Rendiconti *del* Seminario Matematico *della* Università di Padova

Stefano Panizzi

Abstract nonlinear Timoshenko beam equation

Rendiconti del Seminario Matematico della Università di Padova, tome 86 (1991), p. 193-205

<http://www.numdam.org/item?id=RSMUP_1991__86__193_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 86 (1991)

Abstract Nonlinear Timoshenko Beam Equation.

STEFANO PANIZZI (*)

SUMMARY - Let us consider the abstract nonlinear version of the Timoshenko beam equation ($\square_\gamma \stackrel{\text{def}}{=} \partial_{tt} + \gamma A$, where A is a selfadjoint positive definite linear operator):

$$\square_a \square_b u + M(\square_c u) = 0 \qquad (t > 0).$$

Under general assumptions on the nonlinearity M, we prove the existence of a global bounded weak solution for the associated Cauchy problem, provided that the initial data are conveniently small and the propagation speed c satisfies

$$0 < a < c < b.$$

The proof relies on a potential well argument for nonlinear wave equations, essentially due to Sattinger [S]. In order to appreciate the global existence result, we outline some theorem about the local existence of solutions to the Cauchy problem for the equation

$$\square_a \square_b u + G(u, u', u'') = 0 \qquad (t > 0).$$

1. Introduction and statement of the main result.

Let V and H be separable Hilbert spaces over a field $\mathcal{K}$ (which may be indifferently the field $\mathcal{R}$ of real numbers, or the field $\mathcal{C}$ of complex ones), such that $V \subseteq H$ densely and continuously. Denote by V' the (anti)dual space of V: then, thanks to the Riesz identification of H with its

(*) Indirizzo dell'A.: Dipartimento di Matematica, Via D'Azeglio 85/A, 43100 Parma, Italy.

own (anti)dual space, we have

$$V \subseteq H \subseteq V' .$$

We denote the (anti)duality between V and V' by $\langle \cdot, \cdot \rangle$, the inner product in H by $(\cdot, \cdot)$, and the norm of H by $|\cdot|$. Let A: $V \to V'$ be a hermitian positive definite isomorphism, such that

$$\langle Av, v \rangle \geq \lambda_1 \|v\|^2 \qquad (v \in V),$$

for some $\lambda_1 > 0$.

Without lose of generality, we assume that the norm of V is given by $\langle A \cdot, \cdot \rangle^{1/2}$, and we will denote it simply by $\| \|$.

Finally, assume that

(1) *M: $V \to V'$ is a weakly sequentially continuous (nonlinear) operator, which admits a real valued potential F.*

In (1), it is sufficient to intend the notion of potential in the sense of Gateaux: this amounts to say that the functional F: $V \to \mathcal{R}$ satisfies for each v and h in V

$$F(v + \rho h) = F(v) + \rho \operatorname{Re} \langle M(v), h \rangle + o(\rho) \qquad (\mathcal{R} \ni \rho \to 0).$$

To fix ideas, we assume that $F(0) = 0$.

For any real number γ, let us denote $\square_\gamma \overset{\text{def}}{=} \partial_{tt} + \gamma A$. For a, b, c real numbers, with (strict-hyperbolicity condition)

$$0 < a < b,$$

let us consider the fourth order evolution equation

$$\square_a \square_b u + M(\square_c u) = 0 \qquad (t > 0). \tag{2}$$

In the case when $M \equiv m\mathbf{1}_v$ (i.e. $F(v) = m|v|^2/2$) for some $m \in \mathcal{R}$, eq. (2) reduces to

$$\square_a \square_b u + m\square_c u = 0 \qquad (t > 0), \tag{3}$$

which is the abstract version of the Timoshenko beam equation [T]:

$$(\partial_{tt} - a\partial_{xx})(\partial_{tt} - b\partial_{xx})\, u + m(\partial_{tt} - c\partial_{xx})\, u = 0 \quad (0 < x < L,\ \ t > 0). \tag{4}$$

For $m > 0$, eq. (4) describes the small transverse vibrations of a stretched ($c \geq 0$), or compressed ($c \leq 0$) beam in a more accurate way (see [TC], [W], [AE], [CHU]) than the classical Euler-Bernoulli and Rayleigh-Love ones. For an historical survey on eq. (4), we refer the reader to [Kr]. Equation (4) has received in the last years a renewal of interest from the point of view of Control theory, see. e.g. [Ru], [KR], [Sc], [Ko], [IK], [D]. A corresponding two-dimensional model has been given for the plate [M], [U] cf. [LL].

Our main goal consists in establishing the *global existence* and *boundedness* of weak solutions to the Cauchy problem for equations of the type of eq. (2). A relevant feature is that in eq. (2) there is no dissipative mechanism (dissipative versions of eqs. (2), (3) have been studied by [B], [N]).

In the linear case, results in that direction were already given in the work of M.-G. Paoli [Pa]. It is established there that the Cauchy problem for eq. (2) has a (global) globally bounded solution provided that ($[\,]^+ =$ positive part)

$$M = m\mathbf{1}_V \text{ for some } m \geq 0,$$

and

$$-\lambda_1 ab < mc < mb + 4^{-1}\lambda_1 \{[b - a - m\lambda_1^{-1}]^+\}^2 .$$

Actually we prove that the following holds true:

THEOREM 1 (Main result). *Assume that M satisfies condition* (1), *and let us consider eq.* (2) *where*

$$0 < a < c < b. \tag{5}$$

Then the Cauchy problem for eq. (2) *admits at least one* (*global and*) *globally bounded weakly continuous solution in the phase space* $D(A^{3/2}) \times D(A) \times V \times H$, *provided that*

$$\min_{\|v\| \to 0}\lim 2F(v)\|v\|^{-2} > -abc^{-1} \tag{6}$$

and the initial data are suitably small in the norm of the phase space.

The above condition on F is clearly satisfied if the behaviour of F at the origin is superquadratic. Note that the interval of admissibility of the characteristic speed c is wider in the (non-negative) quadratic case of [Pa] than in our superquadratic one. If λ_1 is the largest constant, it is easily seen that the result in the (non-negative) quadratic case is sharp, for example by using spectral decomposition. One can ask if the result in the superquadratic one is sharp too. For the moment, we have not succeeded in answering the question. However we are able to prove the existence of at least one, global (but not necessarily bounded) solution for the case $c = a$ or $c = b$ (Theorem 5).

The idea of the proof is based on the following heuristic consideration. First we remark that $\Box_a u'$ and $\Box_b u'$ $((\cdot)' = \partial_t)$ are the natural multiplicators for eq. (2): when multiplying the equation for $\Box_b u'$ one obtains the following non-autonomous conserved quantity for regular enough solutions of eq. (2):

$$E_1 \stackrel{\text{def}}{=} a\|\Box_b u\|^2 + |\Box_b u'|^2 + 2\,\mathrm{Re}\int_0^t \langle M(\Box_c u), \Box_b u' \rangle\, ds = \text{constant}.$$

One gets an analogous *non-autonomous* conserved quantity E_2 for eq. (2) when multiplying by $\Box_a u'$. By making a suitable *convex* combination of them, with coefficients say λ and μ, one obtains *one autonomous* conserved quantity, that is

$$E_{\text{tot}} \stackrel{\text{def}}{=} a\|\mu^{1/2}\Box_b u\|^2 + b\|\lambda^{1/2}\Box_a u\|^2 + \\ + |\mu^{1/2}\Box_b u'|^2 + |\lambda^{1/2}\Box_a u'|^2 + 2F(\mu^{1/2}\Box_b u + \lambda^{1/2}\Box_a u). \tag{7}$$

This suggests changing variables to

$$x_1 \stackrel{\text{def}}{=} \mu^{1/2}\Box_b u, \qquad x_2 \stackrel{\text{def}}{=} \lambda^{1/2}\Box_a u. \tag{8}$$

In this way eq. (2) is reduced to a second order one with a conserved energy and we may handle it by an argument of *potential-well* type in the spirit of Sattinger [S], see Proposition 1 below.

Plane of the paper: in § 2 we provide some results about local existence; the proof of the main theorem is given in § 3; § 4 contains an application to PDE's.

2. Local existence.

In this section we outline some standard results about local existence of solutions to the Cauchy problem for the equation $((\cdot)' = \partial_t)$

$$\Box_a \Box_b u + G(u, u', u'') = 0 \qquad (t > 0). \tag{9}$$

We can distinguish our results in two category: in the first one there are theorems in the spirit of the classical Peano existence theorem for ODE's, while in the second one there are theorems analogous to Cauchy-Lipschitz' one.

We begin to provide a statement in which we relax the strict-hyperbolicity condition, since we allow also $a = b$. Note that the technique of the proof is not optimal (as we will see later on) in the case $a < b$, indeed we cannot assure that the solution is as regular as its initial data.

THEOREM 2. *If*

$$0 < a \leqslant b$$

and (1) G: $D(A^{3/2}) \times D(A) \times V \to V$ *is a weakly sequentially continuous (non linear) operator, then for every choice of the initial data in the space* $D(A^2) \times D(A^{3/2}) \times D(A) \times V$ *the Cauchy problem for eq.* (9) *admits a local strongly continuous (not necessarily unique) solution in the space* $D(A^{3/2}) \times D(A) \times V \times H$.

PROOF (sketch). Let us consider the Cauchy problem for the linear equation $\square_a \square_b u = 0$ $(t > 0)$. Following [Pa], we solve this problem before with respect to $v \overset{\text{def}}{=} \square_b u$, which satisfies the equation $\square_a v = 0$, then with respect to u. It is not difficult to see that one obtains a unique solution $\square_b u$ strongly continuous in the phase space $D(A) \times V$ and then a unique solution u strongly continuous in the phase space $D(A^{3/2}) \times \times D(A)$. In other words we can say that the Cauchy problem for the linear equation is well-posed in the phase space

$$Y \overset{\text{def}}{=} \{(u_0, u_1, u_2, u_3) \in D(A^{3/2}) \times D(A) \times V \times H: \quad u_2 + Au_0 \in D(A),\ u_3 + Au_1 \in V\}.$$

Thus if we pass to the first order system in the phase space Y, namely

$$\begin{aligned} &U' + \mathcal{A}U = 0 \qquad (t > 0), \\ &U(0) = U_0 \in Y, \end{aligned}$$

$\mathcal{A}$ is the generator of a C^0-semigroup with domain Y. Moreover the Cauchy problem for eq. (9) is equivalent to

$$\begin{cases} U' + \mathcal{A}U + \mathcal{G}(U) = 0 \quad (t > 0), \\ U(0) = U_0 \in Y, \end{cases} \tag{10}$$

where $\mathcal{G}$: $Y \to Y$ is a weakly sequentially continuous operator. The Cauchy problem (10) admits at least one solution as one can easily see by using the Tonelli's compactness argument for solutions of time-delay problems (2). This proves the theorem. ■

(1) For $\alpha \geqslant 0$, $D(A^\alpha)$ is the domain of the α-th power of the selfadjoint operator A: $D(A) \subseteq H \to H$. Note that, in our case, $D(A^{1/2}) = V$.

(2) For time delay problems we mean problems of the type

$$\begin{aligned} U_n' + AU_n + G(U_n(t - n^{-1})) = 0 \quad n > 1\ (t > 0), \\ U_{n|t \leqslant 0} = U_0 \in Y. \end{aligned}$$

REMARK. As a particular case of eq. (9) we may consider the equation $(\Box_b)^2 u + G(\Box_c u) = 0$. Then Theorem 2 assures the local existence for any value of c, provided that G maps V into V is a weakly sequentially continuous fashion. However, it must be noted that in the particular case when $c = b$, the argument of Theorem 2 works well too even if G: $D(A) \to V$.

In the following theorem we establish a sharper result in the strictly-hyperbolic case.

THEOREM 3. *If*

$$0 < a < b \tag{11}$$

and G: $D(A^{3/2}) \times D(A) \times V \to H$ *is a weakly sequentially continuous (non linear) operator, then the Cauchy problem for eq.* (9) *admits a local strongly continuous (not necessarily unique) solution in the phase space* $D(A^{3/2}) \times D(A) \times V \times H$, *for every choice of the initial data in the same phase space.*

PROOF. We reduce eq. (9) to an abstract semilinear wave equation by introducing a suitable pair of variables, that is

$$x_1 \overset{\text{def}}{=} \Box_b u\,, \qquad x_2 \overset{\text{def}}{=} \Box_a u\,, \tag{12}$$

Thanks to assumption (11) we have $u = (b-a)^{-1} A^{-1}(x_1 - x_2)$ and the Cauchy problem for eq. (9) is equivalent to the well-studied one for the second order equation

$$\mathfrak{x}'' + \mathcal{A}\mathfrak{x} + \mathcal{G}(\mathfrak{x}, \mathfrak{x}') = 0\,, \qquad (t > 0) \tag{13}$$

in the phase space $\mathcal{V} \times \mathcal{H}$, where

$$\mathcal{V} \overset{\text{def}}{=} V \times V\,, \qquad \mathcal{H} \overset{\text{def}}{=} H \times H\,, \tag{14}$$

$$\mathcal{A} \overset{\text{def}}{=} \begin{bmatrix} aA & 0 \\ 0 & bA \end{bmatrix}, \tag{15}$$

and $\mathcal{G}$: $\mathcal{V} \to \mathcal{H}$ is a weakly sequentially continuous operator. For the local existence of a strongly continuous solutions to the Cauchy problem for equation (13) we refer to [L] [F]. ■

On the other hand, in the spirit of the Cauchy-Lipschitz existence and uniqueness theorem for ODE's we have the following

THEOREM 4. *If* $0 < a < b$, *and* $G\colon D(A^{3/2}) \times D(A) \times V \to H$ *satisfies the following assumption* (*G Lipschitz continuous on bounded sets*)

$$|G(u_1, v_1, h_1) - G(u_0, v_0, h_0)| \leqslant$$

$$\leqslant L(R)\{|A^{3/2}(u_1 - u_0)| + |A(v_1 - v_0)| + \|h_1 - h_0\|\}$$

for every $(u_i, v_i, h_i) \in D(A^{3/2}) \times D(A) \times V$ *such that* $|A^{3/2}u_i| + |Av_i| + \|h_i\| \leqslant R$ $(i = 0, 1)$, *then the Cauchy problem for eq.* (9) *admits a unique local strongly continuous solution in the phase space* $D(A^{3/2}) \times D(A) \times V \times H$, *for every choice of the initial data in the same phase space.*

3. Proof of the main theorem.

ORIENTATION. Thanks to assumption (5), there exists two positive constants λ, μ such that

$$\lambda a + \mu b = c, \qquad \lambda + \mu = 1.$$

The line of the proof consists in passing to the new variable $\mathfrak{x} \stackrel{\text{def}}{=} (x_1, x_2)$ given by (8) and in exhibiting a conserved quantity in terms of these variable. This is done by considering the equivalent evolution equation satisfied by $\mathfrak{x}$, which is a second order equation in the phase space $\mathcal{V} \times \mathcal{H}$ ($\mathcal{V}$ and $\mathcal{H}$ are defined by (14))

$$\mathfrak{x}'' + \mathcal{A}\mathfrak{x} + \mathcal{M}(\mathfrak{x}) = 0 \qquad (t > 0).$$

Here $\mathcal{A}$ is the linear operator defined in (15) and $\mathcal{M}$ is given by

$$\mathcal{M} \stackrel{\text{def}}{=} \begin{bmatrix} \mu^{1/2} M(\mu^{1/2} x_1 + \lambda^{1/2} x_2) \\ \lambda^{1/2} M(\mu^{1/2} x_1 + \lambda^{1/2} x_2) \end{bmatrix}.$$

A potential for $\mathcal{M}$ is given by the functional

$$\mathcal{F}(x) \stackrel{\text{def}}{=} F(\mu^{1/2} x_1 + \lambda^{1/2} x_2).$$

Therefore one has immediately that (for regular solutions) the following quantity is a conserved one

$$(16)\quad \mathcal{E}(\mathfrak{x}, t) \stackrel{\text{def}}{=} |\mathfrak{x}'(t)|^2 + \langle \mathcal{A}\mathfrak{x}(t), \mathfrak{x}(t)\rangle + 2\mathcal{F}(\mathfrak{x}(t)) = \text{constant} \quad (t \geqslant 0).$$

Then we conclude by using a potential-well argument in the spirit of [S]; more exactly we need the following Proposition, which follows from a general result of [A].

PROPOSITION 1 (cf. Th. 1.2 of [A]). *Let $\mathcal{V}$ and $\mathcal{H}$ be separable Hilbert spaces, $\mathcal{V} \subseteq \mathcal{H}$ densely and continuously and $\mathcal{A}: \mathcal{V} \to \mathcal{V}'$ be a symmetric positive definite isomorphism. Let $\mathcal{M}: \mathcal{V} \to \mathcal{V}'$ be a (nonlinear) weakly sequentially contiuous operator, and consider the second order evolution equation*

$$\mathfrak{x}'' + \mathcal{A}\mathfrak{x} + \mathcal{M}(\mathfrak{x}) = 0 \qquad (t > 0)\,. \tag{17}$$

Assume that there exists a continuous functional

$$\mathcal{F}: \mathcal{V} \to \mathcal{R}\,, \qquad \mathcal{F}(0) = 0\,,$$

such that

i) $\nabla\mathcal{F} = \mathcal{M}$ *(in the Gateaux sense)*

ii) $\min\lim\limits_{\|\mathfrak{x}\| \to 0} \{\langle \mathcal{A}\mathfrak{x}, \mathfrak{x}\rangle + 2\mathcal{F}(\mathfrak{x})\} / (\|\mathfrak{x}\|_{\mathcal{V}})^2 > 0.$

Then the Cauchy problem for eq. (17) *admits at least one (global and) globally bounded weakly continuous solution in the phase space $\mathcal{V} \times \mathcal{H}$, provided that the initial data are suitably small in the norm of the phase space.*

Let us check the hypotheses of the Proposition 1 for the choice

$$\|\mathfrak{x}\|_{\mathcal{V}}^2 \stackrel{\text{def}}{=} a\|x_1\|^2 + b\|x_2\|^2 (= \langle \mathcal{A}\mathfrak{x}, \mathfrak{x}\rangle)\,,$$

$$\mathcal{F}(\mathfrak{x}) \stackrel{\text{def}}{=} F(\mu^{1/2} x_1 + \lambda^{1/2} x_2)\,.$$

The proof of point i) is trivial. Now we prove point ii): from assumption (6) we have that for any β such that

$$-\min\lim_{\|v\| \to 0} 2F(v)\|v\|^{-2} < \beta < abc^{-1}$$

for small enough $\mathfrak{x}$ one has

$$\begin{aligned}
\|\mathfrak{x}\|^2 + 2\mathcal{F}(\mathfrak{x}) = \|\mathfrak{x}\|^2 + 2F(\mu^{1/2} x_1 + \lambda^{1/2} x_2) \geqslant \\
\geqslant \|\mathfrak{x}\|^2 - [\beta]^+ \|\mu^{1/2} x_1 + \lambda^{1/2} x_2\|^2 \geqslant \\
\geqslant \|\mathfrak{x}\|^2 - [\beta]^+ (\mu a^{-1} + \lambda b^{-1})(a\|x_1\|^2 + b\|x_2\|^2) = \\
= \|\mathfrak{x}\|^2 - [\beta]^+ c a^{-1} b^{-1} (a\|x_1\|^2 + b\|x_2\|^2) \geqslant \varepsilon\|\mathfrak{x}\|^2
\end{aligned}$$

where $\varepsilon = 1 - [\beta]^+ / abc^{-1} > 0$. Therefore also point ii) is proved. ∎

REMARK. If in Theorem 1 (resp.: Proposition 1) some additional assumptions are made, then the solution enjoys some additional properties.

i) Let us assume that the operator $\mathcal{M}$ in the statement of Proposition 1 is a *weakly sequentially continuous operator from* $\mathcal{V}$ *to* $\mathcal{H}$. Then, by using a boot-strap argument, the solution provided by the thesis of the same theorem is *strongly continuous.*

ii) Actually one can obtain a sharper condition. Indeed if in Proposition 1 it is assumed that

$\mathcal{F}$ *is weakly sequentially lower semicontinuous on* $\mathcal{V}$,

then the solution $\mathfrak{x}$ of eq. (17) provided by Proposition 1 satisfies

$$\mathcal{E}(\mathfrak{x}, t) \leqslant \mathcal{E}(\mathfrak{x}, 0) \qquad (t \geqslant 0)\,, \tag{18}$$

$(\mathfrak{x}, \mathfrak{x}')$ *is strongly continuous in the phase space* $\mathcal{V} \times \mathcal{H}$ *at* $t = 0^+$.

iii) If uniqueness holds for the Cauchy problem for eq. (17), then one can easily see, by reversing time in eq. (17) and by using inequality (18), that

$$\mathcal{E}(\mathfrak{x}, t) = \text{constant} \qquad (t \geqslant 0)\,,$$

$(\mathfrak{x}, \mathfrak{x}')$ *is strongly continuous in the phase space at each point* $t \geqslant 0$.

iv) As a consequence, if in Theorem 1 it is assumed that

F *is weakly sequentially lower semicontinuous on* V,

then for the solution u of eq. (2) provided by Theorem 1 one has (E_{tot} is defined by (7))

$$E_{\mathrm{tot}}(t) \leqslant E_{\mathrm{tot}}(0) \qquad on\ [0, +\infty[\,,$$

(u, u', u'', u''') *is strongly continuous in the phase space* $D(A^{3/2}) \times \times D(A) \times V \times H$ *at* $t = 0^+$

v) In the case when the Cauchy problem for eq. (2) is uniquely solvable for any choice of the initial data, then

$$E_{\text{tot}} = \text{constant} \qquad on\ [0, +\infty[\,,$$

(u, u', u'', u''') *is strongly continuous in the phase space at each point* $t > 0$.

As we mentioned in § 1, we are able to slightly extend the interval of admissibility of the propagation velocity c, but the nonlinear operator now is requested to map V into H, and nevertheless the solution is not necessarily bounded. Indeed we have the following

THEOREM 5. *Let us assume that* $M\colon V \to H$ *is a weakly sequentially continuous (nonlinear) operator, which admits a real valued potential* F *satisfying condition* ii) *of Proposition* 1. *Then the Cauchy problem for the equation* (2), *where*

$$0 < a < b\,, \qquad c = a \ \ or\ \ c = b\,,$$

admits at least one global (not necessarily bounded) strongly continuous solution in the phase space $D(A^{3/2}) \times D(A) \times V \times H$ *provided that the initial data are suitable small in the same phase space.*

PROOF. Let us assume the $c = b$ (the proof for $c = a$ is analogous). As in the proof of Theorem 3, we pass to the new variables x_1, x_2 defined in (12), and we consider the following second order system

$$\Box_a x_1 + M(x_1) = 0 \qquad (t > 0)\,,$$

$$\Box_b x_2 + M(x_1) = 0 \qquad (t > 0)\,.$$

The Cauchy problem for the first equation admits (for small initial data) at least one strongly continuous solution x_1 in the phase space $V \times H$ (see Remark to Proposition 1, point ii)). Then if we substitute such a solution in the second equation and define $f \overset{\text{def}}{=} M(x_1(\cdot))$, we have $f \in L^1_{\text{loc}}([0, +\infty[; H)$. So we may conclude, by standard linear theory for wave equation, that there exists a unique (in general unbounded) strongly continuous solution x_2 in the phase space $V \times H$. This proves the Theorem. ■

4. An application.

We outline an application of Theorem 1 to a mixed problem for a PDE of hyperbolic type.

Let Ω be a *bounded* regular (C^3) open set of $\mathcal{R}^n$ ($n \geqslant 3$), and consider the Cauchy-Dirichlet problem

$$\text{(19)} \quad \begin{cases} (\partial_{tt} - a\Delta_x)(\partial_{tt} - b\Delta_x)\, u + g|(\partial_{tt} - c\Delta_x)\, u|^{p-1}(\partial_{tt} - c\Delta_x)\, u = 0\,, \\ u(\cdot, t)|_{\partial\Omega} = \Delta_x u(\cdot, t)|_{\partial\Omega} = 0 \qquad (t \geqslant 0) \end{cases}$$

with suitable initial conditions.

Let λ_1 denote the first eigenvalue of minus laplacian ($-\Delta_x$) with respect to the Dirichlet boundary value problem, and assume that the real numbers a, b, c, g, p are subjected to the following limitations:

$$0 < a < c < b\,,$$

$$1 < p \leqslant (n+2)(n-2)^{-1} \qquad \textit{if } n \geqslant 3 \ (1 < p \textit{ if } n \leqslant 2)$$

(*no limitation on* g *is imposed*).

By an application of Theorem 1, the Cauchy-Dirichlet problem (19) results to admit a (global and) globally bounded weakly continuous solution in the phase space

$$(H^3 \cap H_0^1 \cap \{\Delta_x u \in H_0^1\}) \times (H^2 \cap H_0^1) \times H_0^1 \times L^2\,,$$

for initial data which are suitably small in the phase space.

If we exclude the critical case $p = (n+2)(n-2)^{-1}$ for $n \geqslant 3$ (or if we assume that $g \geqslant 0$), then the functional $\mathcal{E}$ defined by (16) satisfies $\mathcal{E}(t) \leqslant \mathcal{E}(0)$, and the solution is strongly continuous in the phase space at $t = 0^+$.

If the exponent p is subjected to the stronger restriction

$$1 < p \leqslant n(n-2)^{-1}\,,$$

then the solution results to be unique: therefore one has that $\mathcal{E} =$ constant, and that the solution is strongly continuous in the phase space at each $t \geqslant 0$.

Acknowledgements. I wish to thank A. Arosio for his fundamental support during this work, R. Natalini and S. Spagnolo for the useful comments.

The research has been supported by a contract with C.N.R. The author is member of the G.N.A.F.A. (C.N.R.).

REFERENCES

[AE] R. APRAHAMIAN - D. A. EVENSEN, J. Appl. Mech.-Trans. ASME (June 1970), pp. 287-291.

[A] A. AROSIO, *Global weak solutions for abstract nonlinear evolution equations of hyperbolic type*, Quad. Dip. Mat. Univ. Parma, **50** (1990).

[APP] A. AROSIO - S. PANIZZI - M. G. PAOLI, *Inhomogeneous Timoshenko beam equations*, short communication at the 5° Symposium on «Control of distributed parameters systems» (Perpignan, June '89); *Temporally inhomogeneous Timoshenko beam equations*, Ann. Mat. Pura Appl. (to appear).

[B] J. BARTAK, *Stability and correctness of abstract differential equations in Hilbert spaces*, Czechoslovak Math. J., **28** (103), No. 4 (1978), pp. 548-593.

[CHU] CREMER - HEEKL - UNGAR, *Structure borne sound*, Springer, Berlin.

[D] Y. DENG, *Exponential uniform stabilization of cantilevered Timoshenko beam with a tip body via boundary feedback control*, 5° Symposium on Control of distributed parameter systems, Perpignan (1989).

[F] A. FLAMMIA, *Il problema di Cauchy per l'equazione* $\Box u = f(u)$ *da un punto di vista astratto, Teorema 1.1*, Tesi di laurea, Univ. Pisa, a.a. 1983/84.

[Fl] W. FLÜGGE, *Die Ausbreitung von biegenvellen in Stäben*, Z. Angew. Math. Mech., 22 (1942), pp. 312-318.

[IK] K. ITO - N. KUNIMATSU, *Boundary control of the Timoshenko beam with viscous internal dampings*, 5° Symposium on Control of distributed parameter systems, Perpignan (1989).

[KR] J. U. KIM - Y. RENARDY, *Boundary control of the Timoshenko beam*, SIAM J. Control Optim., 25, No. 6 (1987), pp. 1417-1429.

[Ko] V. KOMORNIK, *Controlabilité exacte en temps minimal de quelques modèles de placques*, C.R. Acad. Paris, Sér. I Math., **307** (1988), pp. 471-474.

[Kr] M. KRANYS, *Casual theory of evolution and wave propagation in mathematical physics*, ASME J. Appl. Mech., **42**, No. 11 (1989), pp. 305-321.

[LL] J. LAGNESE - J. L. LIONS, *Modelling, analysis and control of thin plates*, Masson, Paris (1988).

[L] J. L. LIONS, *Quelques méthodes de rèsolution des problèmes aux limites on linéaires*, ch. 1, Th. 1.1, Dunod & Gauthier Villars, Paris (1969).

[M] R. D. MINDLIN, *Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates*, J. Appl. Mech., **18** (1951), pp. 31-38.

[N] J. NEUSTUPA, *A contribution to the theory of stability of differential equations in Banach space*, Cwechoslovak Math. J., **29** (104), No. 1 (1979), pp. 27-52.

[Pa] M. G. PAOLI, *Il problema di Cauchy per l'equazione* $(\Box_a \Box_b + a\Box_c)\,u = 0$ *da un punto di vista astratto*, Tesi di Laurea, Univ. Pisa, a.a. 1987/88.

[Ru] D. L. RUSSELL, *Mathematical models for the elastic beam and their con-*

trol-theoretic implications, *Semigroups, theory and applications*, Vol. II, H. Brezis, M. G. Crandall & F. Kappel eds., Longman, Harlow (1986), pp. 177-216.

[S] D. H. SATTINGER, *On global solutions of nonlinear hyperbolic equations*, Arch. Rat. Mech. Anal., **30** (1968), pp. 148-172.

[Sc] J. SCHMIDT, *Entwurf von reglern zur aktiven schwingungdampfung an flexciblen mechanischen strukturen*, Dissertation von Diplom-Ingenieur, Darmstadt (1987).

[T] S. TIMOSHENKO, *On the correction for shear of the differential equation for transverse vibrations of prismatic bars*, Philosophical Magazine, **41** (1921), pp. 744-746; *Vibration problems in engineering*, 3ª ed., Van Nostrand, Princeton (1955).

[TC] T. C. TRAILL-NASH - A. R. COLLAR, *The effects of shear flexibility and rotatory inertia on the binding vibrations of beams*, Quart. J. Mech. Appl. Math., **6** (1953), pp. 186-222.

[U] YA. S. UFLYAND, *The propagation of waves in the transverse vibrations of bar and plates* (in russo), Akad. Nauk SSSR Prikl. Mat. i Meh., **12** (1948), pp. 287-300.

[W] K. WASHIZU, *Variational methods in elasticity and plasticity*, Pergamon, London (1968).

Manoscritto pervenuto in redazione il 6 novembre 1990.