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$*$-multilinear polynomials with invertible values

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*-Multilinear Polynomials with Invertible Values.

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Let R be a ring with involution $*$ and S and K the sets of symmetric and skew elements respectively. Several authors have related the algebraic structure of S or K to that of R. For instance, in [3, Theorem 2.18] the hypothesis that all non zero traces $x + x^*$ are invertible determines the structure of R. Similar results have been obtained for the skew case.

In this paper we will examine a more general situation. In fact we consider the case when all the non zero valuations of a *-multilinear polynomial f are invertible in R.

More precisely, let $X = \{x_1, x_1^*, \ldots, x_n, x_n^*, \ldots\}$ be a countable set of unknows and $F\{X, *\}$ be the free associative algebra with involution $*$ in the x_i's and x_i^*'s. The elements of $F\{X, *\}$ are called *-polynomials. A *-polynomial $f(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*) \in F\{X, *\}$ is multilinear if, for each $i = 1, \ldots, n$, either x_i or x_i^*, but non both, appears in each monomial of f.

We shall denote by D a division ring, $Z(D)$ its center, D_m the ring of $m \times m$ matrices over D and D_m^{op} its opposite ring. Notice that $D_m \oplus D_m^{\mathrm{op}}$ has a natural exchange involution given by $(x, y)^* = (y, x)$.

We shall prove the following result.

THEOREM. *Let F be a field of characteristic different from two such that $|F| > 5$. Let R be a semiprime F-algebra with involution $*$ and let $f = f(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*)$ be a *-multilinear polynomial such that for every $r_1, \ldots, r_n$ in R either $f(r_1, \ldots, r_n, r_1^*, \ldots, r_n^*) = 0$ or $f(r_1, \ldots, r_n, r_1^*, \ldots, r_n^*)$ is invertible in R.*

*If $f(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*)$ is not a *-polynomial identity for R then there exists a division ring D such that R is either*

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1) D_m *where if* $m \geqslant 2$ *then* $\dim_{Z(D)} D$ *is finite and* f *is a *-central polynomial for* $m \geqslant 3$; *or*

2) $D_m \oplus D_m^{\mathrm{op}}$ *with exchange involution, where if* $m \geqslant 2$ *then* $\dim_{Z(D)} D$ *is finite and* f *is a *-central polynomial.*

The conclusion of the Theorem is not surprising, because one cannot to expect that f is a *-central polynomial even if $m \leqslant 2$. Infact the polynomial $f = f(x, x^*) = x - x^*$ is not a *-central polynomial in the ring R of 2×2 matrices over a field F with transpose type involution but it still takes zero or invertible values. The same conclusion holds for f and the ring $D \oplus D^{\mathrm{op}}$ with exchange involution.

We also remark that if R is a ring and f is a multilinear polynomial an analogous theorem was proved in [1].

Throughout this paper F will be a field with more then five elements, char. $F \neq 2$, R will be an associative F-algebra with 1 and $Z = Z(R)$ its center. Also, $f(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*)$ will be a multilinear *-polynomial such that for every $r_1, \ldots, r_n$ in R either $f(r_1, \ldots, r_n, r_1^*, \ldots, r_n^*) = 0$ or $f(r_1, \ldots, r_n, r_1^*, \ldots, r_n^*)$ is invertible in R; moreover we will assume that f is not a *-polynomial identity for R.

We begin by looking the case when R is a simple artinian ring.

In this case $R = D_m$ is the ring of $m \times m$ matrices over a division ring D and two different types of involutions are defined in R:

1) *The transpose type involution*: let $-: D \to D$ be an involution in D and $X = \operatorname{diag}\{c_1, \ldots, c_n\} \in D_m$ such that $0 \neq c_i = \bar{c}_i$ for all i.

If $A = (a_{ij}) \in D_m$ then * is given by

$$A^* = (a_{ij})^* = X(\bar{a}_{ji})X^{-1} .$$

2) *The symplettic type involution*: in this case $D = F$ is a field, $m = 2k$ is even and * is given by $(A_{ij})^* = (A^*_{ji})$, where the A_{ij}'s are 2×2 matrices over F with involution given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Given a sequence $\boldsymbol{u} = (A_1, \ldots, A_n)$ of matrices from D_m, the value of $\boldsymbol{u}$ is defined to be

$$|\boldsymbol{u}| = A_1 A_2 \ldots A_n .$$

Now, let $\mathbb{Z}_2 = \{1,*\}$ be the group with two elements, S_n the symmetric group of n symbols and $H_n = \mathbb{Z}_2 \sim S_n$ the wreath product of $\mathbb{Z}_2$ and S_n.

Also, if $(g, \sigma) = (g_1, \ldots, g_n; \sigma) \in H_n$, we write

$$u^{(g,\sigma)} = (A_{\sigma(1)}^{g_1}, \ldots, A_{\sigma(n)}^{g_n}) \text{ where } A^{g_i} = \begin{cases} A & \text{if } g_i = 1, \\ A^* & \text{if } g_i = * . \end{cases}$$

Let e_{ij} be the usual matrix units of D_m $(i, j = 1, \ldots, m)$. We recall that a sequence $u = (a_1 e_{i_1 j_1}, \ldots, a_n e_{i_n j_n})$ where $a_i \in D$, is called simple. Moreover a simple sequence u is even if there exists $(1, \sigma) \in H_n$ such that $|u^{(1,\sigma)}| = b e_{ii} \neq 0$, for some $b \in D$; u is odd if $|u^{(1,\sigma)}| = b e_{ij} \neq 0$ for some $(1, \sigma) \in H_n$, $b \in D$ and $i \neq j$ (see [5]).

For any simple sequence $u = (a_1 e_{i_1 j_1}, \ldots, a_n e_{i_n j_n})$ write $l(u, t)$ (respectively $r(u, t)$) for the number of occurences of the number t as a left (respectively right) index of one of the unit matrices occuring in u. It is proved in [5] that if u is a simple even sequence then $l(u, t) = r(u, t)$ for every t; and if u is an odd simple sequence then there exist two indices i, j such that $l(u, t) = r(u, t)$ for every $t \neq i, j$ while $l(u, i) = r(u, i) + 1$ and $l(u, j) = r(u, j) - 1$.

Also, we remark that if u is a simple sequence of matrices from D_m with $|u| \neq 0$ then $|l(u, t) - r(u, t)| \leq 1$ for all $t = 1, \ldots, m$; moreover $l(u, t) - r(u, t) = l(u, t') - r(u, t') \neq 0$ implies $t = t'$ or $|u| = 0$.

LEMMA 1. *Let u be a simple sequence from D_m and $(g, \sigma) \in H_n$. Then we have:*

1) *If $|u| = a e_{ii} \neq 0$ then $|u^{(g,\sigma)}| = b e_{jj}$ for some $b \in D$, $1 \leq j \leq m$.*

2) *If $|u| = a e_{ij} \neq 0$, with $i \neq j$, then, for some $b, c \in D$, either $|u^{(g,\sigma)}| = b e_{ij}$ or $|u^{(g,\sigma)}| = c e_{ij}^*$.*

PROOF. If * is of transpose type the conclusion of the Lemma follows by [2, Lemma 1].

Suppose now that * is of symplectic type. Recall that the involution * acts in the following way on the matrix units

$$e_{ij}^* = \begin{cases} -e_{j+1\,i-1} & \text{if } i \text{ is even and } j \text{ is odd}, \\ -e_{j-1\,i+1} & \text{if } i \text{ is odd and } j \text{ is even}, \\ e_{j+1\,i+1} & \text{if } i \text{ and } j \text{ are odd}, \\ e_{j-1\,i-1} & \text{if } i \text{ and } j \text{ are even}. \end{cases}$$

Hence, if we denote by

$$t^* = \begin{cases} t+1 & \text{if } t \text{ is odd}, \\ t-1 & \text{if } t \text{ is even}, \end{cases}$$

then for every simple sequence $\boldsymbol{u}$ and for each $(g,\sigma)\in H_n$ we have

$$\begin{cases} l(\boldsymbol{u}^{(g,\sigma)},t) = l(\boldsymbol{u},t)+d \Leftrightarrow r(\boldsymbol{u}^{(g,\sigma)},t^*) = r(\boldsymbol{u},t^*)-d, \\ r(\boldsymbol{u}^{(g,\sigma)},t) = r(\boldsymbol{u},t)+f \Leftrightarrow l(\boldsymbol{u}^{(g,\sigma)},t^*) = l(\boldsymbol{u},t^*)-f. \end{cases}$$

This says that

$$l(\boldsymbol{v},t)-l(\boldsymbol{u},t) = r(\boldsymbol{u},t^*)-r(\boldsymbol{v},t^*)$$

and

$$r(\boldsymbol{v},t)-r(\boldsymbol{u},t) = l(\boldsymbol{u},t^*)-l(\boldsymbol{v},t^*)$$

where $\boldsymbol{v}=\boldsymbol{u}^{(g,\sigma)}$ for some $(g,\sigma)\in H_n$.
Hence

$$[l(\boldsymbol{v},t)-r(\boldsymbol{v},t)]+[r(\boldsymbol{u},t)-l(\boldsymbol{u},t)] =$$
$$= [r(\boldsymbol{u},t^*)]-l(\boldsymbol{u},t^*)]+[l(\boldsymbol{v},t^*)-r(\boldsymbol{v},t^*)].$$

Now, let $|\boldsymbol{u}| = ae_{ii}\neq 0$, then as we said above $r(\boldsymbol{u},t)-l(\boldsymbol{u},t)=0$, for all $t=1,\dots,m$, hence we can write $l(\boldsymbol{v},t)-r(\boldsymbol{v},t) = l(\boldsymbol{v},t^*)-r(\boldsymbol{v},t^*)$. Since $t\neq t^*$ it follows, by the above remarks, that either $|\boldsymbol{v}| = be_{jj}$, for some $b\in D$, or $|\boldsymbol{v}|=0$.

Suppose now that $|\boldsymbol{u}| = ae_{ij}\neq 0$, and, first, assume that $i^*\neq j$ (hence $i\neq j^*$ too). In this case we have:

$$l(\boldsymbol{v},i)-r(\boldsymbol{v},i)-1 = l(\boldsymbol{v},i^*)-r(\boldsymbol{v},i^*)$$

and

$$l(\boldsymbol{v},j)-r(\boldsymbol{v},j)+1 = l(\boldsymbol{v},j^*)-r(\boldsymbol{v},j^*).$$

Hence, in order to have $|\boldsymbol{v}|\neq 0$ it must happen one of the following case

a) $l(\boldsymbol{v},i)-r(\boldsymbol{v},i)=1$ and $l(\boldsymbol{v},j)-r(\boldsymbol{v},j)=-1$,

b) $l(\boldsymbol{v},i)-r(\boldsymbol{v},i)=0$ (that is $l(\boldsymbol{v},i^*)-r(\boldsymbol{v},i^*)=-1$) and $l(\boldsymbol{v},j)-r(\boldsymbol{v},j)=0$ (that is $l(\boldsymbol{v},j^*)-r(\boldsymbol{v},j^*)=1$).

If *a)* holds then $|\boldsymbol{v}| = be_{ij}$ for some $b\in D$; if *b)* holds it follows that $|\boldsymbol{v}| = ce_{j^*i^*} = c'e_{ij}^*$ for some c and c' in D.

Finally let $i^* = j$. In this case we have

$$l(v, i) - r(v, i) - 1 = 1 + l(v, j) - r(v, j),$$

therefore if $|v| \neq 0$, by above remarks, we must have $l(v, i) - r(v, i) = 1$ and $l(v, j) - r(v, j) = -1$; this implies $|v| = be_{ij}$.

We recall the following definition which is a slight generalization of that given above (see also [2]).

DEFINITION. Let u be a simple sequence. Then u is called even if for some $(g, \sigma) \in H_n$ $|u^{(g,\sigma)}| = be_{ii} \neq 0$, and it is odd if for some $(g, \sigma) \in H_n$ $|u^{(g,\sigma)}| = be_{ij} \neq 0$, where $i \neq j$.

Since $f(x_1, \dots, x_n, x_1^*, \dots, x_n^*)$ is a *-multilinear polynomial we may assume that f is of the following form

$$f(x_1, \dots, x_n, x_1^*, \dots, x_n^*) = \sum \alpha_{(g,\sigma)} x_{\sigma(1)}^{g_1} \dots x_{\sigma(n)}^{g_n}$$

where

$$(g, \sigma) = (g_1, \dots, g_n; \sigma) \in H_n \quad \text{and} \quad x^{g_i} = \begin{cases} x & \text{if } g_i = 1, \\ x^* & \text{if } g_i = * . \end{cases}$$

As a consequence of the previous result we have:

LEMMA 2. *Let $u \in D_m$ be a simple sequence. Then*

1) *If u is even, $f(u, u^*) = \sum_1^m \alpha_i e_{ii}$ with $\alpha_i \in D$.*

2) *If u is odd, for some $a, b \in D$, $f(u, u^*) = ae_{ij} + be_{ij}^*$.*

We are now ready to prove the main result for simple artinian ring.

LEMMA 3. *Let D be a division ring of characteristic different from two and with more then five elements. If $m \geq 3$, then f is a *-central polynomial for D_m*

PROOF. Since all the nonzero valuations of f are invertible in $R = D_m$, by Lemma 2, $f(u, u^*) = 0$ for all odd simple sequences u.

Therefore, by the previous Lemma, for all $A_1, \dots, A_n \in D_m$ we have

$$f(A_1, \dots, A_n, A_1^*, \dots, A_n^*) = \sum a_i f(u_i, u_i^*)$$

where the $\boldsymbol{u}_i'$ are even simple sequences. This says that f takes diagonal values in D_m.

Let W be the subalgebra of D_m generated by all the elements of the form $f(r_1, \dots, r_n, r_1^*, \dots, r_n^*)$, for all $r_1, \dots, r_n \in D_m$. We observe that $xWx^* \subseteq W$ for all x unitary elements of R. Thus, if the involution $*$ on R is symplectic by [4, Theorem 5] we have either $W = 0$ or $W \subseteq Z$, the center of R. The first case is impossible because f is not a $*$-polynomial identity, so $W \subseteq Z$ and f is a $*$-central polynomial. On the other hand, if $*$ is an involution of transpose type, since $m \geq 3$ by [4, Theorem 17] f is a $*$-central polynomial.

LEMMA 4. *Let $R = D_2$. Then D is finite dimensional over its center and, if $*$ is the symplectic involution, f is a $*$-central polynomial.*

PROOF. If $*$ is of transpose type for all $A \in D_2$ we have

$$A^* = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \bar{A}^t \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}^{-1}$$

where $-$ is an involution in D and the c_i' are non zero symmetric elements of D. Let $-^{(1)}: D \to D$ be the involution on D defined by $x \to c_1 \bar{x} c_1^{-1}$.

Then, for all $a_1, \dots, a_n \in D$, we have

$$f(a_1 e_{11}, \dots, a_n e_{11}, (a_1 e_{11})^*, \dots, (a_n e_{11})^*) = f(a_1, \dots, a_n, \bar{a}_1^{(1)}, \dots, \bar{a}_n^{(1)})\, e_{11}\,.$$

Since this values is not invertible in R, then $f(a_1, \dots, a_n, \bar{a}_1^{(1)}, \dots, \bar{a}_n^{(1)})$ is zero in D, so D satisfies a $*$-polynomial identity and D is finite dimensional over its center.

If $*$ is the symplectic involution then $D = F$ is a field. Moreover, if $\boldsymbol{u}$ is an odd simple sequence, $f(\boldsymbol{u}, \boldsymbol{u}^*) = ae_{12} + be_{21}^* = (a - b)\, e_{12}$ and this value is not invertible in R. It follows that $f(\boldsymbol{u}, \boldsymbol{u}^*) = 0$ for all $\boldsymbol{u}$ odd simple sequences and so all the valuations of f are diagonal elements.

As in Lemma 3, the subalgebra W generated by $f(R, R^*)$ is invariant under conjugation by unitary elements of R. In particular, if we consider the unitary $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then, for all $w = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in W$, we have $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -a+b \\ 0 & b \end{pmatrix} \in W$. This implies $a = b$ and so f is a $*$-central polynomial.

We will examine now the general case.

We shall use the notation Z^+ for $Z \cap S$. We have the following:

LEMMA 5. *If R is any ring then Z^+ is a field. Moreover, if R is prime then Z is a field.*

PROOF. Let z be an element of Z^+ and $r_1, \dots, r_n \in R$ such that $f(r_1, \dots, r_n, r_1^*, \dots, r_n^*)$ is invertible.

Then $f(zr_1, \dots, r_n, zr_1^*, \dots, r_n^*) = zf(r_1, \dots, r_n, r_1^*, \dots, r_n^*)$, hence, either $f(zr_1, \dots, r_n, zr_1^*, \dots, r_n^*)$ is invertible and this implies that z is invertible or $f(zr_1, \dots, r_n, zr_1^*, \dots, r_n^*) = 0$ and it follows that $z = 0$.

Now, if R is a prime ring, for all $z \in Z - \{0\}$, $0 \neq zz^* \in Z^+$ and by the above zz^*, and so z, is invertible.

We continue with the following:

LEMMA 6. *If R is semiprime then R is *-simple. Moreover, if R is prime then R is simple.*

PROOF. Let $0 \neq I = I^*$ be a proper ideal of R invariant under the involution *. Since the values of $f(x_1, \dots, x_n, x_1^*, \dots, x_n^*)$ in R are zero or invertible, we have $f(r_1, \dots, r_n, r_1^*, \dots, r_n^*) = 0$ for all $r_1, \dots, r_n \in I$.

Hence f is a *-polynomial identity for I and by [3, Theorem 1.4.2] $Z(I) \neq 0$. Also, by [3, Lemma 1.1.5], $Z(I) \subseteq Z(R)$. Now, if $Z(I) \cap S = 0$ then, for all $z \in Z(I)$, $z + z^* = zz^* = 0$ and this implies $z^2 = 0$, a contradiction as R is semiprime. Hence $0 \neq Z(I) \cap S \subseteq Z(R) \cap S = Z^+$. By Lemma 5, Z^+ is a field and so $I = R$, a contradiction again. Therefore R is *-simple-.

Now, if R is prime, let $I \neq (0)$ be an ideal of R; then II^* is a *-ideal. Since R is *-simple then either $II^* = (0)$ or $II^* = R$ and this implies that $I = R$, that is R is a simple ring.

In the following lemma we study the case when R is a prime ring.

LEMMA 7. *If R is a prime ring,* $\mathrm{char}\, \mathrm{R} \neq 2$, *then*

1) *either R is a division ring, or*

2) *$R \cong D_m$ is a finite dimensional central simple algebra and, if $m \geqslant 3$, $f(x_1, \dots, x_n, x_1^*, \dots, x_n^*)$ is a *-central polynomial.*

PROOF. By the previous Lemma, R is a simple ring. If every symmetric element of R is nilpotent or invertible, by [3, Theorem

2.3.3], then either R is a division ring or the ring of 2×2 matrices over a field and we are done.

Therefore we may assume that there exists $s \in S$ such that s is neither nilpotent or invertible. Let $R_1 = sRs$; for all $r_1, \dots r_n \in R$ we have $f(sr_1 s, \dots, sr_n s, sr_1^* s, \dots, sr_n^* s) = sas$, since s is not invertible $sas = 0$ and so f is a *-polynomial identity for R_1. By [3, Theorem 5.5.1] sRs satisfies an identity, hence R satisfies a generalized polynomial identity.

Since R is a simple ring with 1, R coincides with its central closure and so, by [3, Corollary 2 to the Theorem 1.2.2] either $R \cong D_m$ or, for all $m \geq 1$, R contains a *-invariant subring $R^{(m)}$ such that $R^{(m)} \cong D_m$.

In the first case the conclusion follows by Lemma 3 and Lemma 4. In the second case, by Lemma 3, for all $m \geq 3$, f is a *-central polynomial for D_m. Then, by [3, Lemma 5.1.5] D_m satisfies a polynomial identity of degree $2(\deg f + 1)$ for all $m \geq 3$, a contradiction.

We can now prove the main theorem of this note.

PROOF OF THE THEOREM. By Lemma 6, R is a *-simple ring thus either R is simple or R has a simple homomorphic image R_1 such that $R \cong R_1 \oplus R_1^{\mathrm{op}}$ and * is the exchange involution (see [6, Proposition 2.1.12]).

In the first case the result follows from Lemma 5. We may, therefore, assume that $R = R_1 \oplus R_1^{\mathrm{op}}$ with involution *, where R_1 is a simple ring and * the exchange involution.

By setting

$$x_i = \frac{1}{2}[(x_i + x_i^*) + (x_i - x_i^*)] \quad \text{and} \quad x_i^* = \frac{1}{2}[(x_i + x_i^*) - (x_i - x_i^*)]$$

we can write $f(x_1, \dots, x_n, x_1^*, \dots, x_n^*)$ as a polynomial in the symmetric variables $y_i = x_i + x_i^*$ and in the skew variables $z_i = x_i - x_i^*$.

Let $f = g(y_1, \dots, y_n, z_1, \dots, z_n)$, then $g(y_1, \dots, y_n, z_1, \dots, z_n)$, is a polynomial of degree n in $2n$ unknowns such that, for every monomial M of g we have

$$\deg_{y_i} M + \deg_{z_i} M = 1 \quad \text{and} \quad \deg M = n.$$

Moreover, for all substitutions

$$\begin{cases} y_i \to (a_i, a_i) = \overline{a}_i , \\ z_i \to (b_i, -b_i) = \overline{b}_i , \end{cases}$$

we have that $g(\overline{a}_1, \dots, \overline{a}_n, \dots, \overline{b}_1, \dots, \overline{b}_n)$ is either zero or invertible in R.

Let h be one of the blended components of g; that is h is the sum of all the monomials of g in which appear the variables $y_{i_1}, \ldots, y_{i_t}, z_{j_1}, \ldots, z_{j_s}$ for some partition of $\{1, \ldots, n\}$ in the disjoint subsets $\{i_1, \ldots, i_t\}$ and $\{j_1, \ldots, j_s\}$.

Then

$$h(\bar{a}_{i_1}, \ldots, \bar{a}_{i_t}, \bar{b}_{j_1}, \ldots, \bar{b}_{j_s}) = g(0, \ldots, \bar{a}_{i_1}, \ldots, 0, \ldots, \bar{a}_{i_t}, \bar{b}_{j_1}, 0, \ldots, \bar{b}_{j_s}, 0, \ldots)$$

is zero or invertible in R.

If M is a monomial of h we indicate with M^{op} the opposite monomial of M. Then

$$M(\bar{a}_{i_1}, \ldots, \bar{a}_{i_t}, \bar{b}_{j_1}, \ldots, \bar{b}_{j_s}) =$$
$$= (M(a_{i_1}, \ldots, a_{i_t}, b_{j_1}, \ldots, b_{j_s}), (-1)^s M^{\mathrm{op}}(a_{i_1}, \ldots, a_{i_t}, b_{j_1}, \ldots, b_{j_s}))$$

and so

$$h(\bar{a}_{i_1}, \ldots, \bar{a}_{i_t}, \bar{b}_{j_1}, \ldots, \bar{b}_{j_s}) =$$
$$= (h(a_{i_1}, \ldots, a_{i_t}, b_{j_1}, \ldots, b_{j_s}), (-1)^s h^{\mathrm{op}}(a_{i_1}, \ldots, a_{i_t}, b_{j_1}, \ldots, b_{j_s}).)$$

It follows that h is a multilinear polynomial (without *) that assumes zero or invertible values in R_1

Since R_1 is a simple ring with 1, by [1, Theorem] either R_1 is a division ring or $R_1 \cong D_m$ where $m \geqslant 2$, D is a finite dimensional central division ring and h is a central polynomial in D_m.

This leads to desired conclusion.

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