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DINO SOLA

Nullification number and flyping conjecture

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Nullification Number and Flyping Conjecture.

DINO SOLA (*)

1. Introduction.

In this paper we shall introduce a new invariant for alternating knots and links in $\boldsymbol{R}^3$, namely the nullification number, and apply it to solve the «flyping conjecture» in some particular cases.

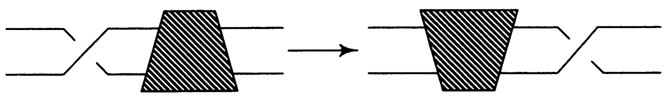
The concept of nullification number is a very natural one: it provides, in one sense, a measure of the complexity of the link. Its definition is quite similar to that of unknotting number (even if the development of the relative theory is completely different), and arises from the concept of nullification of a crossing. The latter notion has come forcefully on to the scene of knot theory thanks to the appearence during the last six years, of a great many polynomial invariants defined through formulas involving operations on diagrams. One of these polynomial invariants is the two-variable polynomial $K(l, m)$ of Lickorish and Millett ([Lickorish and Millett]), ad it is not at all surprising that the nullification number of a minimal alternating projection K is the highest power of m appearing in the polynomial $K(l, m)$. This will be proved in another paper (see [Sola]).

In the second section, we begin by defining the nullification number of a minimal projection, and then show that, for alternating minimal projections, the nullification number can easily be calculated from the projection. We also show that two projections of isotopic links have the same nullification number. The proof of these facts are based on some recent results from the theory of alternating knots. Even if the nullification number is also defined for non-alternating minimal projections, it is still an open problem to understand how to calculate it in this case, and whether any two minimal projections of isotopic non-alternating links have the same nullification number.

(*) Indirizzo dell'A.: Department of Mathematics, University of California, Santa Barbara, CA 93106, U.S.A.

In the third section we apply the invariance of the nullification number to show that certain alternating links admit exactly one minimal projection (up to ambient isotopy) on S^2.

For these links we thus prove the famous *flyping conjecture* of P. G. Tait, which states that any two minimal alternating projections of the same alternating link are related via a finite sequence of *flyping operations*:



We obtain the result of the uniqueness of the minimal projection for those alternating links having nullification number equal to 1, $n-1$, and 2, where n is the crossing number of the links. While the first two classes contain only one link fore each value of the crossing number, the other one is a very large classe. For example, the classical pretzel knots $p(a, b, c)$ with a, b, c all even or all odd, have nullification number equal to 2.

If $\mathcal{K}$ is an alternating link with $3 \leqslant o(\mathcal{K}) \leqslant n(\mathcal{K}) - 2$ (where $o(\mathcal{K})$ denotes the nullification number and $n(\mathcal{K})$ the number of crossings in a minimal projection), $\mathcal{K}$ may have more than one minimal projection. In these cases, given a minimal projection K, there can be some non-trivial flyping operations which transform K into different projections.

2. Nullification number.

If $\mathcal{K}$ is a tame link in $\boldsymbol{R}^3$, we shall always denote by K a minimal projection of $\mathcal{K}$ on S^2, and by $\tilde{K}$ a generic regular projection of $\mathcal{K}$. If $\tilde{K}$ is not minimal, K denotes a minimal projection of the link represented by $\tilde{K}$. All the projections we consider are oriented.

Let us recall that by an oriented *nullification* of a crossing of a regular projection K is meant the following process on the diagram K:

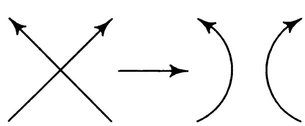
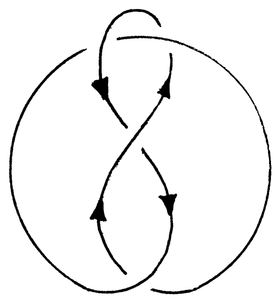


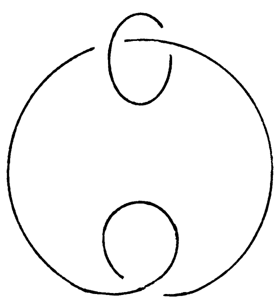
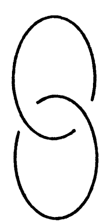
Fig. 1.

(⤧ means ⤧ or ⤨). For a projection $\tilde{K}$, $\tilde{K}_0$ is the projection obtained from K by nullifying a crossing ($\tilde{K}_0$, as well as all the projections we consider, is defined up to ambient isotopy of S^2). Hence $\tilde{K}$ and $\tilde{K}_0$ are projections which are everywhere identical, except in a small disk where they differ as shown in fig. 1.

EXAMPLE. Let K be the following (minimal) projection of the figure-8-knot:

$K =$ 

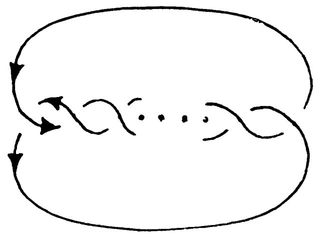
By the above conventions, if we nullify the central crossing we obtain:

$\tilde{K}_0 =$  $K_0 =$ 

Now, let K be a minimal regular projection of a nontrivial link $\mathcal{K}$, $\tilde{K}_{0_1}$ a nullification of K (that is a projection obtained from K nullifying one of its crossings) and K_{0_1} a minimal projection equivalent to $\tilde{K}_{0_1}$, in the sense that $\tilde{K}_{0_1}$ and K_{0_1} are projections of isotopic links. If K_{0_1} does not represent a trivial link, we can nullify a crossing of K_{0_1} and call $K_{0_1 0_2}$ a minimal projection of the nullification of K_{0_1}. This process can go on until $K_{0_1 \ldots 0_r}$ represent a trivial link. The sequence of minimal projections $\{K_{0_1}, K_{0_1 0_2}, \ldots, K_{0_1 \ldots 0_r}\}$ is called a *nullification sequence* for the minimal projection K; r is the length of the sequence. A nullification sequence for K is said to be *minimal* if it is of minimal length among all the nullification sequences for K.

Definition. If K is a minimal projection, the *nullification number* $o(K)$ of K is the length of a minimal nullification sequence for K.

Obviously $o(K)=0$ if and only if K represents a trivial link. If K is the projection of the figure-8-knot of the previous example, it is easy to see that $o(K)=2$. A minimal projection of the trefoil knot has nullification number equal to 2. There are minimal projections K with arbitrarily high crossing number and with $o(K)=1$; for example:



n crossings
(n even).

We shall see that the nullification number of a minimal alternating projection can be easily calculated, and also that minimal alternating projections of isotopic links have the same nullification number. To do that, we shall use some facts about alternating knots. Let us begin with the following fundamental result ([Kauffman]):

Theorem 1. An alternating projection is minimal if and only if it does not contain an isthmus.

An *isthmus* in a projection on S^2 is a crossing such that two of the four local regions of S^2 around that crossing globally belong to the same region. Hence a generic isthmus of a projection on S^2 is in the following situation:

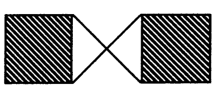


Fig. 2.

Notice that a projection containing isthmi is necessarily the connected sum of two simpler projections, that is:

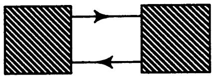


Fig. 3.

Another fact we shall use is the following (Corollary 6 in [Murasugi]):

THEOREM 2. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are alternating links, then

$$n(\mathcal{L}_1 \# \mathcal{L}_2) = n(\mathcal{L}_1) + n(\mathcal{L}_2)$$

where $n(\mathcal{L})$ denotes the crossing number of the link $\mathcal{L}$, and $\#$ denotes the connected sum.

It follows from Theorem 2 that a projection which is a connected sum of two minimal alternating projections is necessarily minimal, even if non-alternating:

COROLLARY 3. Let K be a regular projection such that $K = K_1 \# K_2$, K_1 and K_2 being minimal alternating projections. Then K is also minimal.

PROOF. We have $\mathcal{K} = \mathcal{K}_1 \# \mathcal{K}_2$ and, by Theorem 2, $n(\mathcal{K}) = n(\mathcal{K}_1) + n(\mathcal{K}_2)$. Then:

$$n(K) = n(K_1) + n(K_2) = n(\mathcal{K}_1) + n(\mathcal{K}_2) = n(\mathcal{K}),$$

hence K is a minimal projection. ■

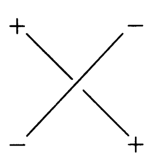
LEMMA 4. If K is an alternating projection then $\tilde{K}_0$ is also alternating.

PROOF. We use the following notation:

——— + means ———┼—

——— − means ———|–

Then an alternating projection «locally» looks like:



No matter which way we nullify a crossing in the preceding picture, we still get an alternating diagram. ■

Now we introduce a useful definition.

DEFINITION. Let $\widetilde{K}$ be a regular projection on S^2. Two crossings P and Q are said to be *facing* if they are in the following situation:

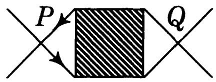


Fig. 4.

The following fact is evident, and will be repeatedly used.

LEMMA 5. Two crossings in a projection K are facing if and only if nullifying one of the two crossings produces an isthmus in the other. ■

If P and Q are facing we write $P \bowtie Q$. To be facing defines an equivalence relation in the set of the crossings of a given projection, once we assume, by convention, that a crossing is always facing to itself. A crossing which is not facing to any other crossing is said to be *isolated*.

If P is a crossing of a minimal projection K, let us suppose that the $\bowtie$-equivalence class of P contains r crossings, $r > 1$. If $\widetilde{K}_0$ is the projection obtained from K by nullifying one of these r crossings, we have $n(K_0) \leq n(K) - r$, where K_0 is minimal a projection equivalento to K. This is clear from fig. 5

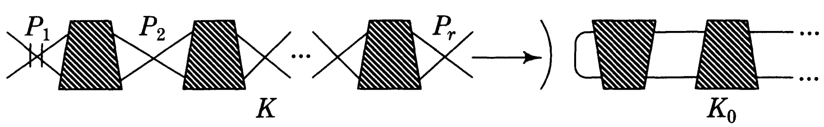


Fig. 5.

If K is a minimal alternating projection the last inequality becomes an equality.

PROPOSITION 6. Let K be a minimal alternating projection, P a crossing of K and $\widetilde{K}_0$ the projection obtained from K by nullifying P. If K_0 is a minimal projection equivalent to $\widetilde{K}_0$, then

$$n(K_0) = n(K) - r$$

where r is the cardinality of the $\bowtie$-equivalence class of P.

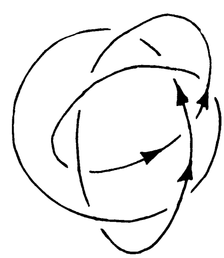
PROOF. Let us first suppose that P is an isolated crossing, i.e. $r=1$. We want to show that $\widetilde{K}_0$ is a minimal projection.

Since $\widetilde{K}_0$ is alternating (Lemma 4), by Theorem 1 it suffices to show that $\widetilde{K}_0$ does not have isthmi. Suppose that $\widetilde{K}_0$ has an isthmus Q: then, by the previous lemma, P and Q would be facing, contradicting the hypothesis that P is isolated. Hence $\widetilde{K}_0$ is minimal.

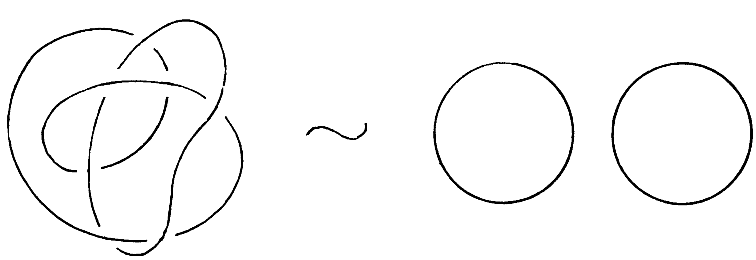
Let us now consider the case in which P is not isolated, that is $r \geqslant 1$. We want to show that K_0 is a minimal projection.

Since K_0 is alternating, (fig. 5), it suffices to show that K_0 does not have istmi. Let us suppose that K_0 has an isthmus Q: then, as we have already pointed out, P and Q would be facing in K (clearly Q can be also considered a crossing of K). But then the $\bowtie$-equivalence class of P would contain more than r elements, a contradiction. Hence K is minimal. ■

The preceeding proposition is false for non-alternating projections; a counterexample is given by projection 8_{20} in the table of Rolfsen's book [Rolfsen].

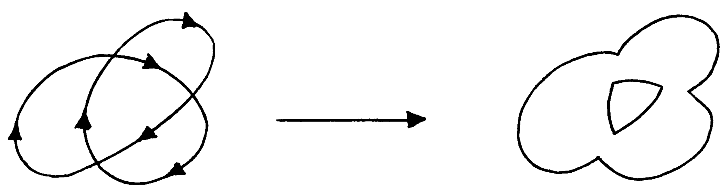
$K =$ 

If we nullify the crossing brought out by the arrows we get:

$K =$ 

Any other *nullification* of $K = 8_{20}$ (that is any projection obtained from K nullifying a crossing different from P) does not represent a trivial link. Hence $o(K) = 1$, but only one nullification sequence is minimal such a situation could not happen for minimal alternating projections: in fact we will prove (Theorem 8) that all the nullification sequences of a minimal alternating projection have the same length.

We are now ready to calculate the nullification number of a minimal alternating projection. We recall that the *Seifert circles* of a regular projection $\tilde{K}$ on S^2 are the disjoint closed curves into which the projection is transformed if we nullify all the crossings simultaneausly. Example:



We will indicate by $s(\tilde{K})$ the number of Seifert circles of the projection $\tilde{K}$. Hence, if K is the minimal projection of the trefoil of the previous picture, we have $s(K) = 2$.

The next observation will be used in the proof of Theorem 8.

LEMMA 8. If K_1 and K_2 are minimal alternating projections of the link $\mathcal{K}$, then $s(K_1) = s(K_2)$.

PROOF. Let $S_{\mathcal{K}}$ denote the Seifert surface for the link $\mathcal{K}$ built by applying Seifert's algorithm ([Rolfsen], p. 120) to the projection K. It is well-know (see for instance [Burde-Zieschang] prop. 13.26) that $S_{\mathcal{K}}$ is a spanning surface of minimal genus for $\mathcal{K}$, this genus being equal to the genus $g(\mathcal{K})$ of $\mathcal{K}$. Now (see [Rolfsen], exercise 10, p. 121) the genus of $S_{\mathcal{K}}$ is given by:

$$g(S_{\mathcal{K}}) = g(\mathcal{K}) = 1 - \frac{s(K) + c(\mathcal{K}) - n(\mathcal{K})}{2}$$

where $c(\mathcal{K}) =$ number of components of $\mathcal{K}$.

The lemma follows immediately from the above formula. ∎

THEOREM 9. If K is a minimal alternating projection, any nullification sequence of K has length $n(K) - s(K) + 1$, therefore is minimal. Hence, the nullification number of a minimal alternating projection is given by the formula:

$$o(K) = n(K) - s(K) + 1 .$$

PROOF. Induction on $n = n(K)$, the number of crossings of K. If $n = 0$ the result is true. Then, let us suppose $n(K) = n > 0$. If $\{K_{0_1}, K_{0_1 0_2}, \dots, K_{0_1 \dots 0_r}\}$ is a nullification sequence of K, let P be a crossing of K that one has nullify to transform K into K_{0_1}. We distinguish two cases.

i) P is an isolated crossing of K. Then $\tilde{K}_{0_1}$ is a minimal projection (Proposition 7) and $n(\tilde{K}_{0_1}) = n(K_{0_1}) = n(K) - 1$. Moreover $s(K) = s(K_{0_1})$, by Lemma 8. Since $o(K_{0_1}) = r - 1$ and $o(K_{0_1}) = n(K_{0_1}) - s(K_{0_1}) + 1$ by the inductive hypothesis, we have $r = n(K) - s(K) + 1$.

ii) P is not isolated. Let $\{P = P_1, P_2, \dots, P_t\}$ be the $\bowtie$-equivalence class of P. If K_0 is the minimal projection equivalent to $\tilde{K}_{0_1}$ obtained as in fig. 5, we have $n(K_0) = n(K) - t$ (Proposition 7). We see from fig. 5 that $s(K) = s(K_0) + t - 1$. Now we have $n(K_{0_1}) = n(K_0)$ and $s(K_{0_1}) = s(K_0)$, by Lemma 7. Since $o(K) = o(K_{0_1}) + 1$, from the inductive hypothesis $o(K_{0_1}) = n(K_{0_1}) - s(K_{0_1}) + 1$, so we still get the result for K. ■

Now we show that the nullification number is an invariant for alternating links.

THEOREM 10. If K and K' are minimal alternating projection of isotopic links $\mathcal{K}$ and $\mathcal{K}'$, then $o(K) = o(K')$.

PROOF. If we compare the formula for $g(\mathcal{K})$ in Lemma 8 with the formula of Theorem 9 we obtain:

$$o(K) = 2g(\mathcal{K}) + c(\mathcal{K}) - 1$$

which proves the invariance of $o(K)$. ■

By Theorem 10, the following definition makes sense:

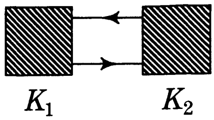
DEFINITION. If $\mathcal{K}$ is an alternating link, the nullification number of $\mathcal{K}$ is the nullification number of a minimal alternating projection of $\mathcal{K}$.

Theorem 9 remains true if K is merely supposed to be a connected sum of minimal alternating projections (then K is also minimal, in view of Corollary 3). To see this let us first show that the nullification number behawes well with respect to connected sums.

PROPOSITION 11. If K_1 and K_2 are minimal alternating projections and $K = K_1 \# K_2$, then

$$o(K) = n(K) - s(K) + 1 .$$

PROOF. The situation is the following:

$K=$ 

Let us recall that , by Corollary 3, if K' and K'' are minimal alternating projections, then $K' \# K''$ is also minimal. It readily follows that, in the hypothesis of the proposition, $o(K) = o(K_1) + o(K_2)$. Now applying Theorem 9 to K_1 and K_2 yields:

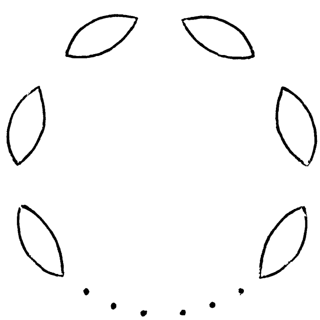
$$o(K) = o(K_1) + o(K_2) = n(K_1) - s(K_1) + 1 + n(K_2) - s(K_2) + 1 =$$
$$= n(K) - s(K) + 1 . \quad \blacksquare$$

3. Application to the flyping conjecture.

Now we want to apply the invariance of the nullification number to show that certain alternating links admit only one minimal projection on S^2.

Alternating links $\mathcal{K}$ with $o(\mathcal{K}) = 1$.

If K is a minimal alternating projection of $\mathcal{K}$ with $o(K) = 1$, from the formula $o(K) = n(K) - s(K) + 1$ we get $n(K) = s(K)$. Now, we can look at the Seifert circles as «polygons», where the vertices correspond to crossings of the projection. If a projection has no isthmi, all its Seifert polygons have at least two vertices. If $s(K) = n(K)$, every Seifert polygon must have exactly two vertices; hence, if $o(K) = 1$, the family of the Seifert polygons of K on S is the following



($n(K)$ 2-gons)

Hence the projection K must be of this form:

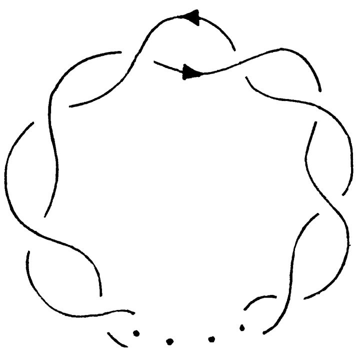
$K =$  $(n(K)$ crossings; $n(K)$ even$)$

Fig. 6.

Alternating links $\mathcal{K}$ with $o(\mathcal{K}) = n(\mathcal{K}) - 1$.

If K is a minimal alternating projection of with $o(K) = n(K) + 1$, then $s(K) = 2$, so that every Seifert polygon has $n(K)$ vertices. Thus there is only one possible minimal alternating projection, namely:

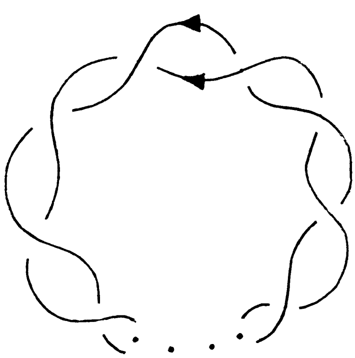
 $(n(K)$ crossings$)$

Fig. 7.

Alternating links $\mathcal{K}$ with $o(\mathcal{K}) = 2$.

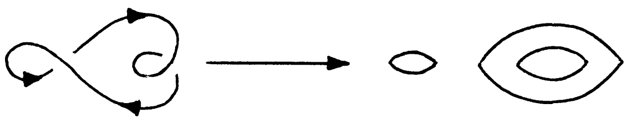
For all minimal alternating projections K of $\mathcal{K}$ we have $s(K) = n(K) + 1$. Since the total number of vertices of the Seifert polygons of K is $2 \cdot n(K)$, there are only two possibilities:

i) there are $n(K) - 3$ 2-gons and two 3-gons.

ii) there are $n(K) - 2$ 2-gons and one 4-gons.

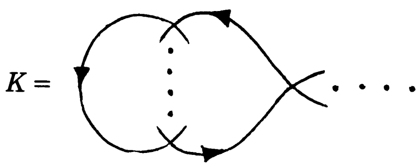
It is now worth recalling that the *Seifert circles of first type* are those which bound a simply connected region of $S^2-(K)$, where (K) is the union of the Seifert circles of K. A Seifert circle which is not of first type is said a *Seifert circle of second type.*

It is clear that if a 2-gon is of second type its vertices are isthmi:



Hence the 2-gons of K are of first type.

Case i) Let us see that both the 3-gons are necessarily of first type. In fact, if one of the 3-gons were of second type, we would have the following situation for K:



Since such a projection contains isthmi, it cannot be minimal. Hence in case i) all the Seifert polygons are of first type, and there are only two possible kinds of minimal projections for every each value of $n(K)$:

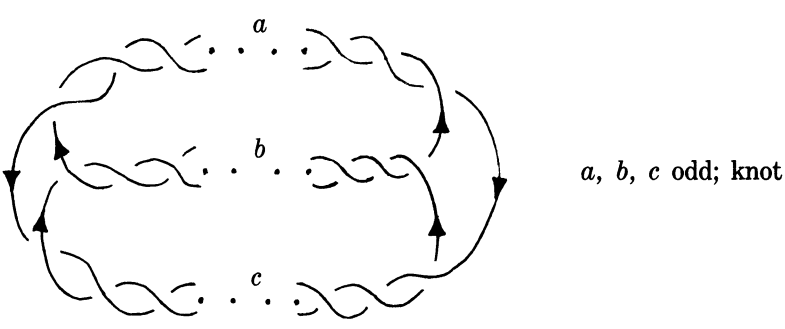
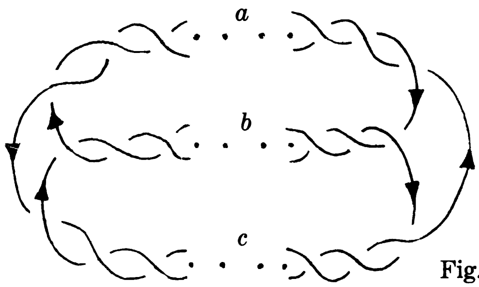


Fig. 8.



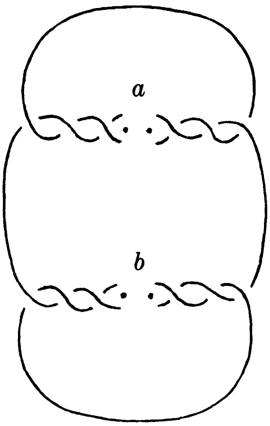
3-component link

a, b, c even and $\neq 0$

Fig. 9.

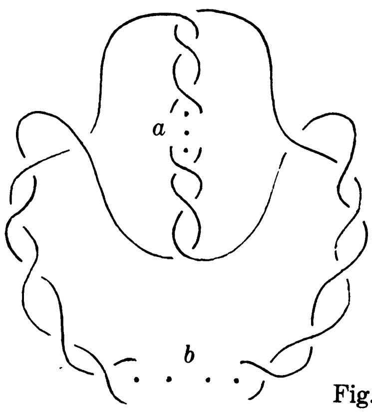
Note that the preceding projections represent pretzel links $p(a, b, c)$, where the integers a, b, c have the same sign.

Case ii) According to whether the 4-gon is of first or of second type, there are two possible projections:



a and b even and $\neq 0$;
non-prime link with 3 components

Fig. 10.



a and b even and $\neq 0$; knot.

Fig. 11.

So we have classified all the minimal alternating projections K with $o(K)=2$ and (and $n(K)\geqslant 4$). Now we want to show that a link represented by one of these projections admits only one minimal projection. First of all we observe that projections which belong to different types among the four we have listed cannot represent isotopic links: this follows easily from a calculation of crossing numbers and of the number of components, together with the fact that pretzel links are prime.

It only remains to prove that two projections belonging to the same type represent isotopic links if and only if they are the same projection (in S^2). A proof of this is given in the next section, where the various projections are distinguished by means of their Kauffman's bracket polynomial.

4. Calculations.

In this section we shall see that each of the projections in fig. 8 through 11 are uniquely determined by the set of coefficients involved in our notation. This is enough to conclude that two projections belonging to the same type of figure represent isotopic links if and only if they are the same projection on S^2. By section 3, this proves the uniqueness of the minimal projection for the alternating links with nullification number 2.

Our tool will be the bracket polynominal defined in [Kauffman]. If K is an unoriented link diagram, the bracket of K is the element $\langle K\rangle$ of $Z[A, A^{-1}]$ defined by means of the following rules:

(i) $\langle \bigcirc \rangle = 1$,

(ii) $\langle \bigcirc \cup K\rangle = (-A^{-2}+A^2)\langle K\rangle$, K non empty,

(iii) $\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$,

where, in formulas (i) and (ii), $\bigcirc$ denotes the unknot.

If each crossing of an oriented knot projection is given a value $+1$ or -1, where signs are chosen according to fig. 1, the twist number (or writhe) $w(K)$ of K is defined as the sum of the values of the crossing of K. Then the Laurent polynomial $f[K]$ defined by

$$f[K] = (-A)^{-3w(K)}\langle K\rangle$$

is an ambient isotopy invariant for oriented links (see [Kaufmann]).

Let us denote by (n) the projection of fig. 6 (or 7), where n is the number of crossings. We have, for $n > 3$,

$$\langle (n) \rangle = A^{-1} + A \langle (n-1) \rangle = A^{-1} + 1 + A + \ldots + A^{n-2} \langle (2) \rangle =$$

$$= A^{-1} + 1 + A + \ldots + A^{n-3} + A^{n-1}$$

since $\langle (2) \rangle = A + A^{-1}$.

If $p(a, b, c)$ denotes the unoriented projections of fig. 8 or 9, we have:

$$\langle p(a,b,c) \rangle = A^{-1} \langle (b+c) \rangle + A \langle p(a-1,b,c) \rangle =$$

$$= (A^{-1} + 1 + A + \ldots + A^{a-3}) \langle (b+o) \rangle + A^{a-1} \langle p(1,b,c) \rangle =$$

$$= (A^{-1} + 1 + A + \ldots + A^{a-2}) \langle (b+c) \rangle + A^{a} \langle (b) \# (c) \rangle$$

since $\langle p(a,b,c) \rangle = A^{-1} \langle (b+c) \rangle + A \langle (b) \# (c) \rangle$. Now $\langle (b) \# (c) \rangle = \langle (b) \rangle \cdot \langle (c) \rangle$; hence

$$\langle p(a,b,c) \rangle = (A^{-1} + 1 + A + \ldots + A^{a-2}) \cdot$$

$$\cdot (A^{-1} + 1 + A + \ldots + A^{b+c-3} + A^{b+c-1}) +$$

$$+ A^{a} (A^{-1} + 1 + A + \ldots + A^{b-3} + A^{b-1})(A^{-1} + 1 + A + \ldots + A^{c-3} + A^{c-1}).$$

Now observe that $f[p(a,b,c)] = f[p(a',b',c')]$ if and only if $\langle p(a,b,c) \rangle = \langle p(a',b',c') \rangle$. It is easy to see that the coefficient of A^{a-2} in the above expression for $\langle p(a,b,c) \rangle$ is $a+1$. Calculating $\langle p(a,b,c) \rangle$ in a different way we obtain that the coefficient of A^{b-2} and of A^{c-2} are $b+1$ and $c+1$ respectively. This shows that $p(a, b, c)$ depends on the (non-oriented) set $\{a, b, c\}$, once we have noted that $p(a,b,c) = p(b,c,a) \doteq p(c,a,b) = p(b,a,c)$.

Let us now observe that the projection of fig. 10 is $(a) \# (b)$. Since $\mathcal{K}_1 \# \mathcal{K}_2 = \mathcal{L}_1 \# \mathcal{L}_2$ if and only if $\mathcal{K}_1 = \mathcal{L}_1$ and $\mathcal{K}_2 = \mathcal{L}_2$ (up to reordering), we deduce the uniqueness of the minimal projection for the links represented by fig. 10.

To obtain the same result for the projections of fig. 11, let us calculate the bracket polynomial of those projections, which we denote by (a, b). Since $(a, b) = (b, a)$ it will be enough to show that $\langle (a,b) \rangle$ depends only on the set $\{a, b\}$. We have:

$$\langle (a,b) \rangle = A^{-1} \langle (b) \rangle + A \langle (a-1,b) \rangle =$$

$$= (A^{-1} + 1 + A + \ldots + A^{a-4}) \langle (b) \rangle + A^{a-2} \langle (2,b) \rangle =$$

$$= (A^{-1} + 1 + A + \ldots + A^{a-3}) \langle (b) \rangle + A^{a-1} \langle (b+1) \rangle =$$

$$= (A^{-1} + 1 + A + \ldots + A^{a-3})(A^{-2} + 1 + A + \ldots + A^{b-3} + A^{b-1}) +$$

$$+ A^{a-1}(A^{-1} + 1 + A + \ldots + A^{b-2} + A^{b}) =$$

$$= (A^{-1} + 1 + A + \ldots + A^{a-3})(A^{-1} + 1 + A + \ldots + A^{b-3}) +$$

$$+ A^{b-1}(A^{-1} + 1 + A + \ldots + A^{a-3}) + A^{a-1}(A^{-1} + 1 + A + \ldots + A^{b-3}) +$$

$$+ A^{a+b-3} + A^{a+b-1}.$$

Last expression depends on the set $\{a, b\}$ (not only on the sum $a + b$), as required.

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