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Extensions of Compact Continuous Maps into Decomposable Sets.

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SUMMARY - From the known fact that a compact map from a closed subset of a metric space, with values in a decomposable set can be extended with values in the same set, here is presented a construction that ensures also the compactness of the image of the extension.

1. Introduction.

Any continuous function which maps a subset A of a metric X into a totally bounded set of a normed space E can be extended to the whole space X keeping the values in a totally bounded set [4]. In fact the range of the extension, the convex hull of a totally bounded subset of a normed space, is totally bounded.

Purpose of this note is to present a similar result for maps into $L^1(T, E)$ that uses the concept of decomposable hull instead of that of a convex hull. It is well known that decomposable sets are absolute retracts [1]; however knowing that a totally bounded map with values in a decomposable set can be extended with values in this set, does not imply by itself that the extension will have values in a totally bounded set. In fact the decomposable hull of a set cannot be totally bounded unless it is a singleton [3].

The range of the extension proposed here is a totally bounded subset of the decomposable hull of the original image; apparently it cannot be characterized in simple terms like convexity.

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A special case for $X = L_1(I, K)$, where $I \subset \mathfrak{R}$ is an interval and K a closed subset of $\mathfrak{R}^n$, has been presented in [2].

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2. Notations and definitions.

Throughout this paper, $(T, \mathcal{F}, \mu)$ denotes a measure space with a σ-algebra $\mathcal{F}$ of subsets of T and a positive measure μ. Given a μ-integrable function $f: T \to \mathfrak{R}$, $f \cdot \mu$ denote the measure having density f with respect to μ. When E is a Banach space with norm $\|\cdot\|_E$, M denotes the vector space of those functions $u: T \to E$, which are measurable with respect to $\mathcal{F}$ and to the Borel subsets of E, while $L_1(T, E)$ is the Banach space of those functions $u \in M$ such that $\|u\|_E \in L_1(T, \mathfrak{R})$, with norm $\|u\|_1 = \int_T \|u\|_E \, d\mu$ (See [8], p. 132).

The open unit ball of $L_1(T, E)$ is denoted by B_1. For every $x, y \in L_1(T, E)$, set $d(x, y) = \|x - y\|$ and $d(x, A) = \inf_{a \in A} \|x - a\|$, where $A \subset L_1(T, E)$.

Let $\nu: \mathcal{F} \to \mathfrak{R}^n$ be a vector measure, whose components have no atoms. A family $(A_\alpha)_{\alpha \in [0,1]}$, $A_\alpha \in \mathcal{F}$, is called increasing if $A_\alpha \subset A_\beta$ when $\alpha \leq \beta$. An increasing family is called refining $A \in \mathcal{F}$ with respect to the measure ν if $A_0 = \emptyset$, $A_1 = A$ and $\nu(A_\alpha) = \alpha\nu(A)$ for every $\alpha \in [0, 1]$. Let ν be a vector measure absolutely continuous with respect to μ; then if μ is nonatomic there exists a family $(A_\alpha)_{\alpha \in [0,1]}$ refining T with respect to (ν, μ) (see [5]). From this point on, we assume μ nonatomic. For the following concept one can refer to [7].

DEFINITION 1. *A set $K \subset M$ is decomposable if*

$$u \cdot \mathcal{X}_A + v \cdot \mathcal{X}_{T \setminus A} \in K \quad \textit{whenever } u, v \in K, \; A \in \mathcal{F}.$$

The collection of all nonempty closed decomposable subsets of a subspace L of M is denoted by $D(L)$. For any set $H \subset L$, the decomposable hull of H in L is

$$\mathrm{dec}_L(H) = \bigcap \{K \in D(L): H \subset K\}.$$

Clearly, $\mathrm{dec}_L(H)$ represents the smallest decomposable subset of L which contains H.

It will be useful the following,

PROPOSITION 2. *If $f_1, f_2, \ldots, f_n \in L_1(T, E)$, and $(A_\alpha)_{\alpha \in [0,1]}$ is an increasing family refining the measure $f_1 \cdot \mu, f_2 \cdot \mu, \ldots, f_n \cdot \mu$, then the* *set*

$$S = \left\{ \sum_{j=1}^{n} f_j \mathcal{X}_{B(\lambda, j)} \right\},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ *is such that* $\lambda_h \geq 0$, $\sum_{h=1}^{n} \lambda_h = 1$, *and* $B(\lambda, j) = A_{l(j)} \setminus A_{l(j-1)}$ *with* $l(j) = \sum_{h=1}^{j} \lambda_h$ $(l(0) = 0)$, *is compact.*

3. Main result.

THEOREM 3. *Let* $A \subset L_1(T, E)$ *be totally bounded and* $i: A \to A$ *be the identity. Then there exists a totally bounded set* B, *with* $A \subset B \subset \operatorname{dec}(A)$, *and a continuous function* $\hat{i}: L_1(T, E) \to B$ *such that* $\hat{i}|_A = i$.

PROOF: Set $X = L_1(T, E)$. It is not restrictive to assume A closed. The proof is divided into several steps.

a) Let $(A_n)_{n \geq 1}$ be the open sets defined by

$$A_1 = \{x \in X: d(x, A) > 1\}$$

$$A_2 = \left\{x \in X: \frac{1}{2} < d(x, A) < \frac{3}{2}\right\}$$

$$\ldots\ldots$$

$$A_n = \left\{x \in X: \frac{1}{2^{n-1}} < d(x, A) < \frac{3}{2^{n-1}}\right\}$$

$$\ldots\ldots$$

We have: $X \setminus A = \bigcup_{n \geq 1} A_n$.

Set $\varepsilon_n = 1/2^n$, $n \geq 1$, and let $N_n = \{a_0^n, \ldots, a_{j_n}^n\}$ be an ε_n-net of A. Let $\pi: X \to A$ be a function such that $d(x, \pi x) = d(x, A)$ (π is any selection of the projection of minimal distance). Put

$$\mathcal{U}_j^n = A_n \cap (\pi^{-1}(a_j^n + \varepsilon_n B_1) + \varepsilon_n B_1).$$

Consider the pairs (n, j); $n \geq 1$, $j = 0, \ldots, j_n$, in the lexicographic order; the pair (n, j) is identified with a natural h by the relation $h = \sum_{l=1}^{n-1} (j_l + 1) + j + 1$. If h corresponds to the pair (n, j), a^h e $\mathcal{U}^h$ will denote respectively a_j^n and $\mathcal{U}_j^n$.

Let $\{q^h(x)\}$ be a continuous partition of unity subordinate to $\{\mathcal{U}^h\}$. Set $\theta^0(x) = 0$ and define $\theta^h(x) = \sum_{l=1}^{h} q^h(x)$.

Denote by $s_k^1(t); k = 0, \ldots, (j_1 + j_2)$ the elements in the set $N_1 \cup N_2 \cup \{0\}$.

Let $(E_\lambda)_{\lambda \in [0,1]}$ be an increasing family refining T with respect to the measures generated by the densities $g_{l,m}^1(t) = \|s_l^1(t) - s_m^1(t)\|_E$, where $l, m = 0, \ldots, (j_1 + j_2)$.

Define a continuous function i_1 on A_1 by setting

$$i_1(x) = \sum_h a^h \mathcal{X}_{E_{\theta^h(x)} \setminus E_{\theta^{h-1}(x)}}.$$

b) Let $R_1 = \{b_m^1\}_{m=0,\ldots,m_1}$ be an ε_3-net of the totally bounded set $i_1(A_1)$. It is easy to verify that there exists a finite decomposition of $T, E^\beta; \beta = 0, \ldots, \beta_1$, i.e. $E^\alpha \cap E^\beta \neq \emptyset$ if $\alpha \neq \beta$ and $T = \bigcup_{\beta=0}^{\beta_1} E^\beta$, such that b_m^1 coincides on E^β with an element $y_{m,\beta}^1$ of the set $N_1 \cup N_2$.

Let f_1 be a function mapping each x belonging to $i_1(A_1)$ into an element of R_1, whose distance from x is less than ε_3. Define the open sets

$$\mathcal{V}_{j,m}^1 = \mathcal{U}_j^1 \cap (f_1^{-1}(b_m^1) + \varepsilon_4 B_1), \qquad j = 0, \ldots, j_1; m = 0, \ldots, m_1$$

$$\mathcal{V}_{j,0}^n = \mathcal{U}_j^n, \qquad j = 0, \ldots, j_n; n \geq 2.$$

Consider the triples $(n, j, m); n \geq 1, j = 0, \ldots, j_n, m = 0, \ldots, m_n$ (set $m_n = 0$ if $n \neq 1$), in the lexicografic order; the triple (n, j, m) is identified with a natural h by the relation $h = \sum_{l=1}^{n-1} (j_l + 1)(m_l + 1) + (j + 1) \cdot (m + 1)$. Denote with h_n the index corresponding to the triple (n, j_n, m_n). If h corresponds with the triple (n, j, m); $y_\beta^h, a^h, \mathcal{V}^h$ will denote respectiveley $y_{m,\beta}^n$, a_j^n and $\mathcal{V}_{j,m}^n$.

Let $\{q^h(x)\}$ be a continuos partition of unity subordinate to $\{\mathcal{V}^h\}$. Set $\gamma^0(x) = 0$ and define $\gamma^h(x) = \sum_{l=1}^{h} q^l(x)$.

Denote by $s_k^2; k = 0, \ldots, k_2$ the elements in the set $N_1 \cup N_2 \cup N_3 \cup \{0\}$.

Let $(E_\lambda^\beta)_{\lambda \in [0,1]}$ be an increasing family refining E^β with respect to the measures generated by the densities $g_{l,m}^2(t) = \|s_l^2(t) - s_m^2(t)\|_E$;

$l, m = 0, \dots, k_2$. Define a continuous function i_2 on $A_1 \cup A_2$ by setting

$$i_2(x) = \sum_\beta \left(\sum_{h=1}^{h_1} y_\beta^h \chi_{E_\gamma^{\beta_h(x)} \setminus E_\gamma^{\beta_{h-1}(x)}} + \sum_{h=h_1+1}^{+\infty} a^h \chi_{E_\gamma^{\beta_h(x)} \setminus E_\gamma^{\beta_{h-1}(x)}} \right).$$

Further, from the definition of $\{b_m^1\}$, for every $x \in A_1 \setminus \overline{A}_2$, we have

$$\|i_1(x) - i_2(x)\|_1 \leqslant \|i_1(x) - b_{\overline{m}}^1\| + \|b_{\overline{m}}^1 - i_2(x)\| \leqslant 4\varepsilon_3 = \varepsilon_1 .$$

c) Let us proceed by induction. Suppose that we have defined continuous functions i_j on $\bigcup_{l \leqslant j} A_l$ such that

$$(@) \qquad \|i_{j-1}(x) - i_j(x)\|_1 \leqslant \varepsilon_{j-1} \qquad \text{on } (\bigcup_{l \leqslant j-1} A_l) \setminus \overline{A}_j \qquad \text{for } j = 2, \dots, n.$$

Then there exists i_{n+1} such that (@) holds for $j = n+1$. In fact, let $R_n = \{b_m^n\}_{m=0,\dots,m_n}$ be an ε_{n+2}-net of the totally bounded set $i_n(\bigcup_{m \leqslant n} A_m)$. Then, there exists a finite decomposition of T, $(E^\beta)_{\beta=0,\dots,\beta_n}$, such that b_m^n coincides on each E^β with an element $y_{m,\beta}^n$ of the set $\bigcup_{j=1}^{n+1} N_j$.

Let f_n be a function that maps each x belonging to $i_n(\bigcup_{l \leqslant n} A_l)$ into an element of R_n, whose distance from x is less than ε_{n+2}. Then, define the open sets

$$\mathcal{V}_{h,m}^k = \mathcal{U}_h^k \cap (f_n^{-1}(b_m^n) + \varepsilon_{n+3} B_1); h = 0, \dots, j_k; m = 0, \dots, m_n; k = 0, \dots, n-1,$$

$$\mathcal{V}_{h,0}^k = \mathcal{U}_h^k, \qquad h = 0, \dots, j_k; k \geqslant n.$$

Let $\{q^h(x)\}$ be a continuous partition of unity subordinate to $\{\mathcal{V}^h\}$. Denote by $s_k^n; k = 0, \dots, k_n$ the elements in the set $\bigcup_{j=1}^{n+2} N_j \cup \{0\}$. Let $(E_\lambda^\beta)_{\lambda \in [0,1]}$ be an increasing family refining E^β with respect to the measures generated by the densities, $g_{l,m}^n(t) = \|s_l^n(t) - s_m^n(t)\|_E$; $l, m = 0, \dots, k_n$.

Then, define a continuous function i_{n+1} on $\bigcup_{j=1}^{n+1} A_j$, by setting

$$i_{n+1}(x) = \sum_\beta \left(\sum_{h=1}^{h_n} y_\beta^h \chi_{E_\gamma^{\beta_h(x)} \setminus E_\gamma^{\beta_{h-1}(x)}} + \sum_{h=h_n+1}^{+\infty} a^h \chi_{E_\gamma^{\beta_h(x)} \setminus E_\gamma^{\beta_{h-1}(x)}} \right).$$

Further, from the definition of $\{b_m^n\}$, for every $x \in \bigcup_{j=1}^{n} A_j \setminus \overline{A_{n+1}}$, we have

$$\|i_n(x) - i_{n+1}(x)\|_1 \leq \|i_n(x) - b_m^n\|_1 + \|b_m^n - i_{n+1}(x)\|_1 \leq 4\varepsilon_{n+2} = \varepsilon_n.$$

d) Define a function $\hat{i}: X \to X$ by setting, for every $x \in A_n$,

$$\hat{i}(x) = \lim_{m \geq n} i_m(x)$$

and $\hat{i}(x) = i(x)$ for every $x \in A$. Since the image of each i_m is contained in $\operatorname{dec} A$, then also $\hat{i}(X) \subset \operatorname{dec} A$.

From the relation

$$\|i_p(x) - i_q(x)\|_1 \leq \sum_{j=p}^{q+1} \varepsilon_j, \qquad p < q, \qquad x \in \bigcup_{h=1}^{p} A_h \setminus \overline{A}_{p+1}$$

it is easy to verify that $\hat{i}$ is continuous on $X \setminus A$. Let us check the continuity on A. Fix $\varepsilon > 0$ and $a \in A$; there exists a $\delta > 0$, $\delta < \varepsilon$, such that if $b \in A$ with $\|a - b\|_1 < \delta$ then $\|\hat{i}(a) - \hat{i}(b)\|_1 < \varepsilon/4$. Now, if $x \in X \setminus A$ and $\|x - a\|_1 < \delta/4$, then x belongs to some $\mathcal{U}_{j_0}^n$, with n sufficently large. Indeed, $d(\pi x, a_{j_0}^n) < \varepsilon_n + \varepsilon_n = \varepsilon_{n-1}$.

Therefore, if $q_j^n(x) \neq 0$, $d(a, a_j^n) < d(a, x) + d(x, \pi x) + d(\pi x, a_j^n) < 3d(a, x) < \delta$, and so $\|\hat{i}(a) - \hat{i}(a_i^n)\|_1 < \varepsilon/4$; then $\|i_n(x) - i(a_{j_0}^n)\|_1 \leq \sup_{\{j: q_j^n(x) \neq 0\}} \|i(a_j^n) - i(a_{j_0}^n)\|_1 \leq \varepsilon/2$, and so $d(i_n(x), \hat{i}(a)) \leq d(i_n(x), i(a_{j_0}^n)) + d(i(a_{j_0}^n), \hat{i}(a)) < \varepsilon/2 + \varepsilon/4$.

Because of the relation

$$(*) \qquad \|\hat{i}(x) - i_n(x)\|_1 < \sum_{j=n}^{\infty} \varepsilon_j \leq \varepsilon_n < \frac{\delta}{4} < \frac{\varepsilon}{4},$$

$$\text{for every } x \in \bigcup_{h=1}^{n} A_h \setminus \overline{A}_{n+1},$$

we have, $\|\hat{i}(x) - i(a)\|_1 < \varepsilon$, for every $x \in X$ with $d(x, a) < \delta/4$.

It is left to show that $\hat{i}(X)$ is totally bounded. Fix $\varepsilon > 0$. Since $\hat{i}$ is continuous, and A is compact, there exists $\delta > 0$ such that $\hat{i}(A + \delta B_1) \subset i(A) + (\varepsilon/2) B_1$. Since A is totally bounded, then $\hat{i}(A + \delta B_1)$ can be covered by a finite number of balls of radius ε. Choose m so that $\{A_j : j = 1, \dots, m\}$ cover $X \setminus [A + \delta B_1]$ while A_{m+1} has empty intersection with it. Since each $i_j(\bigcup_{l=1}^{m} A_l)$, $j \geq m$ is totally bounded, and $(*)$ holds, we have that whenever j satisfies $\varepsilon_j < \varepsilon/2$, an $(\varepsilon/2)$-net of $i_j(\bigcup_{l=1}^{m} A_l)$ is also an ε-net of $\hat{i}(\bigcup_{l=1}^{m} A_l)$.

Hence we have found a finite ε-net for the set $\hat{i}(X)$. $\triangle$

As an application of theorem 3, the following result give in particular a new proof of a result of Fryszkowski [6].

THEOREM 4. *Let K be any closed, decomposable subset of $L_1(T,E)$, and let $F: K \to K$ be continuous with $F(K)$ totally bounded. Then F has a fixed point in K.*

PROOF. Set $A = \overline{F(K)}$. Following the notations of theorem 3, define the function $\hat{F}: L_1(T,E) \to L_1(T,E)$ by $\hat{F}(x) = F(\hat{i}(x))$.

For every $x \in L_1(T,E)$, $\hat{F}(x) \subset F(B) \subset A$; in particular $\hat{F}$ maps $\overline{co}(A)$ into itself.

Let x^* be a fixed point of $\hat{F}$. Then $x^* = F(\hat{i}(x^*)) \in A$, hence, $\hat{F}(x^*) = F(x^*)$. $\triangle$

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