2\right)^{1/p} =$$

$$= \|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t |k(s)|\,ds \int_s^t dt_1 \int_{t_1}^t (t_2-t_1)^{-1-\sigma p}\|w(t_2-s)-w(t_1-s)\|^p dt_2\right)^{1/p} =$$

$$= \|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t |k(s)|\,ds \int_0^{t-s} dt_1 \int_{t_1}^{t-s} (t_2-t_1)^{-1-\sigma p}\|w(t_2)-w(t_1)\|^p dt_2\right)^{1/p} \leqslant$$

$$\leqslant 2^{-1/p}\|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t |k(s)|\,|w|^p_{W^{\sigma,p}((0,t-s);X)}\,ds\right)^{1/p} \leqslant$$

$$\leqslant 2^{-1/p}\|k\|_{L^1(0,t)}\,|w|_{W^{\sigma,p}((0,t);X)}, \qquad \forall t \in (0,T].$$

As far as function F_1 is concerned, first we observe that

$$F_1(t_1,t_2) \leqslant \|k\|_{L^1(0,t)}^{1/p'} \left(\int_{t_1}^{t_2} |k(s)|\,\|w(t_2-s)\|^p ds\right)^{1/p}, \qquad 0 \leqslant t_1 < t_2 \leqslant T.$$

Proceeding as above, from (2.6) we get the following estimates

$$\left(\int_0^t dt_2 \int_0^{t_2} (t_2 - t_1)^{-1-\sigma p} \|F_1(t_1, t_2)\|^p dt_2\right)^{1/p} \leqslant \tag{2.17}$$

$$\leqslant \|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t dt_2 \int_0^{t_2} (t_2 - t_1)^{-1-\sigma p} dt_1 \int_{t_1}^{t_2} |k(s)| \|w(t_2 - s)\|^p ds\right)^{1/p} =$$

$$= \|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t |k(s)| ds \int_s^t \|w(t_2 - s)\|^p dt_2 \int_0^s (t_2 - t_1)^{-1-\sigma p} dt_1\right)^{1/p} \leqslant$$

$$\leqslant (\sigma p)^{-1/p} \|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t |k(s)| ds \int_s^t (t_2 - s)^{-\sigma p} \|w(t_2 - s)\|^p dt_2\right)^{1/p} =$$

$$= (\sigma p)^{-1/p} \|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t |k(s)| ds \int_0^{t-s} t_2^{-\sigma p} \|w(t_2)\|^p dt_2\right)^{1/p} \leqslant$$

$$\leqslant c_3(\sigma, p) \|k\|_{L^1(0,t)}^{1/p'} \left(\int_0^t |k(s)| \, |w|_{W^{\sigma,p}((0,t-s);X)}^p ds\right)^{1/p} \leqslant$$

$$\leqslant c_3(\sigma, p) \|k\|_{L^1(0,t)} |w|_{W^{\sigma,p}((0,t);X)}, \qquad \forall t \in (0, T].$$

Finally from (2.15)-(2.17) we deduce immediately estimates (2.12).

To prove (2.13) we consider the identity

$$(k * w)(t_2) - (k * w)(t_1) = \left(\int_{t_1}^{t_2} k(s) ds\right) w(0) + \tag{2.18}$$

$$+ \int_{t_1}^{t_2} k(s)[w(t_2 - s) - w(0)] ds + \int_0^{t_1} k(s)[w(t_2 - s) - w(t_1 - s)] ds =$$

$$= F_0(t_1, t_2) + F_1(t_1, t_2) + F_2(t_1, t_2), \qquad 0 \leqslant t_1 < t_2 \leqslant T.$$

The same arguments as above show that the following estimates for F_1

and F_2 hold true

$$\sum_{j=1}^{2}\left(\int_0^t dt_2\int_0^t |t_2-t_1|^{-1-\sigma p}\|F_j(t_1,t_2)\|^p dt_1\right)^{1/p} \leqslant \tag{2.19}$$

$$\leqslant c_4(p,\sigma)\|k\|_{L^1(0,t)}^{1/p'}\left(\int_0^t |k(s)|\,|w|^p_{W^{\sigma,p}((0,t-s);X)}ds\right)^{1/p} \leqslant$$

$$\leqslant c_4(\sigma,p)\|k\|_{L^1(0,t)}|w|_{W^{\sigma,p}((0,t);X)}, \qquad \forall t\in(0,T].$$

From Lemma 2.3 we derive the estimate

$$\left(\int_0^t dt_2\int_0^{t_2}(t_2-t_1)^{-1-\sigma p}\|F_0(t_1,t_2)\|^p dt_1\right)^{1/p} \leqslant \tag{2.20}$$

$$\leqslant 2^{1/p}[p(1-\sigma)]^{-1/p}\sigma^{-1}\|w(0)\|\|t^{1-\sigma}k\|_{L^p(0,t)}, \qquad \forall t\in(0,T].$$

Finally from (2.18)-(2.20) we derive estimate (2.13). ■

LEMMA 2.5. *Let $k\in L^q(0,T)$ $(1<q\leqslant+\infty)$. Then the following estimates hold:*

$$\|(k*)^m\|_{L^1(0,t)}\leqslant 2^{-1/q}(m!)^{-1/q'}\|k\|^m_{L^q(0,t)}\, t\in(0,T],\ m\in\mathbb{N}\setminus\{0,1\}, \tag{2.21}$$

$(k*)^m$ denoting the convolution $k*\ldots*k$ (k is convolved by itself m times).

PROOF. From the estimate

$$\|k*k\|_{L^1(0,t)}=\int_0^t|k*k(s)|ds\leqslant\int_0^t(|k|*|k|)(s)\,ds= \tag{2.22}$$

$$=\frac{1}{2}\left(\int_0^t|k(s)|ds\right)^2\leqslant\frac{1}{2}t^{2/q'}\|k\|^2_{L^q(0,t)}, \qquad \text{for a.e. } t\in(0,T),$$

we get

$$(2.23)\qquad \|(k*)^m\|_{L^1(0,t)} \leqslant \|(|k|*)\|_{L^1(0,t)} = 1*(|k|*)^m(t) =$$

$$= [(|k|*)^{m-2} * (1*|k|*|k|)(t) \leqslant$$

$$\leqslant \frac{1}{2}\|k\|^2_{L^q(0,t)} \int_0^t (t-s)^{2/q'}(|k|*)^{m-2}(s)\,ds := \psi_m(t),$$

for a.e. $t \in (0, T]$, $\forall m \in \mathbb{N} \setminus \{0, 1, 2\}$.

Then we deduce

$$(2.24)\qquad \psi_{m+1}(t) = |k| * \psi_m(t) \leqslant \|k\|_{L^q(0,t)} \left(\int_0^t |\psi_m(s)|^{q'} ds \right)^{1/q'},$$

for a.e. $t \in (0, T]$, $\forall m \in \mathbb{N} \setminus \{0, 1\}$.

Since

$$\psi_3(t) \leqslant 2^{-1/q'}(3!)^{-1/q'} t^{3/q'} \|k\|^3_{L^q(0,t)} \qquad \text{for a.e. } t \in (0, T],$$

from (2.24) we deduce (by induction) the estimates

$$(2.25)\qquad \psi_m(t) \leqslant 2^{-1/q'}(m!)^{-1/q'} t^{m/q'} \|k\|^m_{L^q(0,t)},$$

$\forall t \in (0, T]$, $\forall m \in \mathbb{N} \setminus \{0, 1, 2\}$.

From (2.23), (2.25) we easily derive estimates (2.21). ■

PROOF OF THEOREM 1.1. First we rewrite equation (1.3) in the following fixed-point form:

$$(2.26)\qquad N(w) = w,$$

where operator B is defined by the right hand-side in (1.3).

Our aim consists in showing that N satisfies the hypotheses of generalized Banach's contraction principle. We assume first that $\sigma \in (0, 1/p)$ and we observe that N maps the Banach space $W^{\sigma,p}((0, T); X)$ into itself according to Lemmas 2.1, 2.2, 2.4.

Hence it remains to prove that N^k is a contraction for large enough k.

From Lemmas 2.1, 2.2, 2.4 and (0.6) we infer the following estimates valid for any $t \in (0, T]$:

$$(2.27) \quad \|N(w_1) - N(w_2)\|_{L^p((0,t);X)} =$$

$$= \|N_2(k * BA^{-1}w_1) - N_2(k * BA^{-1}w_2)\|_{L^p((0,t);X)} \leqslant$$

$$\leqslant (1 + T^\sigma) c_3(\sigma, p, M_0, M_1) M_2 \|k\|_{L^1(0,t)}^{1/p'} \cdot$$

$$\cdot \left(\int_0^t |k(s)| \|w_2 - w_1\|_{W^{\sigma,p}((0,t-s);X)}^p ds \right)^{1/p},$$

$$(2.28) \quad |N(w_1) - N(w_2)|_{W^{\sigma,p}((0,t);X)} =$$

$$= |N_2(k * BA^{-1}w_1) - N_2(k * BA^{-1}w_2)|_{W^{\sigma,p}((0,t);X)} \leqslant$$

$$\leqslant c_{10}(\sigma, p, M_0, M_1) M_2 \|k\|_{L^1(0,t)}^{1/p'} \cdot \left(\int_0^t |k(s)| \, |w_1 - w_2|_{W^{\sigma,p}((0,t-s);X)}^p ds \right)^{1/p}.$$

From (2.27) and (2.28) we immediately derive the estimates

$$(2.29) \quad \|N(w_1) - N(w_2)\|_{W^{\sigma,p}((0,t);X)}^p \leqslant c_{11}(\sigma, p, M_0, M_1, M_2) \cdot$$

$$\cdot (1 + T^\sigma)^p \|k\|_{L^1(0,t)}^{p-1} \cdot \int_0^t |k(s)| \|w_2 - w_1\|_{W^{\sigma,p}((0,t-s);X)}^p ds, \qquad \forall t \in (0, T].$$

Assume now $m \in \mathbb{N} \setminus \{0,1\}$: from (2.29) we deduce the integral inequalities

$$(2.30) \quad \phi_{m+1}(t) \leqslant c_{11}(\sigma, p, M_0, M_1, M_2) \cdot$$

$$\cdot (1 + T^\sigma)^p \|k\|_{L^1(0,t)}^{p-1} \int_0^t |k(s)| \phi_m(t-s) \, ds, \qquad \forall t \in (0, T],$$

where we have set

$$(2.31) \quad \phi_m(t) := \|N^m(w_1) - N^m(w_2)\|_{W^{\sigma,p}((0,t);X)}^p, \qquad \forall m \in \mathbb{N}, \ \forall t \in (0, T].$$

Since ϕ_0 is a non-decreasing function, from (2.30) we easily get

(2.32) $\phi_m(t) \leq c_{11}(\sigma, p, M_0, M_1, M_2)^m (1 + T^{\sigma})^{mp} \|k\|_{L^1(0,T)}^{m(p-1)} (k *)^m * \phi_0(t) \leq$

$\leq c_{11}(\sigma, p, M_0, M_1, M_2)^m (1 + T^{\sigma})^{mp} \|k\|_{L^1(0,T)}^{m(p-1)} \|(|k| *)^m\|_{L^1(0,t)} \phi_0(t),$

$\forall m \in \mathbb{N}, \ \forall t \in (0, T].$

From (2.31) and (2.32) we derive the desired estimates

(2.33) $\|N^m(w_1) - N^m(w_2)\|_{W^{\sigma,p}((0,t);X)} \leq c_{11}(\sigma, p, M_0, M_1 M_2)^{m/p} \cdot$

$\cdot (1 + T^{\sigma})^m \cdot \|k\|_{L^1(0,T)}^{m/p'} \|(|k| *)^m\|_{L^1(0,t)}^{1/p} \|w_1 - w_2\|_{W^{\sigma,p}((0,t);X)},$

$\forall m \in \mathbb{N} \setminus \{0\}, \ \forall t \in (0, T].$

From Lemma 2.5 we deduce that N^m is a contraction in $W^{\sigma,p}((0,t);X)$ for large enough m.

Assume now that $\sigma \in (1/p, 1)$. Using the embedding $D_A(\sigma + 1/p', p) \hookrightarrow D_A$, from (1.3) we get the fixed-point equation

(2.34) $w = Au_0 + N_4[Au_0 + f(0)] + N_2[f - f(0)] +$

$+ N_3\{k * [BA^{-1}(w - w(0) + b - b(0))]\} + N_3[\tilde{k}(BA^{-1}w(0) + b(0))],$

where $N_4(x)(t) = (e^{tA} - I)x$ and $\tilde{k}(t) = \int_0^t k(s)\,ds$ for any $t \in [0, T]$.

Observe now that, according to Theorem 8 in [1], we deduce that $N_4[Au_0 + f(0)] \in W^{\sigma,p}((0,t);X)$ and $N_4[Au_0 + f(0)](0) = 0$. Using Lemmas 2.1-2.4 and the embedding $W^{\sigma,p}((0,T);X) \hookrightarrow C([0,T];X)$ ($\sigma \in (1/p, 1)$) it is easy to check that N maps the complete metric space $\mathcal{W}^{\sigma,p} = \{w \in W^{\sigma,p}((0,T);X): w(0) = Au_0\}$ into itself.

Finally, reasoning as in the previous case, from Lemma 2.1-2.4 we deduce again estimate (2.32). Hence N^m is a contradiction in $\mathcal{W}^{\sigma,p}$ for large enough m. Summing up, fixed-point equation (1.3) admits, for any $\sigma \in (0,1) \setminus \{1/p\}$, a unique solution $w \in W^{\sigma,p}((0,T);X)$. Consequently the function u defined by (1.1) (with Bu replaced by $BA^{-1}w$) belongs to $W^{1+\sigma,p}((0,T);X) \cap W^{\sigma,p}((0,T);Y)$ accordingly to our assumptions on data, to Lemma 2.4 and Theorem 30 in [1]. ■

3. Proof of Theorem 1.2.

Also in this section we need prove some preliminary lemmas. To this purpose we introduce the following complete metric spaces defined by the equations:

(3.1) $\mathcal{K}(M_3) = \{k \in L^q(0, T): \|k_{L^q(0,T)} \leq M_3\}$, if $\sigma \in (0, 1/p)(M_3 \in \mathbb{R}_+)$,

(3.2) $\mathcal{K}(M_3) = \{k \in L^p((0, T), t^{1-\sigma}dt): \|t^{1-\sigma}k\|_{L^p(0,T)} \leq M_3\}$,

if $\sigma \in (1/p, 1)(M_3 \in \mathbb{R}_+)$,

(3.3) $\mathcal{W}^{\sigma,p} = W^{\sigma,p}((0, T); X)$, if $\sigma \in (0, 1/p)$,

(3.4) $\mathcal{W}^{\sigma,p} = \{w \in W^{\sigma,p}((0, T); X): w(0) = Au_0\}$, if $\sigma \in (1/p, 1)$.

Consider then the (nonlinear) operator

$$W: D(W) \subset L^1(0, T) \times W^{\sigma,p}((0, T); X) \times W^{\sigma,p}((0, T); X) \times$$

$$\times D_A(\sigma + 1/p', p) \hookrightarrow \mathcal{W}^{\sigma,p}((0, T); Y)$$

defined by equation

(3.5) $$W(k, b, f, u_0) = w,$$

w being the solution to problem (1.3) corresponding to the quadruplet of data (k, b, f, u_0) satisfying properties listed in the statement of Theorem 1.1.

From now on $c(\sigma, p, M_0, M_1, M_2, \ldots, T)$ will be denoting a positive constant depending continuously and increasingly on M_0, M_1, $M_2, \ldots, T$.

LEMMA 3.1. *Assume* $k \in \mathcal{K}(M_3)$ *and the triplet* (b, f, u_0) *fulfil properties listed in the statement of Theorem* 1.1. *Then operator* W *defined by equation* (3.5) *satisfies the following estimates for any* $t \in (0, T]$:

(3.6) $$\|W(k, b, f, u_0)\|_{W^{\sigma,p}((0,t);X)} \leq c_{12}(\sigma, p, M_0, M_1, M_2, M_3, T) \cdot$$
$$\cdot (\|b\|_{W^{\sigma,p}((0,t);X)} + \|f\|_{W^{\sigma,p}((0,t);X)} + \|u_0\|_{\sigma+1/p',p}),$$
$$\forall t \in (0, T], \text{ if } \sigma \in (0, 1/p),$$

(3.7) $$\|W(k, b, f, u_0)\|_{W^{\sigma,p}((0,t);X)} \leq c_{13}(\sigma, p, M_0, M_1, M_2, M_3, T) \cdot$$
$$\cdot (\|b\|_{W^{\sigma,p}((0,t);X)} + \|f\|_{W^{\sigma,p}((0,t);X)} + \|Au_0\| + \|Au_0 + f(0)\|_{\sigma-1/p,p}),$$
$$\forall t \in (0, T], \text{ if } \sigma \in (1/p, 1).$$

PROOF. Assume first $\sigma \in (0, 1/p)$. From equation (1.3), Lemmas

2.1-2.4 we derive the following chain of estimates, where we have set $w = W(k, b, f, u_0)$:

$$(3.8) \quad \|w\|_{W^{\sigma,p}((0,t);X)} \le c_{14}(\sigma, p, M_0, M_1)[\|u_0\|_{\sigma+1/p',p} +$$

$$+(1+T^{\sigma})(\|b\|_{W^{\sigma,p}((0,t);X)} + \|f\|_{W^{\sigma,p}((0,t);X)} + \|k*BA^{-1}w\|_{W^{\sigma,p}((0,t);X)})] \le$$

$$\le c_{14}(\sigma, p, M_0, M_1)\Bigg\{\|u_0\|_{\sigma+1/p',p} + (1+T^{\sigma})\Bigg[\|b\|_{W^{\sigma,p}((0,t);X)} + \|f\|_{W^{\sigma,p}((0,t);X)} +$$

$$+M_2\|k\|_{L^1(0,T)}^{1/p'}\left(\int_0^t |k(s)|\,|w|_{W^{\sigma,p}((0,t-s);X)}^p\,ds\right)^{1/p}\Bigg]\Bigg\}, \qquad \forall t \in (0, T].$$

Consider then the case $\sigma \in (1/p, 1)$. Proceeding as above, from formula (2.34), Lemma 2.4 and the membership of w in $\mathcal{W}^{\sigma,p}$ we weasily derive the estimate

$$(3.9) \quad \|w\|_{W^{\sigma,p}((0,t);X)} \le \|Au_0\| + c_{15}(\sigma, p, M_0, M_1, M_2)\cdot$$

$$\cdot\Bigg\{\|Au_0 + f(0)\|_{\sigma+1/p,p} + (1+T^{\sigma})\Bigg[\|k\|_{L^1(0,T)}\|b\|_{W^{\sigma,p}((0,t);X)} +$$

$$+\|f\|_{W^{\sigma,p}((0,t);X)} + \|k\|_{L^1(0,T)}^{1/p'}\left(\int_0^t |k(s)|\,\|w\|_{W^{\sigma,p}((0,t-s);X)}^p\,ds\right)^{1/p}\Bigg]\Bigg\}, \quad \forall t \in (0, T].$$

Thus we have proved that w satisfies the following inequality for any $t \in (0, T]$, any $\sigma \in (0,1)\setminus\{1/p\}$ and suitable functions $G \in L^{\infty}(0, T)$:

$$(3.10) \quad \|w\|_{W^{\sigma,p}((0,t);X)}^p \le 2^{p-1}G(t) + 2^{p-1}C^p\int_0^t |k(s)|\,\|w\|_{W^{\sigma,p}((0,t-s);X)}^p\,ds.$$

Set then

$$\phi(t) = \|w\|_{W^{\sigma,p}((0,t);X)}^p, \qquad \forall t \in (0, T].$$

Hence (3.10) can be rewritten as a convolution inequality

$$(3.11) \qquad \phi(t) \le 2^{p-1}G(t) + 2^{p-1}C^p|k| * \phi(t), \qquad \forall t \in (0, T].$$

From (3.11) we easily derive the estimate

$$\phi(t) \leqslant 2^{p-1} \sum_{n=0}^{\infty} 2^{n(p-1)} C^{np} (|k| *)^n * G(t), \qquad \forall t \in (0, T].$$

It, together with (2.21), implies

$$\|\phi\|_{L^\infty(0,t)} \leqslant 2^{p-1} \sum_{n=0}^{\infty} 2^{n(p-1)} C^{np} \|(|k|*)^n\|_{L^1(0,t)} \|G\|_{L^\infty(0,t)} \leqslant$$

$$\leqslant 2^{p-1-1/q} \sum_{n=0}^{\infty} 2^{n(p-1)} C^{np} (n!)^{-1/q'} t^{n/q'} \|k\|^n_{L^q(0,t)} \|G\|_{L^\infty(0,t)}, \qquad \forall t \in (0, T].$$

Since $k \in \mathcal{K}(M_3)$ (see also Remark 1.2), this proves estimates (3.6), (3.7). ■

LEMMA 3.2. *Assume that* $k_j \in \mathcal{K}(M_3)$ $(j=1,2)$ *and the triplets* $(b_j, f_j, u_{0,j})$ $(j=1,2)$ *fulfil properties listed in the statement of Theorem* 1.2. *Then operator* W *satisfies the following estimates*:

$$(3.12) \qquad \|W(k_2, b_2, f_2, u_{0,2}) - W(k_1, b_1, f_1, u_{0,1})\|_{W^{\sigma,p}((0,t);X)} \leqslant$$

$$\leqslant c_{16}(\sigma, p, M_0, M_1, M_2, M_3, T)(1 + M_4)[\|k_2 - k_1\|_{L^1(0,t)} +$$

$$+ \|b_2 - b_1\|_{W^{\sigma,p}((0,t);X)} + \|f_2 - f_1\|_{W^{\sigma,p}((0,t);X)} + \|u_{0,2} - u_{0,1}\|_{\sigma + 1/p', p}],$$

$$\forall t \in (0, T], \ \text{if } \sigma \in (0, 1/p),$$

$$(3.13) \qquad \|W(k_2, b_2, f_2, u_{0,2}) - W(k_1, b_1, f_1, u_{0,1})\|_{W^{\sigma,p}((0,T);X)} \leqslant$$

$$\leqslant c_{17}(\sigma, p, M_0, M_1, M_2, M_3, T)(1 + M_4)[\|t^{1-\sigma}(k_2 - k_1)\|_{L^p(0,t)} +$$

$$+ \|b_2 - b_1\|_{W^{\sigma,p}((0,t);X)} + \|b_2(0) - b_1(0)\| + \|f_2 - f_1\|_{W^{\sigma,p}((0,t);X)} +$$

$$+ \|Au_{0,2} - Au_{0,1}\| + \|Au_{0,2} - Au_{0,1} + f_2(0) - f_1(0)\|_{\sigma - 1/p, p}],$$

$$\forall t \in (0, T], \ \sigma \in (1/p, 1).$$

PROOF. We limit ourselves to proving the case when $\sigma \in (0, 1/p)$, since the case when $\sigma \in (1/p, 1)$ cam be proved similarly taking equation (2.34) (with $w(0) = Au_0$) into account.

For the sake of simplicity we set $w_j = W(k_j, b_j, f_j, u_{0,j})$ $(j=1,2)$. From equation (1.3) and Lemmas 2.1-2.4 we deduce the estimates valid for any $t \in (0, T]$:

$$\|w_2 - w_1\|_{W^{\sigma,p}((0,t);X)} \leqslant \|N_1(u_{0,2} - u_{0,1})\|_{W^{\sigma,p}((0,t);X)} + \tag{3.14}$$

$$+ \|N_2(f_2 - f_1)\|_{W^{\sigma,p}((0,t);X)} + \|N_2[(k_2 - k_1) * (BA^{-1}w_1 + b_1)]\|_{W^{\sigma,p}((0,t);X)} +$$

$$+ \|N_2\{k_2 * [BA^{-1}(w_2 - w_1) + (b_2 - b_1]\}\|_{W^{\sigma,p}((0,t);X)} \leqslant$$

$$\leqslant c_4(\sigma, p, M_0, M_1, M_2, M_3, T) \cdot$$

$$\cdot \Bigg\{ \|u_{0,2} - u_{0,1}\|_{\sigma + 1/p', p} + \|f_2 - f_1\|_{W^{\sigma,p}((0,t);X)} + M_4 \|k_2 - k_1\|_{L^1(0,t)} +$$

$$M_3^{1/p'} \left(\int_0^t |k_2(s)| \|w_2 - w_1\|^p_{W^{\sigma,p}((0,t-s);X)} ds + \|b_2 - b_1\|_{W^{\sigma,p}((0,t);X)} \right)^{1/p} \Bigg\}.$$

Reasoning as in the proof of Lemma 3.1 and recalling that $k_2 \in \mathcal{K}(M_3)$, we easily deduce estimate (3.12). ■

PROOF OF THEOREM 1.2. It is an immediate consequence of Lemmas 3.2, 3.3, formula (1.1), Theorem 30 and the stability results in [1]. ■

4. Proofs of Theorems 1.3, 1.4.

First we prove the following

LEMMA 4.1. *The function $N_3(f)$ defined in the right-hand side in (1.22) belongs to $W^{\sigma,p}((0,T);X)$ for any $f \in W^{\sigma,p}((0,T);D_A(\theta,p))$ ($\sigma \in (0,1) \setminus \{1/p\}$, $\theta \in (0,1)$) such that $f(0) \in D_A(\sigma - 1/p, p)$ when $\sigma \in (1/p, 1)$. Moreover $N_3(f)$ satisfies the following estimate, where Y_+ denotes Heaviside's function:*

$$\|N_3(f)\|_{W^{\sigma,p}((0,t);X)} \leqslant c_{18}(\sigma, p, \theta, \varepsilon, M_0, M_1) T^{\theta - \varepsilon} \cdot \tag{4.1}$$

$$\cdot \left(\int_0^t (t-s)^{-1+\varepsilon p} \|f\|^p_{W^{\sigma,p}((0,s);D_A(\theta,p))} ds + \|f(0)\|^p_{\sigma - 1/p, p} Y_+(\sigma - 1/p) \right)^{1/p} \leqslant$$

$$\leqslant c_{18}(\sigma, p, \theta, \varepsilon, M_0, M_1) T^{\theta - \varepsilon} [(\varepsilon p)^{-1} T^{\varepsilon p} \|f\|^p_{W^{\sigma,p}((0,T);D_A(\theta,p))} +$$

$$+ \|f(0)\|^p_{\sigma - 1/p, p} Y_+(\sigma - 1/p)]^{1/p}, \qquad \forall t \in (0, T], \ \forall \varepsilon \in (0, \theta).$$

PROOF. Assume first $\sigma \in (0, 1/p)$. We consider the inequalities valid for any $f \in W^{\sigma, p}((0, T); D_A)$, any $t \in (0, T]$ and any $\varepsilon \in (0, \theta)$:

$$\|N_3(f)(t)\| \leqslant \int_0^t s^{\theta - \varepsilon - 1/p'} s^{1 + \varepsilon - \theta - 1/p} \|Ae^{sA} f(t-s)\| ds \leqslant$$

$$\leqslant [(\theta - \varepsilon) p']^{-1/p'} t^{\theta - \varepsilon} \left(\int_0^t s^{(1 + \varepsilon - \theta)p - 1} \|Ae^{sA} f(t-s)\|^p ds \right)^{1/p}.$$

They imply

$$(4.2) \qquad [(\theta - \varepsilon) p']^{p/p'} t^{-(\theta - \varepsilon)p} \|N_3(f)\|^p_{L^p((0,t);X)} \leqslant$$

$$\leqslant \int_0^t s^{(1+\varepsilon-\theta)-p} ds \int_s^t \|Ae^{sA} f(t-s)\|^p ds = \int_0^t s^{(1+\varepsilon-\theta)p} ds \int_0^{t-s} \|Ae^{sA} f(r)\|^p dr =$$

$$= \int_0^t dr \int_0^{t-r} s^{(1+\varepsilon-\theta)-p} \|Ae^{sA} f(r)\|^p ds = \int_0^t (t-r)^{\varepsilon p} \|f(r)\|^p_{\theta, p} dr =$$

$$= \varepsilon p \int_0^t (t-r)^{\varepsilon p - 1} \|f\|^p_{L^p((0,r); D_A(\theta, p))} dr, \qquad \forall t \in (0, T].$$

We note that the last equality has been obtained by an integration by parts. Consider then the following identities, where $0 \leqslant t_1 < t_2 \leqslant T$:

$$(4.3) \qquad N_3(f)(t_2) - N_3(f)(t_1) = \int_{t_1}^{t_2} Ae^{sA} f(t_2 - s) ds +$$

$$+ \int_0^{t_1} Ae^{sA} [f(t_2 - s) - f(t_1 - s)] ds = N_{3,1}(f)(t_2, t_1) + N_{3,2}(f)(t_2, t_1).$$

Interchanging the order of integrations and performing simple changes of variables, from (4.3) and (2.4) we deduce

$$(4.4) \qquad [(\theta - \varepsilon) p']^{p/p'} t^{-(\theta - \varepsilon)p} \int_0^t dt_2 \int_0^{t_2} (t_2 - t_1)^{-1-\sigma p} \|N_{3,1}(t_1, t_2)\|^p dt_1 \leqslant$$

$$\leqslant \int_0^t dt_2 \int_0^{t_2} (t_2 - t_1)^{-1-\sigma p} dt_1 \int_{t_1}^{t_2} s^{(1+\varepsilon-\theta)p - 1} \|Ae^{sA} f(t_2 - s)\|^p ds \leqslant$$

$$\leqslant \int_0^t s^{(1+\varepsilon-\theta)p-1}ds \int_s^t \|Ae^{sA}f(t_2-s)\|^p dt_2 \int_0^s (t_2-t_1)^{-1-\sigma p} dt_1 \leqslant$$

$$\leqslant (\sigma p)^{-1} \int_0^t s^{(1+\varepsilon-\theta)p-1}ds \int_s^t (t_2-s)^{-\sigma p} \|Ae^{sA}f(t_2-s)\|^p dt_2 \leqslant$$

$$\leqslant (\sigma p)^{-1} \int_0^t s^{(1+\varepsilon-\theta)p-1}ds \int_0^{t-s} t_2^{-\sigma p} \|Ae^{sA}f(t_2)\|^p dt_2 =$$

$$= (\sigma p)^{-1} \int_0^t t_2^{-\sigma p} dt_2 \int_0^{t-t_2} s^{(1+\varepsilon-\theta)p-1} \|Ae^{sA}f(t_2)\|^p ds \leqslant$$

$$\leqslant (\sigma p)^{-1} \int_0^t (t-t_2)^{\varepsilon p} t_2^{-\sigma p} \|f(t_2)\|^p_{\theta,p} dt_2 =$$

$$= \varepsilon\sigma^{-1} \int_0^t (t-t_2)^{\varepsilon p-1} \|\tau^{-\sigma} f\|^p_{L^p((0,t_2);D_A(\theta,p))} dt_2 \leqslant$$

$$\leqslant \varepsilon\sigma^{-1} c_3(\sigma,p) \int_0^t (t-t_2)^{\varepsilon p-1} \|f\|^p_{W^{\sigma,p}((0,t_2);D_A(\theta,p))} dt_2, \qquad \forall t \in (0,T].$$

To derive the last equality we have again integrated by parts.

By a similar procedure we get the chain of inequalities

$$\text{(4.5)} \quad [(\theta-\varepsilon)p']^{p/p'} t^{-(\theta-\varepsilon)p} \int_0^t dt_2 \int_0^{t_2} (t_2-t_1)^{-1-\sigma p} \|N_{3,2}(t_1,t_2)\|^p dt_1 \leqslant$$

$$\leqslant \int_0^t s^{(1+\varepsilon-\theta)p-1} ds \int_s^t dt_2 \int_0^{t_2} (t_2-t_1)^{-1-\sigma p} \|Ae^{sA}[f(t_2-s)-f(t_1-s)]\|^p dt_1 =$$

$$= \int_0^t s^{(1+\varepsilon-\theta)p-1} ds \int_0^{t-s} dt_2 \int_0^{t_2} (t_2-t_1)^{-1-\sigma p} \|Ae^{sA}[f(t_2)-f(t_1)]\|^p dt_1 \leqslant$$

$$\leqslant \int_0^t dt_2 \int_0^{t_2} (t_2-t_1)^{-1-\sigma p} dt_1 \int_0^{t-t_2} s^{(1+\varepsilon-\theta)p-1} \|Ae^{sA}[f(t_2)-f(t_1)]\|^p ds \leqslant$$

$$\leqslant \int_0^t (t-t_2)^{\varepsilon p} dt_2 \int_0^{t_1} (t_2-t_1)^{-1-\sigma p} \|f(t_2)-f(t_1)\|_{\theta,p}^p ds \leqslant$$

$$\leqslant \varepsilon\sigma^{-1} \int_0^t (t-t_2)^{\varepsilon p-1} \|\tau^{-\sigma} f\|_{L^p((0,t_2);D_A(\theta,p))}^p dt_2 \leqslant$$

$$\leqslant \frac{1}{2}\varepsilon p \int_0^t (t-t_2)^{\varepsilon p-1} \|f\|_{W^{\sigma,p}((0,t_2);D_A(\theta,p))}^p dt_2, \qquad \forall t \in (0,T].$$

From (4.2), (4.4) and (4.5) we easily deduce (4.1) when $\sigma \in (0, 1/p)$ and $f \in W^{\sigma,p}((0,T); D_A)$. A density argument proves the general case.

When $\sigma \in (1/p, 1)$ we have to consider the additional identiy

$$(4.6) \qquad N_3(f)(t) = N_3[f - f(0)](t) + (e^{tA} - I)f(0), \qquad \text{for a.e. } t \in (0,T).$$

Apply then Lemma 2.2 and the same procedure as above. ■

We observe now that, by virtue of Remark 1.4 and Lemma 4.1, the integral equation (1.22) is uniquely solvable in $\mathcal{W}^{\sigma,p}$ for any $k \in W^{\sigma,p}(0,T)$ and any triplet (b, f, u_0) fulfilling the hypotheses listed in the statement of Theorem 1.3.

We note then that, by virtue of Remark 1.4 and Lemma 4.1, the integral equation (1.22) is uniquely solvable in $\mathcal{W}^{\sigma,p}$ for any $k \in W^{\sigma,p}(0,T)$ and any triplet (b, f, u_0) fulfilling the hypotheses listed in the statement of Theorem 1.3. In fact, the quadruplet $(\tilde{k}, \tilde{b}, \tilde{f}, \tilde{u}_0) = (k, b', f', Au_0 + f(0))$ belongs to $L^p((0,T); t^{(1-\sigma)p} dt) \times W^{\sigma,p}((0,T); X) \times W^{\sigma,p}((0,T); X) \times D_A(\sigma + 1/p', p)$ and function $l = k[Au_0 + f(0)]$ satisfies the assumptions of Lemma 4.1 according to the equation

$$(4.7) \qquad A\tilde{u}_0 + \tilde{f}(0) = E(b, f, g, u_0)$$

and hypothesis iv) in Theorem 1.3.

Hence we can introduce the (nonlinear) operator

$$W\colon D(W) \subset W^{\sigma,p}(0,T) \times W^{\sigma,p}((0,T);X) \times W^{\sigma,p}((0,T);X) \times$$
$$\times D(\sigma + 1/p', p) \hookrightarrow \mathcal{W}^{\sigma,p}((0,T);Y)$$

defined by equation

$$(4.8) \qquad W(k, b, f, u_0) = w,$$

w being the solution to problem (1.22) corresponding to the quadruplet of data (k, b, f, u_0) with the stated properties.

Moreover from Lemmas 3.1, 3.2, 4.1 it is immediate to deduce that W enjoys the properties listed in the following Lemma 4.2, where

$$(4.9)\quad \mathcal{K}(M_3) = \{k \in W^{\sigma,p}(0,T): \|k\|_{W^{\sigma,p}(0,T)} \leq M_3\}, \qquad (M_3 \in \mathbb{R}_+).$$

To this purpose we use the following estimate, implied by (1.28), where $\chi_j^{-1} = \Phi[Au_{0,j} + b_j(0)]$ $(j=1,2)$:

$$|k_2(0) - k_1(0)| \leq |\chi_1\chi_2| \|\Phi\|_{X^*}[\|Au_{0,2} - Au_{0,1}\| + \|b_2(0) - b_1(0)\|] +$$

$$+ |\chi_1| \|\Phi\|_{X^*}[|g_2''(0) - g_1''(0)| + \|E(b_2, f_2, g_2, u_{0,2}) - E(b_1, f_1, g_1, u_{0,1})\|].$$

LEMMA 4.2. *Assume that* $k, k_1, k_2 \in \mathcal{K}(M_3)$ *and the triplets* (b, f, u_0), $(b_1, f_1, u_{0,1})$, $(b_2, f_2, u_{0,2})$ *fulfil properties listed in the statement of Theorem* 1.3. *Then operator* W *defined by equation* (4.8) *satisfies the following estimates for any* $t \in (0, T]$:

$$(4.10)\quad \|W(k, b, f, u_0)\|_{W^{\sigma,p}((0,t);X)} \leq c_{19}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, M_3, T)\cdot$$

$$\cdot[\|b\|_{W^{1+\sigma,p}((0,t);X)} + \|f\|_{W^{1+\sigma,p}((0,t);X)} + \|Au_0 + f(0)\|_{\sigma+1/p',p} +$$

$$+ \|Bu_0 + b(0)\|_{\theta,p}] := c_{19}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, M_3, T) M_5(b, f, u_0)(t),$$

$$\forall t \in (0, T],\ \forall \varepsilon \in (0, \theta),\ \text{if } \sigma \in (0, 1/p),$$

$$(4.11)\quad \|W(k, b, f, u_0)\|_{W^{\sigma,p}((0,t);X)} \leq c_{20}(\sigma, p, \varepsilon, M_0, M_1, M_2, M_3, T)\cdot$$

$$\cdot[\|b\|_{W^{1+\sigma,p}((0,t);X)} + \|f\|_{W^{1+\sigma,p}((0,t);X)} + \|b'(0)\| + \|Bu_0 + b(0)\|_{\sigma-1/p,p} +$$

$$+ \|Au_0\| + \|E(b, f, g, u_0)\|_{\sigma-1/p,p}] :=$$

$$:= c_{20}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, M_3, T) M_5(b, f, u_0)(t),$$

$$\forall t \in (0, T],\ \forall \varepsilon \in (0, \sigma - 1/p),\ \text{if } \sigma \in (1/p, 1),$$

$$(4.12)\quad \|W(k_2, b_2, f_2, u_{0,2}) - W(k_1, b_1, f_1, u_{0,1})\|_{W^{\sigma,p}((0,t);X)} \leq$$

$$\leq c_{21}\Big(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, M_3, \|\Phi\|_{X^*}, \max_{1\leq j\leq 2}|\chi_j|, T\Big)\cdot$$

$$\cdot\Big[1 + \max_{1\leq j\leq 2} M_5(b_j, f_j, u_{0,j})(T_0)\Big]\cdot$$

$$\cdot\Bigg[\|k_2-k_1\|_{L^1(0,t)}+\left(\int_0^t (t-s)^{-1+\varepsilon p}\|k_2-k_1\|^p_{W^{\sigma,p}(0,s)}ds\right)^{1/p}+$$

$$+\|b_2-b_1\|_{W^{1+\sigma,p}((0,t);X)}+\|f_2-f_1\|_{W^{1+\sigma,p}((0,t);X)}+$$

$$+\|u_{0,2}-u_{0,1}\|_{\sigma+1/p',p}+|Bu_{0,2}-Bu_{0,1}+b_2(0)-b_1(0)|_{\theta,p}\Bigg]$$

$$\forall t\in(0,T],\ \forall\varepsilon\in(0,\theta),\ \text{if } \sigma\in(0,1/p),$$

$$(4.13)\qquad \|W(k_2,b_2,f_2,u_{0,2})-W(k_1,b_1,f_1,u_{0,1})\|_{W^{\sigma,p}((0,T);X)}\leqslant$$

$$\leqslant c_{22}\Big(\sigma,p,\theta,\varepsilon,M_0,M_1,M_2,M_3,\|\Phi\|_{X^*},\max_{1\leqslant j\leqslant 2}|\chi_j|,T\Big)\cdot$$

$$\cdot\Big[1+\max_{1\leqslant j\leqslant 2}M_5(b_j,f_j,u_{0,j})(T_0)\Big]\cdot$$

$$\cdot\Bigg[\|t^{1-\sigma}(k_2-k_1)\|_{L^p(0,t)}+\left(\int_0^t (t-s)^{-1+\varepsilon p}\|k_2-k_1\|^p_{W^{\sigma,p}(0,s)}ds\right)^{1/p}+$$

$$+\|b_2-b_1\|_{W^{1+\sigma,p}((0,t);X)}+\|f_2-f_1\|_{W^{1+\sigma,p}((0,t);X)}+$$

$$+|g_2''(0)-g_1''(0)|+\|b_2(0)-b_1(0)\|+\|b_2'(0)-b_1'(0)\|+$$

$$+\|Bu_{0,2}-Bu_{0,1}+b_2(0)-b_1(0)\|_{\sigma-1/p,,p}+\|Au_{0,2}-Au_{0,1}\|+$$

$$+\|E(b_2,f_2,g_2,u_{0,2})-E(b_1,f_1,g_1,u_{0,1})\|\Bigg],$$

$$\forall t\in(0,T],\ \forall\varepsilon\in(0,\sigma-1/p),\ \text{if } \sigma\in(1/p,1).$$

PROOF OF THEOREM 1.3. First we replace the first function w appearing in the second term in the right-hand side in (1.23) by the right-hand side in (1.22). Then we substitute $W(k, b, f, u_0)$ for w. Thus we get the following (equivalent) fixed-point equation for k:

$$(4.14)\qquad S(k,b,f,g,u_0)=k,$$

where operator S is defined by

$$(4.15)\qquad S(k, b, f, g, u_0) = \chi g'' + \chi\Phi\{f' + N_1[Au_0 + f(0)] + N_2(f')\} +$$
$$- \chi\Phi\{N_2[k*(BA^{-1}W(k, b, f, u_0) + b')] + N_3[k(Bu_0 + b(0))] +$$
$$+ k*(BA^{-1}W(k, b, f, u_0) + b')\} := S_1(k, b, f, g, u_0) + S_2(b, f, g, u_0).$$

Then we prove that S satisfies the hypotheses of Banach's contraction principle for small enough $T \in (0, T_0]$. To this purpose we introduce a positive constant M_6 such that (see estimates (4.10), (4.11))

$$(4.16)\qquad M_5(b, f, u_0)(T_0) + \|g\|_{W^{2+\sigma, p}(0, T_0)} \leq M_6.$$

Then, from Lemmas 2.1-2.4, 4.1, 4.2 we deduce the following estimate, where for the sake of simplicity we drop out the dependence of S on data:

$$(4.17)\qquad \|S(k)\|_{W^{\sigma, p}(0, T)} \leq$$
$$\leq c_{23}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, M_3, T)|\chi|(1 + M_6)\|\Phi\|_{X^*}\cdot$$
$$\cdot\max(T^{1/p'}, T^{\varepsilon}) + c_{24}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, T_0)|\chi|(1 + M_5),$$
$$\forall k \in \mathcal{K}(M_2).$$

We choose now $M_3 > c_{24}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, T_0)|\chi|(1 + M_6)$. Since the positive constant c_{24} is bounded as $T \to 0^+$, we deduce that S maps $\mathcal{K}(M_3)$ into itself for small enough T.

In order to show that S is a contraction for small enough T we consider first the following identity, where we drop out also the dependence of W on data:

$$(4.18)\qquad S(k_2) - S(k_1) = -\chi\Phi\{N_2[(k_2 - k_1)*(BA^{-1}W(k_2) + b')] +$$
$$+ N_2[k_1*BA^{-1}(W_1(k_2) - W_1(k_1))] + N_3[(k_2 - k_1)(Au_{0,2} + f_2(0))] +$$
$$+ (k_2 - k_1)*BA^{-1}(W(k_2) + b') + k_1*BA^{-1}(W(k_2) - W(k_1))\}.$$

Then we use again the above mentioned Lemmas. They easily imply the following estimate, which proves our assertion:

$$(4.19)\qquad \|S(k_2) - S(k_1)\|_{W^{\sigma, p}(0, T)} \leq c_{25}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, M_3, T)\cdot$$
$$\cdot|\chi|\|\Phi\|_{X^*}(1 + M_6)\max(T^{1/p'}, T^{\varepsilon})\|k_2 - k_1\|_{W^{\sigma, p}(0, T)},$$
$$\forall k_1, k_2 \in \mathcal{K}(M_3). \qquad \blacksquare$$

REMARK 4.1. From the previous proof we deduce that the solution k to fixed-point equation (4.14) exists in the interval $[0, T] \subset [0, T_0]$, where T_0 satisfies the inequality

$$(4.20) \quad c_{23}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, M_3, T)|\chi|(1 + M_6)\|\Phi\|_{X^*} \cdot$$

$$\cdot \max(T^{1/p'}, T^{\varepsilon}) < M_3 - c_{24}(\sigma, p, \theta, \varepsilon, M_0, M_1, M_2, T_0)|\chi|(1 + M_6).$$

We note that T depends on data only through $(M_6, |\chi|)$.

PROOF OF THEOREM 1.4. Consider the identity

$$(4.21) \quad k_2 - k_1 = S_1(k_2, b_2, f_2, g_2, u_{0,2}) - S_1(k_1, b_2, f_2, g_2, u_{0,2}) +$$
$$+ S_1(k_1, b_2, f_2, g_2, u_{0,2}) - S_1(k_1, b_1, f_1, g_1, u_{0,1}) +$$
$$+ S_2(b_2, f_2, g_2, u_{0,2}) - S_2(b_1, f_1, g_1, u_{0,1}).$$

From Lemmas 2.1-2.4, 4.1, 4.2 and estimates (4.18), (4.19) we easily derive estimates (1.29), (1.30). To this purpose we use the following estimate valid when $\sigma \in (1/p, 1)$:

$$\|f(0)\| \leq c(\sigma, p)\, T^{-1/p}(1 + T^{\sigma p})^{1/p}\|f\|_{W^{\sigma,p}((0,t);X)},$$
$$\forall f \in W^{\sigma,p}((0, T); X). \quad \blacksquare$$

5. Some applications.

Let Ω be an open bounded set in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class $C^{2+\gamma}$ for some $\gamma \in (0,1)$ and let $a_{h,j}, a_h, a_0, b_{h,j}, b_h, b_0: \overline{\Omega} \to \mathbb{R}$ $(h, j = = 1, 2, \ldots, n)$ be continuous functions.

Consider then the differential operators

$$\mathcal{A} = \sum_{h,j=1}^{n} a_{h,j}(x) D_h D_j + \sum_{h=1}^{n} a_h(x) D_h + a_0(x),$$

$$\mathcal{B} = \sum_{h,j=1}^{n} b_{h,j}(x) D_h D_j + \sum_{h=1}^{n} b_h(x) D_h + b_0(x).$$

In addition assume that $\mathcal{A}$ is uniformly elliptic in $\overline{\Omega}$, i.e.

$$\sum_{h,j=1}^{n} a_{h,j}(x)\xi_h \xi_j > 0, \qquad \forall x \in \mathbb{R}^n, \ \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given the functions $b, l: [0, T] \times \Omega \to \mathbb{R}$, $z_0: \Omega \to \mathbb{R}$, $k: [0, T] \to \mathbb{R}$, consider the following direct problem: *determine a function* $z: [0, T] \times$

$\times \Omega \to \mathbb{R}$ *such that*

$$(5.1)\qquad D_t z(t, x) - \mathcal{A}z(t, x) - \int_0^t k(t-s)\,\mathcal{B}z(s, x)\,ds = l(t, x),\qquad (t, x) \in (0, T) \times \Omega,$$

$$(5.2)\qquad z(0, x) = z_0(x),\qquad x \in \Omega,$$

$$(5.3)\qquad z(t, x) = b(t, x),\qquad (t, x) \in (0, T) \times \partial\Omega.$$

Introduce now the new unknown $u = z - b$. It is immediate to realize that u solves the following problem:

$$(5.4)\qquad D_t u(t, x) - \mathcal{A}u(t, x) - \int_0^t k(t-s)[\mathcal{B}z(s, x) + \mathcal{B}b(s, x)]\,ds = f(t, x),\qquad (t, x) \in (0, T) \times \Omega,$$

$$(5.5)\qquad u(0, x) = u_0(x),\qquad x \in \Omega,$$

$$(5.6)\qquad u(t, x) = 0,\qquad (t, x) \in (0, T) \times \partial\Omega,$$

where we have set

$$(5.7)\qquad f(t, x) = l(t, x) - D_t b(t, x) + \mathcal{A}b(t, x),\qquad (t, x) \in (0, T) \times \Omega,$$

$$(5.8)\qquad u_0(x) = z_0(x) - b(0, x),\qquad x \in \Omega.$$

To write (5.4)-(5.6) in the abstract form (0.1), (0.2) we choose $X = L^p(\Omega)$ $(1 < p < +\infty)$, $D_A = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $D_B = W^{2,p}(\Omega)$ and $A = \mathcal{A}$, $B = \mathcal{B}$. Such a choice implies the equations (see [1, Section 9])

$$(5.9)\qquad D_A(\theta, p) = W^{2\theta, p}(\Omega),\qquad \text{if } \theta \in (0, 1/2p) \text{ and } p \in (1, +\infty),$$

$$(5.10)\qquad D_A(\theta, p) = \{w \in W^{2\theta, p}(\Omega) \colon w = 0 \text{ on } \partial\Omega\},\qquad \text{if } \theta \in (1/2p, 1)\setminus\{1/2\} \text{ and } p \in (1, +\infty).$$

From the results states in Section 1 we derive

THEOREM 5.1. *Assume that* $b \in W^{1+\sigma, p}((0, T); L^p(\Omega)) \cap W^{\sigma, p}((0, T); W^{2,p}(\Omega))$, $l \in W^{\sigma, p}((0, T); L^p(\Omega))$ *and* $z_0 \in W^{2(\sigma + 1/p'), p}(\Omega)$ *for some* $\sigma \in (0, 1)\setminus\{1/p, 1/p - 1/2\}$ *and* $p \in (1, +\infty)$ *satisfy the following equations*:

$$(5.11)\qquad b(0, \cdot) = z_0,\qquad \text{on } \partial\Omega \text{ in the sense of traces, if } \sigma \in (1/p, 1),$$

(5.12) $D_t b(0, \cdot) = \mathcal{A} z_0 + l(0, \cdot)$,

on $\partial\Omega$ in the sense of traces, if $p > 3/2$ and $\sigma \in (3/2p, 1)$.

Assume further that:

i) *$k \in L^q(0, T)$ for some $q \in (1, +\infty]$ when $\sigma \in (0, 1/p)$;*

ii) *$k \in L^p((0, T); t^{p(1-\sigma)} dt)$ when $\sigma \in (1/p, 1)$.*

Then problem (5.1)-(5.3) admits a unique solution $z \in W^{1+\sigma, p}((0, T); L^p(\Omega)) \cap W^{\sigma, p}((0, T); W^{2, p}(\Omega))$ depending continuously on data (k, b, l, z_0) (belonging to the previous class of admissible data) with respect to the norms pointed out (see estimates (1.7), (1.8)).

REMARK 5.1. Theorem 5.1 holds also when $\sigma = 1/p - 1/2$ ($p \in (1, 2)$): in this case we have to assume that $z_0 \in B^{1, p}(\Omega)$, the Besov space of order 1 and exponent p.

Consider now the problem of identifying kernel k. We assume that we are given the following additional information

$$\int_\Omega h(x) z(t, x)\, dx = g(t), \qquad 0 \leq t \leq T, \tag{5.13}$$

where $h: \Omega \to \mathbb{R}$ is a fixed function and $g: [0, T] \to \mathbb{R}$ is our measurement.

Since in our case we have to replace b by $\mathcal{B} b$ in equation (0.1), the corresponding result is reported in the following Theorem 5.2, where

$$\begin{aligned} \mathcal{E}(b, f, g, u_0) = {} & \mathcal{A}[A z_0 + l(0, \cdot) - D_t b(0, \cdot)] + \\ & + D_t l(0, \cdot) - D_t^2 b(0, \cdot) + D_t \mathcal{A} b(0, \cdot) + \left(\int_\Omega h(x)\, \mathcal{B} z_0(x)\, dx \right)^{-1} \cdot \\ & \cdot \Bigg\{ g''(0) - \int_\Omega h(x) \{ \mathcal{A}[A z_0 + l(0, z) - D_t b(0, x)] + D_t l(0, x) + \\ & - D_t^2 b(0, x) + D_t \mathcal{A} b(0, x) \}\, dx \Bigg\} \mathcal{B} z_0 . \end{aligned} \tag{5.14}$$

THEOREM 5.2. *Assume that for some* $\sigma \in (0, 1) \setminus \{1/p, 1/p - 1/2\}$, $\theta \in (0, 1/2)$ *and* $p \in (1, +\infty)$ *the following properties hold*:

i) $h \in L^{p'}(\Omega)$;

ii) $b \in W^{2+\sigma, p}((0, T); L^p(\Omega)) \cap W^{1+\sigma, p}((0, T); W^{2, p}(\Omega))$;

iii) $l \in W^{1+\sigma, p}((0, T); L^p(\Omega))$;

iv) $g \in W^{2+\sigma, p}(0, T)$;

v) $z_0 \in W^{2+2\theta, p}(\Omega)$ *if* $\sigma \in (0, 1/p)$;

vi) $z_0 \in W^{2+2(\sigma - 1/p), p}(\Omega)$, $\mathcal{A}z_0 + l(0, \cdot) - D_t b(0, \cdot) \in W^{2(\sigma + 1/p'), p}(\Omega)$ *and* $\mathcal{E}(b, f, g, u_0) \in W^{2(\sigma - 1/p), p}(\Omega)$ *if* $\sigma \in (1/p, 1)$.

Assume further that data (b, l, g, z_0) *satisfy the following conditions*:

(5.15) $b(0, \cdot) = z_0$, *on* $\partial\Omega$ *in the sense of traces*,

(5.16) $\mathcal{A}b(0, \cdot) = \mathcal{A}z_0 - f(0, \cdot)$, *on* $\partial\Omega$ *in the sense of traces, if* $1 < p < 3/2$ *and* $\sigma \in (3/2p - 1, 1) \setminus \{1/p\}$ *or* $p > 3/2$ *and any* $\sigma \in (0, 1) \setminus \{1/p\}$,

(5.17) $\mathcal{B}z_0 = 0$, *on* $\partial\Omega$ *in the sense of traces, if* $p > 3/2$ *and* $\sigma \in (3/2p, 1)$,

(5.18) $$\int_\Omega h(x)\, \mathcal{B}z_0(x)\, dx \neq 0,$$

(5.19) $\mathcal{E}(b, f, g, u_0) = 0$, *on* $\partial\Omega$ *in the sense of traces, if* $p > 3/2$ *and* $\sigma \in (3/2p, 1) \setminus \{1/2 + 1/p\}$,

(5.20) $$\int_\Omega h(x)[z_0(x) - b(0, x)]\, dx = g(0),$$

(5.21) $$\int_\Omega h(x)[\mathcal{A}z_0(x) - D_t b(0, x) + l(0, x)]\, dx = g'(0).$$

Then problem (5.1)-(5.3) *admits a unique solution*

$$(z, k) \in [W^{2+\sigma, p}((0, T); L^p(\Omega)) \cap W^{1+\sigma, p}((0, T); W^{2, p}(\Omega))] \times W^{\sigma, p}(0, T)$$

depending continuously on (b, l, g, z_0) (*belonging to the previous class of admissible data*) *with respect to the norms pointed out* (*see estimates* (1.29), (1.30)).

REMARK 5.2. Theorem 5.2 holds also when $\sigma = 1/p - 1/2$ ($p \in (1, 2)$): in this case we have to assume that $\mathcal{A}z_0 + l(0, \cdot) - D_t(0, \cdot) \in B^{1, p}(\Omega)$, the Besov space of order 1 and exponent p.

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