RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

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**Density theorems for local minimizers of area-type functionals**

*Rendiconti del Seminario Matematico della Università di Padova*, tome 85 (1991), p. 217-248
<http://www.numdam.org/item?id=RSMUP_1991__85__217_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 85 (1991)

# Density Theorems for Local Minimizers of Area-Type Functionals.

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## 1. Introduction.

The single most important result of the present work is perhaps Theorem 4.7 at the end of the paper. According to it, the set $S_u$ of essential discontinuities (*jump set*) of a simple $BV$ function of the type $u = \sum_j t_j \phi_{U_j}$ (where $\{t_j\}$ is a *finite* set of real numbers and $\{U_j\}$ a *finite partition* of a given open set $\Omega \subset \mathbf{R}^n$) is closed in $\Omega$ and satisfies a nice density property, *provided* it has «minimal area» in a certain extended sense.

This result can be viewed as a first step in the regularization program of «optimal partitions», a question of considerable interest both from the theoretical point of view and for possible applications.

We remark that problems modelled on classes of partitions of a given domain and whose solutions tend to minimize the total area of the separating interfaces (with various weights and constraints), have been considered since the time of Plateau's experiments with soap films and bubbles; see e.g. [2], [26]. Recently, the subject has gained further stimulus from Computer Vision Theory; here, a central problem concerns «image segmentation», i.e. the problem of decomposing a given domain in uniform regions, separated by sharp contours, in order to single out the most significant features of the image, trying to eliminate noise and fussy details. See e.g. [23].

Results analogous to Theorem 4.7 can be proved for «optimal segmentations»: see e.g. [30]. However, we think that the major

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contribution of this work centers around the *method* we have been developing, in order to obtain that particular result.

Working with quite general assumptions and using basically isoperimetric estimates, we succeeded in proving some powerful «decay lemmas», which enable us to derive quite easily density estimates at boundary points of the optimal partitioning sets. Moreover, as will be shown in a subsequent paper [31], under appropriate assumptions they can be further improved so as to provide (in combination with other techniques) regularity results for optimal segmentations.

As the content of the article is rather technical, we begin with an informal description of our method in a particularly simple situation, i.e. when $u = \phi_E$ is the characteristic function of a set of finite perimeter (clearly, this corresponds to decomposing $\Omega$ in just two parts: $E$ and its complement $\Omega - E$).

We introduce a functional «of the type of the area»:

$$\mathcal{F}(E,A) = \int_{\partial^* E \cap A} \phi((x, \nu_E(x))\, dH^{n-1}(x)$$

where $A$ is open in $\Omega$, $\partial^* E$ is the reduced boundary of $E$, $\nu_E(x)$ is the inner normal vector at $x \in \partial^* E$, and where the integrand $\phi\colon \Omega \times S^{n-1} \to \mathbf{R}$ is assumed to satisfy:

$$0 < c_1 \leqslant \phi(x, \nu) \leqslant c_2 < +\infty\,.$$

Thus, if in particular $c_1 = c_2 = 1$, $\mathcal{F}$ reduces to the *perimeter functional*:

$$P(E,A) = H^{n-1}(\partial^* E \cap A)$$

(here and above, $H^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$).

We then consider *local minimizers* of $\mathcal{F}$, i.e. sets $E$ satisfying

$$\mathcal{F}(E,A) \leqslant \mathcal{F}(F,A) \qquad \forall A \subset \Omega,\ \ \forall F \text{ s.t. } F \Delta E \subset\subset A$$

and prove that if in a certain ring $A_{r,s} = \{x \in \mathbf{R}^n\colon r < |x| < r+s\}$ the Lebesgue measure of $E$ is «relatively small», in the following sense:

$$s^{-n} |E \cap A_{r,s}| < \eta$$

for an appropriate constant $\eta > 0$, then in a suitable subring $A_{r_1, s_1} \subset\subset A_{r,s}$ it is «even smaller», i.e.

$$s_1^{-n} |E \cap A_{r_1, s_1}| < \eta/2\,.$$

This is the essence of our «decay lemma» (Lemma 3.2). By iterated application, we find immediately a «fracture result», according to which the set $E$ splits into two parts separated by a spherical cup (Lemma 3.1). Actually, the preceding results have a much more general validity: firstly, we can add a perturbation to the functional $\mathcal{F}$, i.e. assume that $E$ be a local minimizer of $\mathcal{F}$ plus e.g. a volume term, or subject to various constraints; secondly, it suffices to assume a «unilateral» minimality condition, i.e. that $E$ be a local minimizer only with respect to subsets $F$ of $E$ itself. This is important in the applications to solutions of least area or of prescribed mean curvature with obstacles, volume constraints, and so on.

The general assumptions under which our decay and fracture lemmas will be proved are the following (see Lemma 3.1 and 3.2):

$$(\mathrm{H}_1) \qquad \mathcal{F}(E, A_{r,s}) \leq \mathcal{F}(F, A_{r,s}) + c_3 |E \setminus F|^{(n-1)/n} \qquad \forall F \subset E \text{ s.t. } E \setminus F \subset\subset A_{r,s},$$

$$(\mathrm{H}_2) \qquad s^{-n} |E \cap A_{r,s}| < \eta,$$

where $c_3 \geq 0$ is given and where $\eta$ is a suitable (explicitly computable) positive constant depending only on $n, c_1, c_2, c_3$. Having obtained the fracture result, i.e. the existence of $\bar{r} \in (r, r+s)$ such that $\partial B_{\bar{r}}$ splits $E$ in two pieces (actually, there are several such fractures in $A_{r,s}$), we investigate the possibility of «eliminating» $E$ in between two adjacent fractures. We show that this is precisely the case whenever $c_3 < n\omega_n^{1/n} c_1$ (Lemma 3.4). On the other hand, Example 3.3 ($\mathrm{E}_3$) shows that the hypothesis $c_3 > n\omega_n^{1/n} c_1$ can lead to a quite different conclusion. Finally, if equality holds, then all but at most one rings $A_{\bar{r},\bar{s}}$ between any two consecutive fractures are empty (i.e. $|A_{\bar{r},\bar{s}} \cap E| = 0$).

Density results are then easily derived (Theorem 3.5).

In the second part of the paper we extend the previous results to simple $BV$ functions $u = \sum_j t_j \phi_{U_j}$ minimizing functionals given by an «area term» (controlled by the $(n-1)$-dimensional Hausdorff measure of the jump set $S_u$ of $u$) plus a «perturbation» with bounded $L^{n/(n-1)}$-norm. We prove a first decay lemma (Lemma 4.3), which is an actual generalization of Lemma 3.2 discussed before. Some isoperimetric inequalities of direct and inverse type (Lemma 4.2 and 4.4) allow us to obtain a second decay lemma (Lemma 4.1), which is formulated in terms of the $H^{n-1}$-measure of $S_u$.

Fracture and elimination results (Lemma 4.5 and 4.6) then follow as easy consequences, again with explicit determination of the «optimal threshold» for $c_3$. Finally, Theorem 4.7 giving the closure and density estimate of $S_u$ is proved. Here is an outline of the content of the paper: in Section 2 we recall known properties of sets of finite perimeter, define a class of functionals «of the type of the area», and state a few preliminary results (among them, an isoperimetric inequality in circular rings).

Section 3 is devoted to the study of minimizing sets: we state the decay lemma, deduce fracture and elimination results, discuss by examples the validity of the minimality assumption and the optimality of the threshold $c_3 = n\omega_n^{1/n} c_1$, and prove a typical density result. Applications are only sketched: we refer the interested reader to the paper [8] where several examples are discussed in detail.

Finally, in the Section 4 we recall the basic facts about simple $BV$ functions and associated finite partitions of the given domain, provide detailed proofs of (two versions of) the decay lemma, and derive the announced consequences.

## 2. Notation, definitions and preliminary results.

If $\Omega$ is an open subset of $\mathbf{R}^n$ (in the following we shall always have $n \geq 2$), we denote, as usual, by $BV(\Omega)$ the space of integrable functions defined on $\Omega$ whose distributional derivatives are measures of bounded total variation in $\Omega$ itself (see e.g. [17]).

Moreover, we denote by $|\cdot|$ the Lebesgue measure in $\mathbf{R}^n$, by $H^d$ the $d$-dimensional Hausdorff measure in $\mathbf{R}^n$ (see [14], [16]) and by $\mathcal{B}(\mathbf{R}^n)$ the family of Borel sets of $\mathbf{R}^n$.

If $E \in \mathcal{B}(\mathbf{R}^n)$ and if $\alpha \in [0, 1]$, we denote by $E(\alpha)$ the set of points of density $\alpha$ of $E$ (with respect to Lebesgue measure):

$$E(\alpha) = \left\{ x \in \mathbf{R}^n \colon \lim_{\rho \to 0} \frac{|E \cap B_{x,\rho}|}{|B_{x,\rho}|} = \alpha \right\}.$$

Here, $B_{x,\rho}$ is the $n$-dimensional open ball with centre $x \in \mathbf{R}^n$ and with radius $\rho > 0$: $B_{x,\rho} = \{y \in \mathbf{R}^n \colon |x - y| < \rho\}$; when $x = 0$ we shall write for short $B_\rho$ instead of $B_{0,\rho}$ and set $\omega_n = |B_1|$.

$S^{n-1}$ will denote the $(n-1)$-dimensional unit sphere:

$$S^{n-1} = \partial B_1 = \{x \in \mathbf{R}^n \colon |x| = 1\};$$

finally we put

$$A_{r,s} = B_{r+s} - \overline{B}_r = \{x \in \mathbf{R}^n \colon r < |x| < r + s\} \quad \text{if } r > 0,\ s > 0,$$

$$A_{r,s} = B_s \quad \text{if } r = 0,\ s > 0.$$

The notation $E \subset\subset \Omega$ means that $\overline{E}$ is a compact subset of $\Omega$. $E \in \mathcal{B}(\mathbf{R}^n)$ will be said to have *finite perimeter* in the open set $\Omega$ iff $\phi_E \in BV(\Omega)$, where $\phi_E$ is the characteristic function of $E$; we denote by $P(E,\Omega)$ the perimeter of $E$ in $\Omega$ defined by: $P(E,\Omega) = |D\phi_E|(\Omega)$ (total variation of the measure $D\phi_E$ in $\Omega$). When $\Omega$ coincides with $\mathbf{R}^n$, we shall write more briefly $P(E)$ instead of $P(E,\mathbf{R}^n)$ (see [14], [17], [19]).

We say that $E$ is a *Caccioppoli set* and we write $E \in \mathcal{C}(\mathbf{R}^n)$ iff $E \in \mathcal{B}(\mathbf{R}^n)$ and $P(E,\Omega) < +\infty$ for every $\Omega \subset\subset \mathbf{R}^n$.

We recall that if $E$ has a finite perimeter in $\Omega$ we can always assume (possibly modifying $E$ by sets of measure zero) that

$$0 < |E \cap B_{x,\rho}| < |B_{x,\rho}| \qquad \forall x \in \partial E \cap \Omega, \ \forall \rho > 0 .$$

Moreover, if $E$ has a finite perimeter in $\Omega$, then at $H^{n-1}$-almost every point $x \in \partial E \cap \Omega$, we can define the *approximate inner normal* $\nu_E(x)$:

$$\nu_E(x) = \lim_{\rho \to 0} \frac{D\phi_E(B_{x,\rho})}{|D\phi_E|(B_{x,\rho})} .$$

$\partial^* E$ will denote the *reduced boundary* of $E$:

$$\partial^* E = \{x \in \partial E \colon |\nu_E(x)| = 1\} .$$

Finally $\phi_E^+$, $\phi_E^-$ will denote the *inner* and the *outer traces* of $\phi_E$ on spherical surfaces (see [17]).

Let us consider now a functional

$$\mathcal{F}\colon \mathcal{C}(\mathbf{R}^n) \times \mathcal{B}(\mathbf{R}^n) \to [0, +\infty]$$

such that

($P_1$) *there exist* $c_1, c_2$ *with* $0 < c_1 \leq c_2 < +\infty$ *s.t.*

$$c_1 \leq P(E,\Omega) \leq \mathcal{F}(E,\Omega) \leq c_2 P(E,\Omega) \qquad \forall E \in \mathcal{C}(\mathbf{R}^n) \text{ and } \forall \Omega \subset \mathbf{R}^n .$$

($P_2$) $\mathcal{F}(E,\cdot)$ *is a positive measure on* $\mathcal{B}(\mathbf{R}^n)$, $\quad \forall E \in \mathcal{C}(\mathbf{R}^n)$.

($P_3$) $\mathcal{F}(E,\Omega) = \mathcal{F}(F,\Omega)$ *whenever* $E \cap \Omega = F \cap \Omega$.

The family of all functionals verifying the previous requirements ($P_1$)-($P_3$) will be briefly denoted by $F_{c_1,c_2}$.

The simplest functional of this type is the perimeter itself, for which $c_1 = c_2 = 1$; a meaningful and more general example is given by:

$$\mathcal{F}(E,\Omega) = \int_{\partial^* E \cap \Omega} \phi(x, \nu_E(x))\, dH^{n-1}(x)$$

where $\phi$: $\mathbf{R}^n \times S^{n-1} \to [c_1, c_2]$ is Borel measurable.

In the following chapters we shall use the arithmetic lemma given below.

LEMMA 2.1. *For any set of $k^2$ real numbers $\{a_j: j = 1, \ldots, k^2\}$ satisfying*

$$a_1 \leq a_2 \leq \ldots \leq a_{k^2} \qquad (k \in \mathbf{N})$$

*at least one of the following two statements holds*:

(A$_1$) $\quad 2^k a_1 < a_{k^2}$;

(A$_2$) $\quad \exists j \in \{1, \ldots, k\}$ *s.t.* $a_{jk} \leq 2a_{(j-1)k+1}$.

Indeed, notice that by grouping together the numbers $a_j$ in $k$ groups of $k$ elements each, and by keeping the given order, (A$_2$) expresses the fact that at least one of such groups is «balanced» in the sense that its last element does not exceed the double of the first one. If this does not hold, then the extreme points $a_1, a_{k^2}$ are «far apart» in the sense of (A$_1$).

A few isoperimetric inequalities will also be useful, and for convenience we group them in the following way: *there exist positive constants $\alpha_1, \ldots, \alpha_4$ which depend only on the dimension $n$ and such that*

(D$_1$) $\quad \alpha_1(n)\,|E|^{(n-1)/n} \leq P(E) \quad \forall E \subset \mathbf{R}^n$ *s.t.* $|E| < +\infty$,

(D$_2$) $\quad \alpha_2(n)\,|E|^{(n-1)/n} \leq |D\phi_E|(B_1) \quad \forall E \subset B_1$ *s.t.* $|E| \leq \dfrac{\omega_n}{2} = \dfrac{|B_1|}{2}$,

(D$_3$) $\quad \alpha_4(n)\,|E|^{(n-1)/n} \leq |D\phi_E|(A_{1,s}) \quad \forall s \in (0,1],\ \forall E \subset A_{1,s}$ *s.t.*

$$|E| \leq \frac{|A_{1,s}|}{2}, \qquad |D\phi_E|(A_{1,s}) < \alpha_3(n)\, s^{n-1}.$$

(D$_1$) and (D$_2$) are well known (see, e.g., [11], [19], [20]) and the corresponding «optimal constants» are known to be:

$$\alpha_1 = n\omega_n^{1/n}, \qquad \alpha_2 = \frac{2^{(n-1)/n}\,\omega_{n-1}}{\omega_n^{1-1/n}}.$$

We sketch instead the proof of ($D_3$).

Let then $E \subset A_{1,s}$ with $0 < s \leqslant 1$ be s.t.

$$|E| \leqslant \frac{|A_{1,s}|}{2}.$$

If, for $\rho \in (1, 1+s)$,

$$E_\rho = \{\omega \in S^{n-1} : \rho\omega \in E\}$$

then there exists $\rho_1$ s.t. the traces $\phi_E^+, \phi_E^-$ coincide on $\partial B_{\rho_1}$ and in addition

$$H^{n-1}(E_{\rho_1}) \leqslant \frac{n\omega_n}{2} = \frac{H^{n-1}(S^{n-1})}{2}.$$

Since

(i) $$H^{n-1}(E_\rho) \leqslant H^{n-1}(E_{\rho'}) + |D\phi_E|(A_{1,s})$$

$\forall \rho, \rho' \in (1, 1+s)$ such that the traces of $\phi_E$ coincide on the respective spherical surfaces then, under the assumption

$$|D\phi_E|(A_{1,s}) \leqslant \frac{n\omega_n}{4} s^{n-1},$$

we have

$$H^{n-1}(E_\rho) \leqslant \frac{3}{4} n\omega_n \qquad \text{for a. e. } \rho \in (1, 1+s)$$

whence

(ii) $$\alpha_0(n)\,[H^{n-1}(E_\rho)]^{(n-2)/(n-1)} \leqslant P_{n-2}(E_\rho, S^{n-1})$$

for a. e. $\rho \in (1, 1+s)$

by virtue of the isoperimetric inequality on spherical surfaces (the right hand side denotes the ($n$-2)-dimensional perimeter of $E_\rho \subset S^{n-1}$ in the unit sphere; see, e.g., [18]).

Under the assumption

$$\text{(iii)} \qquad 0 < |D\phi_E|(A_{1,s}) < \min\left\{\frac{n\omega_n}{4}, \alpha_0(n)^{n-1}\right\} s^{n-1}$$

there must then exist $\rho_2 \in (1, 1+s)$ s.t. the traces of $\phi_E$ coincide on $\partial B_{\rho_2}$ and moreover

$$\text{(iv)} \qquad H^{n-1}(E_{\rho_2}) < |D\phi_E|(A_{1,s}).$$

For, assuming that $H^{n-1}(E_\rho) \geqslant |D\phi_E|(A_{1,s})$ for a.e. $\rho$, then from (ii) and from the inequality:

$$\text{(v)} \qquad |D\phi_E|(A_{1,s}) \geqslant \int_1^{1+s} \rho^{n-2} P_{n-2}(E_\rho, S^{n-1})\, d\rho$$

(see [18], Prop. 3) we would get a contradiction to (iii).

Therefore, from (i) and (iv) we obtain for a.e. $\rho$: $H^{n-1}(E_\rho) \leqslant 2|D\phi_E|(A_{1,s})$ which, together with (ii) yields

$$\alpha_0(n) H^{n-1}(E_\rho) \leqslant \left[2|D\phi_E|(A_{1,s})\right]^{1/(n-1)} \cdot P_{n-2}(E_\rho, S^{n-1})$$

that is (since $s \leqslant 1$)

$$\alpha_0(n) \rho^{n-1} H^{n-1}(E_\rho) \leqslant 2^{n/(n-1)} |D\phi_E|(A_{1,s})^{1/(n-1)} \rho^{n-2} P_{n-2}(E_\rho, S^{n-1})$$

for a.e. $\rho \in (1, 1+s)$. By integration (see (v)) we finally obtain

$$\alpha_0(n)^{(n-1)/n} |E|^{(n-1)/n} \leqslant 2|D\phi_E|(A_{1,s}),$$

which concludes the proof of ($D_3$).

By the same technique, an inequality of type ($D_3$) can also be obtained under the assumption of «small measure» rather than of small perimeter (i.e., assuming $|E| < \alpha_3 s^n$ instead of $|D\phi_E|(A_{1,s}) < \alpha_3 s^{n-1}$). Analogously, by recourse to ($D_1$) and to the Fubini type theorem for partial perimeters (see Theorem 2, Section 2.2.1 in [19]), one proves that there exist $\alpha_3'(n)$, $\alpha_4'(n) > 0$ s.t.

$$\alpha_4' |E|^{(n-1)/n} \leqslant |D\phi_E|(\Omega) \qquad \forall E \subset \Omega = \mathbf{R}^{n-1} \times (0, s) \ s.t.$$

$$|E| < +\infty, \ |D\phi_E|(\Omega) < \alpha_3' \cdot s^{n-1} \ (\text{or } \forall E \subset \Omega \ s.t. \ |E| < \alpha_3' s^n).$$

Finally, we recall the following elementary inequality:

$$(D_4) \qquad \left(\sum_{i=1}^{k} a_i\right)^{(n-1)/n} \leqslant \sum_{i=1}^{k} a_i^{(n-1)/n} \qquad (a_i \in \mathbf{R},\ a_i \geqslant 0)$$

which holds $\forall k \in \mathbf{N}$. When

$$a \leqslant a_i \leqslant \tau a \qquad \forall i = 1, \ldots, k$$

with a $\geqslant 0$ and $\tau \geqslant 1$ independent of $i$, instead of ($D_4$) we have:

$$(D_5) \qquad \left(\sum_{i=1}^{k} a_i\right)^{(n-1)/n} \leqslant k^{-1/n}\, \tau^{(n-1)/n} \left(\sum_{i=1}^{k} a_i^{(n-1)/n}\right).$$

## 3. Density results for minimizing sets.

In this section we present a general result about Caccioppoli sets (Lemma 3.1), which finds application in different situations. It is based on an appropriate «decay estimate» stated in Lemma 3.2.

Essentially, the result tells us that if a set $E$ satisfies a certain minimality condition (see ($H_1$) below) with respect to a functional $\mathcal{F}$ of the class $F_{c_1, c_2}$ defined in Section 2, and is «very thin» (see ($H_2$)) then it splits necessarily into two parts separated by a spherical surface.

Notice that the calculations are done in an annulus $A_{r,s}$ with $r \geqslant 0$ and $s > 0$, that $c_1$ and $c_2$ are the constants estimating the ratio $\mathcal{F}/P$ ($P$ is the perimeter), and that $c_3$ controls the «perturbation» added to $\mathcal{F}$.

Also notice that the minimality of $E$ is intended in a local «unilateral sense», i.e. only with respect to subsets $F$ of $E$ itself coinciding with $E$ outside a compact subset of $A_{r,s}$.

In order to give a general formulation of the result, we decided to introduce further constants $\gamma$, $r_0$ and $s_0$, which determine the position and thickness of the initial annulus where the process takes place, thus helping in locating the spherical surface which splits $E$; when $\gamma = 1$ we have in particular $r_0 = r$, $s_0 = s$.

LEMMA 3.1. *For every $n \geqslant 2$, for every $c_1, c_2, c_3$ s.t. $0 < c_1 \leqslant c_2 < +\infty$ and $0 \leqslant c_3 < +\infty$, and for every $\gamma \in (0, 1]$ there exists a constant $\eta = \eta(n, c_1, c_2, c_3, \gamma) > 0$ such that, if in $A_{r,s}$ ($r \geqslant 0$, $s > 0$) the set $E \in \mathcal{C}(\mathbf{R}^n)$ verifies the following two conditions:*

$$(H_1) \qquad \mathcal{F}(E, A_{r,s}) \leqslant \mathcal{F}(F, A_{r,s}) + c_3 |E - F|^{(n-1)/n}$$

$$\forall F \subset E \ \ s.t.\ \ E - F \subset\subset A_{r,s} \ \text{ and for a certain } \mathcal{F} \in F_{c_1, c_2};$$

(H$_2$) $|E \cap A_{r,s}| < \eta s^n$;

*then for every* $r_0, s_0$ *satisfying*

$$\begin{cases} r \leqslant r_0 < r_0 + s_0 \leqslant r + s \\ s_0 \geqslant \gamma s \end{cases}$$

*there exists*

$$\bar{r} \in \left(r_0 + \frac{s_0}{3},\ r_0 + \frac{2s_0}{3}\right)$$

*s.t.*

$$\partial B_{\bar{r}} \subset E(0).$$

Lemma 3.1 will be proved by repeated application of the following *decay Lemma*:

LEMMA 3.2. *For every* $n, c_1, c_2, c_3, \gamma$ *as in Lemma* 3.1 *there exist two constants* $\eta = \eta(n, c_1, c_2, c_3, \gamma) > 0$ *and* $\sigma = \sigma(n, c_1, c_3, \gamma) \in (0, 1/6]$ *such that, if E fulfils assumptions* (H$_1$), (H$_2$) *above, then, for every* $r_0, s_0$ *as above, there exists* $r_1$ *such that, setting* $s_1 = \sigma s_0$, *one has*

$$r_0 + \frac{s_0}{3} < r_1 < r_1 + s_1 < r_0 + \frac{2s_0}{3},$$

$$s_1^{-n} |E \cap A_{r_1, s_1}| \leqslant \frac{1}{2} s^{-n} |E \cap A_{r,s}|.$$

The proof of Lemma 3.2 is postponed until the next chapter where such a result will be obtained in a much more general context (see Lemma 4.3 and the remark preceding it).

We now give the proof of Lemma 3.1.

By repeatedly applying Lemma 3.2 we construct two sequences $r_j \to \bar{r}$ and $s_j = \sigma^j s_0 \to 0$ such that $\forall j \geqslant 0$:

$$(*) \qquad \begin{cases} r_j + \dfrac{s_j}{3} < \bar{r} < r_j + \dfrac{2s_j}{3} \\ |E \cap A_{r_j, s_j}| \leqslant 2^{-j} \eta s_j^n. \end{cases}$$

Let then $x \in \partial B_{\bar{r}}$, $\rho \in (0, s_1/3)$ and let $j = j(\rho)$ be the unique index such that $s_{j+1}/3 \leqslant \rho < s_j/3$; by obvious inclusions

$$\rho^{-n} |E \cap B_{x,\rho}| \leqslant \left(\frac{\sigma^{j+1} s_0}{3}\right)^{-n} \cdot |E \cap A_{r_j, s_j}| = 3^n \sigma^{-n} s_j^{-n} |E \cap A_{r_j, s_j}|$$

whence

$$\limsup_{\rho \to 0} \rho^{-n} |E \cap B_{x,\rho}| = 0, \text{ by virtue of } (*). \quad \text{Q.E.D.}$$

Now let us examine some simple examples that will shed light on the results we have been obtaining.

EXAMPLE 3.3.

($E_1$) Let $E = B_R$ be a ball contained in the circular ring $A_{r,1}$. Then (see [6], [24])

$$P(B_R) \leq P(F) + n\omega_n^{1/n} |B_R - F|^{(n-1)/n} \qquad \forall F \subset B_R$$

and hence $E = B_R$ verifies assumption ($H_1$) of Lemma 3.1 with $c_1 = c_2 = 1$ (i.e. $\mathcal{F} =$ perimeter) and $c_3 = n\omega_n^{1/n}$. Choosing (for $\gamma = 1$) $\eta = 6^{-n}\omega_n$, we immediately check the validity of Lemma 3.1.

($E_2$) By increasing the number $p$ of balls (mutually disjoint and with the same radius $R = 1/2p$ and centres $x_i$ on a fixed coordinate axis) contained in the circular ring $A_{r,1}$, it is obviously possible to make the overall measure as small as we please. Nevertheless in such a case

$$E = \bigcup_{i=1}^{p} B_{x_i,R}$$

verifies ($H_1$) with $c_3 \geq p^{1/n} n\omega_n^{1/n}$ (see [6]); as $p$ increases, $c_3$ explodes and consequently the constant $\eta$ of Lemma 3.1 goes to 0 (see steps 3 and 4 in the proof of Lemma 4.3); Lemma 3.1 does not apply ...!

($E_3$) For $n = 2$ and $a > 2$ we set

$$E = \bigcup_{i=0}^{\infty} B_i,$$

where $B_i$ is the circle of radius $r_i = a^{-i-2}$ with centre on the horizontal axis, at the point of abscissa $2^{-i}$; moreover let $\Omega = B_R$ be a circle of suitably large radius, so as to have $E \subset\subset \Omega$. Then

$$0 \in \Omega \cap \partial E \cap E(0).$$

For $\delta \geq 1$, we denote by $\phi_\delta$: $\mathbf{R}^2 \to \mathbf{R}$ the function that equals 1 on $\partial E$ and $\delta$ elsewhere ($\phi_\delta$ is thus lower semicontinuous) and by $\mathcal{F}_\delta$ the functional

$$\mathcal{F}_\delta(F, B) = \int_{\partial^* F \cap B} \phi_\delta(x)\, dH^1(x).$$

Obviously $\mathcal{F}_\delta \in F_{c_1,c_2}$ with $c_1 = 1$, $c_2 = \delta$. Let then $M_\delta$ be a minimizer of $\mathcal{F}_\delta(\cdot, \mathrm{R}^2)$ in the class of sets $G \subset \Omega$ such that $|G| = |E|$:

$$(*) \qquad \int_{\partial^* M_\delta} \phi_\delta dH^1 \leq \int_{\partial^* G} \phi_\delta dH^1 \qquad \forall G \subset \Omega \text{ s.t. } |G| = |E|.$$

We claim that:

$$(**) \qquad \textit{if } \left(\frac{a+1}{a-1}\right)^{1/2} < \delta \leq 2 \textit{ then } M_\delta = E.$$

As a consequence of (**), that we shall shortly prove, we have that if $F \subset E$ and $G = F \cup B_\rho$ with $B_\rho \subset\subset \Omega - \overline{E}$ s.t. $|B_\rho| = |E - F|$, then by virtue of (*):

$$(***) \qquad \mathcal{F}_\delta(E, \Omega) \leq \mathcal{F}_\delta(G, \Omega) = \mathcal{F}_\delta(F, \Omega) + 2\pi^{1/2}\delta|E - F|^{1/2}.$$

We are therefore under the assumptions of Lemma 3.1 with $c_1 = 1$, $c_2 = \delta$, $c_3 = 2\pi^{1/2}\delta$ ($r = 0$, $0 < s < R$). Notice that $\delta$ may be taken arbitrarily close to 1, by choosing the constant $a$ large enough (see (**)); moreover, with small changes (e.g. by inserting sequences of very small circles between $B_{i+1}$ and $B_i$) it is possible to construct a set $E$ that verifies again (***) above and with $0 \in \partial E \cap E(0)$ but such that $|E \cap A_{r,s}| > 0$ $\forall r, s > 0$ with $r + s < 1$.

We now sketch a proof of (**).

First, it is easily seen that

$$M_\delta = \left(\bigcup_h B_{i_h}\right) \cup B_\rho$$

where $B_{i_h}$ are *certain* circles chosen among the $B_i'$ s that form $E$ and where $B_\rho$ is a suitable circle $\subset\subset \Omega - \overline{E}$ that «compensates» for the *missing* $B_i'$ s, in such a way that $|M_\delta| = |E|$; e.g., if $\delta = 1$ then obviously $M_1 = B_\rho$. Now let

$$F = E - \bigcup_{h=1}^{k} B_{i_h}, \quad F' = F - B_{i_0} \text{ where } \begin{cases} i_1 > \ldots > i_k \\ i_0 \notin \{i_1, \ldots, i_k\} \end{cases}$$

and, as above, let

$$G = F \cup B_\rho, \qquad G' = F' \cup B_\rho,$$

with $|G| = |G'| = |E|$.

Then

$$\mathcal{F}_\delta(G') < \mathcal{F}_\delta(G)$$

if and only if

$$2\delta\left(\sum_{h=1}^{k} r_{i_h}^2\right)^{1/2} > (\delta^2-1)\, r_{i_0}$$

and this last relation is certainly true if $\delta \leq 2$ and if $i_0 > i_1$.

Summing up, it suffices to compare $E$ with

$$G = \left(E - \bigcup_{i=j}^{\infty} B_i\right) \cup B_\rho :$$

one finds

$$\mathcal{F}_\delta(G) - \mathcal{F}_\delta(E) = 2\pi a^{-j-1}(a^2-1)^{-1/2}\left\{\delta - \left(\frac{a+1}{a-1}\right)^{1/2}\right\}$$

whence (**).

(E$_4$) Let $E$ be a local supersolution in $A_{r,1}$ of a functional that is sum of the perimeter and of a curvature term, i.e.

$$|D\phi_E|(A) + \int_{E\cap A} H(x)\,dx \leq |D\phi_F|(A) + \int_{F\cap A} H(x)\,dx$$

for every open $A \subset\subset A_{r,1}$ and for every $F \subset E$ such that $E - F \subset\subset A$ (see e.g. [8], [24]). If $H \in L^n(A_{r,1})$ then by Hölder inequality one has

$$|D\phi_E|(A_{r,1}) \leq |D\phi_F|(A_{r,1}) + \|H\|_{L^n(A)}|E-F|^{(n-1)/n}$$

for every $F \subset E$ such that $E - F \subset\subset A \subset\subset A_{r,1}$; hence (H$_1$) holds with $c_1 = c_2 = 1$, $c_3 = \|H\|_{L^n(A_{r,1})}$. Moreover, since in such a case $\|H\|_{L^n(A)} \to 0$ for $|A| \to 0$, by suitably localizing the argument it is possible to make the constant $c_3$ arbitrarily small. An analogous argument holds when $H \in L^p(A_{r,1})$ with $n < p \leq +\infty$.

(E$_5$) Let $E$ be a local minimizer in $A_{r,1}$ of the perimeter functional with a volume constraint, i.e.

$$(*) \qquad |D\phi_E|(A_{r,1}) \leq |D\phi_G|(A_{r,1})$$

$\forall G$ such that $E\Delta G \equiv (E-G)\cup(G-E) \subset\subset A_{r,1}$ and $|G \cap A_{r,1}| = |E \cap A_{r,1}|$ (see e.g. [8], [24]). We suppose that there exist $x \in \mathbf{R}^n$ and $R > 0$ such that $B_{x,R} \subset A_{r,1} - E$ and consider $F \subset E$ such that $|E - F| = v \leq \omega_n R^n$.

Let $G = F \cup B_{x,\rho}$ with $\rho = (v/\omega_n)^{1/n}$, so that $|G \cap A_{r,1}| = |E \cap A_{r,1}|$.

Because of (*) we obtain:

$$|D\phi_E|(A_{r,1}) \leqslant |D\phi_F|(A_{r,1}) + n\omega_n^{1/n}|E-F|^{(n-1)/n}.$$

In this situation we then have ($H_1$) with $c_1 = c_2 = 1$, $c_3 = n\omega_n^{1/n}$.

At this stage we think it convenient to deepen the meaning of Lemma 3.1, according to which it is always possible to «break» a set $E$ that verifies assumptions ($H_1$), ($H_2$) in the circular ring $A_{r,s}$.

Provided $\gamma$ is chosen small enough it is, in fact, possible to create an arbitrarily large number of fractures. However, as it was seen in example ($E_3$), the measure of $E \cap A'$ may be strictly positive in every circular subring $A'$ of $A_{r,s}$.

The following Lemma shows that the situation is better when the constant $c_3$ is below a well-determined threshold.

LEMMA 3.4. *Under the hypotheses of Lemma* 3.1, *we suppose that* $\bar{r}_1, \ldots, \bar{r}_{k+1}$ *verify*

$$r < \bar{r}_1 < \ldots < \bar{r}_{k+1} < r+s$$

$$(*) \qquad \partial B_{\bar{r}_i} \subset E(0) \qquad \forall i = 1, \ldots, k+1$$

*and we set* $\bar{s}_i = \bar{r}_{i+1} - \bar{r}_i$, $i = 1, \ldots, k$.

*It follows that:*

(i) *if* $c_3 < n\omega_n^{1/n} c_1$ *then* $|E \cap A_{\bar{r}_i, \bar{s}_i}| = 0 \ \forall i = 1, \ldots, k$,

(ii) *if* $c_3 = n\omega_n^{1/n} c_1$ *then there exists at most one index*

$$i_0 \in \{1, \ldots, k\} \ s.t. \ |E \cap A_{\bar{r}_{i_0}, \bar{s}_{i_0}}| > 0.$$

PROOF. We denote by $A$ any one of the given circular rings $A_{\bar{r}_i, \bar{s}_i}$ and set $F = E - \overline{A}$. From assumption ($H_1$) we then get, by virtue of properties ($P_1$)-($P_3$) of the functional $\mathcal{F}$ (see Section 2) and because of (*):

$$c_1 P(E \cap A) \leqslant c_3 |E \cap A|^{(n-1)/n}.$$

Now if $c_3 \leqslant n\omega_n^{1/n} c_1$, because of the isoperimetric property of the ball (see ($D_1$), Chapter 2) we obtain $E \cap A = B_{x,R}$, with suitable $x \in \mathbb{R}^n$ and $R \geqslant 0$; moreover if $c_3 < n\omega_n^{1/n} c_1$ then obviously $R = 0$, whence (i). In order to prove (ii), we denote by $A_1, A_2$ any two distinct rings chosen among the $A_{\bar{r}_i, \bar{s}_i}$'s. Because of what has just been said, we have

$$E \cap A_i = B_{x_i, R_i} \qquad i = 1, 2.$$

We may assume $R_1 > 0$, $R_2 = \tau^{1/n} R_1$ with $\tau \geq 0$. If $F = E - \overline{(A_1 \cup A_2)}$ then, arguing as above, we have:

$$c_1 P(E, A_1 \cup A_2) \leq c_3 |E \cap (A_1 \cup A_2)|^{(n-1)/n}$$

namely, since $c_3 = n\omega_n^{1/n} c_1$:

$$0 \leq n\omega_n c_1 R_1^{n-1} \{(1+\tau)^{(n-1)/n} - (1 + \tau^{(n-1)/n})\}$$

hence necessarily $\tau = 0$, which proves (ii). Q.E.D.

We notice that examples ($E_1$) and ($E_4$) above may shed light on the results (ii) and (i), respectively, of Lemma 3.4.

The threshold introduced in Lemma 3.4 is pivotal also for the results to follow. Indeed, if $E$ fulfils the minimality condition ($H_1$) and if $c_3 \leq n\omega_n^{1/n} c_1$, then we have good estimates of the density of $E$ at its boundary points (see Theorem 3.5 (i)). If, on the contrary, $c_3 > n\omega_n^{1/n} c_1$, then $\partial E$ may have points of zero density for $E$ itself (see again ex. ($E_3$)). However these «highly singular points» can be «separated» from $\partial E$, in the sense of the following:

DEFINITION. *For a given measurable subset $E \subset \mathbf{R}^n$, we say that the point $x \in \mathbf{R}^n$ is separable from $\partial E$ when there exists a sequence $\rho_j \to 0$ such that $\partial B_{x,\rho_j} \subset E(0)$ $\forall j$ (or, s.t. $\partial B_{x,\rho_j} \subset E(1)$ $\forall j$).*

In example ($E_3$) above we actually had 0 separable from $\partial E$.

THEOREM 3.5. *Let $\mathcal{F} \in F_{c_1, c_2}$, let $\Omega$ be open $\subset \mathbf{R}^n$, $0 \leq c_3 < +\infty$ and assume that $E$ verifies*

$$(\mathrm{H}_1) \qquad \mathcal{F}(E, A) \leq \mathcal{F}(F, A) + c_3 |E - F|^{(n-1)/n}$$

*for every open $A \subset\subset \Omega$ and for every $F \subset E$ s.t. $E - F \subset\subset A$.*

*We denote by $\eta$ the constant of Lemma 3.1, which corresponds to the choice $\gamma = 1$, namely $\eta = \eta(n, c_1, c_2, c_3, 1)$ and by $\Theta_*(E, x)$ the lower Lebesgue density of $E$ at the point $x \in \mathbf{R}^n$:*

$$\Theta_*(E, x) = \liminf_{\rho \to 0} \omega_n^{-1} \rho^{-n} |E \cap B_{x,\rho}|.$$

*It follows that*

(i) *if $c_3 \leq n\omega_n^{1/n} c_1$, then $\Theta_*(E, x) \geq \omega_n^{-1} \eta$ $\forall x \in \partial E \cap \Omega$;*

(ii) *on the other hand, if $x \in \Omega$ and $\Theta_*(E,x) < \omega_n^{-1}\eta$, then $x$ is separable from $\partial E$, for any $c_3 < +\infty$.*

PROOF. For simplicity, we suppose that $0 \in \Omega$ and $\Theta_*(E,0) < \omega_n^{-1}\eta$; then there exists a sequence $\rho_j$ decreasing to 0 and such that

$$B_{\rho_1} \subset\subset \Omega \quad \text{and} \quad |E \cap B_{\rho_j}| < \eta\rho_j^n \qquad \forall j \geq 1 .$$

Because of Lemma 3.1 (with $\gamma = 1$, $r = 0$, $s = \rho_j$) we can find $\bar{r}_j \in (\rho_j/3, 2\rho_j/3)$ such that $\partial B_{\bar{r}_j} \subset E(0)$, $\forall j \geq 1$, whence (ii).

Now if $0 \in \partial E \cap \Omega$ and $c_3 \leq n\omega_n^{1/n} c_1$, then (i) must be true: in fact, if $\Theta_*(E,0) < \omega_n^{-1}\eta$ were to hold, part (ii) just established together with Lemma 3.4 would imply the existence of $\bar{\rho} > 0$ s.t. $|E \cap B_{\bar{\rho}}| = 0$, contrary to the assumption $0 \in \partial E$. Q.E.D.

REMARK. If $c_3 < n\omega_n^{1/n} c_1$, the density result established in Theorem 3.5 (i) above, can also be obtained through an elementary procedure that avoids the use of Lemma 3.2 and which derives its inspiration from the methods introduced by De Giorgi in [10]. The details of this proof are presented in [8], along with various applications to specific problems, like, for instance, the problem of minimal boundaries with obstacles and that of surfaces of prescribed mean curvature in $L^n$.

For completeness, we think it advisable to reproduce below the argument on a case of relevance.

To this end let $\Omega$ be an open set $\subset \mathbf{R}^n$, $L \in \mathcal{B}(\Omega)$ be the obstacle, $H \in L^n_{loc}(\Omega)$ be the prescribed curvature; we suppose that $G \supset L$ verifies

$$(*) \qquad |D\phi_G|(A) + \int_{G \cap A} H(x)\,dx \leq |D\phi_F|(A) + \int_{F \cap A} H(x)\,dx$$

for every open $A \subset\subset \Omega$, and for every $F \supset L$ s.t. $F \Delta G \subset\subset A$. Let $x \in \partial G \cap \Omega$ and $r > 0$ s.t. $B = B_{x,r} \subset\subset \Omega$ and $\|H\|_{L^n(B)} < n\omega_n^{1/n}$.

If $E = G^c$, from $(*)$ we easily obtain (see $(E_4)$):

$$|D\phi_E|(B) \leq |D\phi_F|(B) + \|H\|_{L^n(B)}|E - F|^{(n-1)/n}$$

$\forall F \subset E$ s.t. $E - F \subset\subset B$, that is $(H_1)$ with $c_1 = c_2 = 1$, $c_3 < n\omega_n^{1/n}$.

It follows from Theorem 3.5 (i) that $\Theta_*(E, x) \geq \omega_n^{-1} \eta$, i.e.:

$$\Theta^*(G, x) = \limsup_{\rho \to 0} \omega_n^{-1} \rho^{-n} |G \cap B_{x,\rho}| \leq 1 - \omega_n^{-1}/\eta \qquad \forall x \in \partial G \cap \Omega .$$

## 4. Extension to simple functions.

The results obtained in the preceding section concern Caccioppoli sets or, equivalently, $BV$ functions with values in $\{0, 1\}$. We now extend them to the case of simple $BV$ functions, taking on a finite number $m$ of values, with $m \geq 2$; when $m = 2$ we thus revert to the case already examined.

As usual let $\Omega$ be an open set $\subset \mathbf{R}^n$, $n \geq 2$ and let $T$ be a finite subset of $\mathbf{R}^p$ of diameter $d$:

$$T = \{t_1, \ldots, t_m\}, \qquad t_i \neq t_j \text{ if } i \neq j, \quad d = \operatorname{diam} T .$$

Let $u \in BV(\Omega, T)$ i.e. $u = (u_1, \ldots, u_p)$ with $u_i \in BV(\Omega, \mathbf{R})$ $\forall i = 1, \ldots, p$ and $u(x) \in T$ $\forall x \in \Omega$.

We recall the definition of the jump set $S_u$ of $u$:

$$S_u = \{x \in \Omega\text{: the approximate limit of } u \text{ at } x \text{ does not exist}\}$$

(see [12], [16], for a definition of approximate limit).

As in what follows the dimension $p$ of the image space is irrelevant, we shall assume for convenience that $p = 1$; in this case (see [12], [16])

$$S_u = \{x \in \Omega\text{: } \operatorname*{aplim\,inf}_{y \to x} u(y) < \operatorname*{aplim\,sup}_{y \to x} u(y)\}$$

where

$$\operatorname*{aplim\,inf}_{y \to x} u(y) = \sup\{t \in \mathbf{R}\text{: } \Theta(u^{-1}(-\infty, t), x) = 0\},$$

$$\operatorname*{aplim\,sup}_{y \to x} u(y) = \inf\{t \in \mathbf{R}\text{: } \Theta(u^{-1}(t, +\infty), x) = 0\}$$

and where $\Theta(E, x)$ denotes the density of the set $E$ at the point $x$, i.e.:

$$\Theta(E, x) = \lim_{\rho \to 0} \omega_n^{-1} \rho^{-n} |E \cap B_{x,\rho}| \qquad \forall E \in \mathcal{B}(\Omega) .$$

It is well known (see [17]) that the levels $\{x \in \Omega\text{: } v(x) > t\}$ of a function $v \in BV(\Omega)$ have finite perimeter in $\Omega$, for almost every $t \in \mathbf{R}$.

Therefore, if $u \in BV(\Omega, T)$ and $U_j = \{x \in \Omega : u(x) = t_j\}$ then

$$|D\phi_{U_j}|(\Omega) < +\infty \qquad \forall j = 1, \dots, m \text{ and } u = \sum_{j=1}^{m} t_j \phi_{U_j}$$

where $\{U_j\}_{j=1,\dots,m}$ is a partition of $\Omega$.

We notice that in the particular case $m=2$, $T=\{0,1\}$, setting $E = U_2$ (so that $u = \phi_E$) one has:

$$\partial^* E \cap \Omega \subset S_u = \partial E \cap (\Omega - [E(0) \cup E(1)]),$$

$$H^{n-1}(S_u) = H^{n-1}(\partial^* E \cap \Omega).$$

This can be deduced from the following properties of the reduced boundary of sets of finite perimeter, for which we refer to [14], Cap. 4:

$$\partial^* E \cap \Omega \subset E\left(\frac{1}{2}\right) \cap \Omega; \qquad H^{n-1}\left(E\left(\frac{1}{2}\right) \cap \Omega - \partial^* E\right) = 0;$$

$$P(E, \Omega) = H^{n-1}(\partial^* E \cap \Omega) = H^{n-1}\left(E\left(\frac{1}{2}\right) \cap \Omega\right);$$

$$H^{n-1}\left(\Omega - \left[E(0) \cup E(1) \cup E\left(\frac{1}{2}\right)\right]\right) = 0.$$

The previous relations can easily be extended to the case $m>2$: *if*

$$u = \sum_{j=1}^{m} t_j \phi_{U_j} \in BV(\Omega, T),$$

*then*

$$\bigcup_{j=1}^{m} (\partial^* U_j \cap \Omega) \subset S_u = \bigcup_{j=1}^{m} \{(\partial U_j \cap \Omega) - [U_j(0) \cup U_j(1)]\}$$

and

$$\text{(I)} \qquad 2H^{n-1}(S_u) = \sum_{j=1}^{m} H^{n-1}(\partial^* U_j \cap \Omega) = \sum_{j=1}^{m} P(U_j, \Omega) < +\infty.$$

We notice that these last relations (on which the majority of the results in the present section are based), are essentially connected to partitions of the open set $\Omega$ in a *finite* number of sets $\{U_1, \dots, U_m\}$ with finite perimeter in $\Omega$ (see [29]); they do *not* necessarily hold in the case of *countable* partitions of $\Omega$. However they *can* be suitably extended to the countable case

$$u = \sum_{j=1}^{\infty} t_j \phi_{U_j}$$

provided further assumptions are made on $u$ (of the type: $u \in SBV(\Omega) \cap L^{\infty}(\Omega)$ and $H^{n-1}(S_u) < +\infty$)). This will allow us to extend the results of the present section also to the case in which $T$ is countable and $d < +\infty$; however, this will be the object of a subsequent paper (see [30]).

At this stage we introduce a new class of functionals that generalizes the corresponding class $F_{c_1, c_2}$ defined in chapter 2.

*Let* $\mathcal{F}: BV(\Omega, T) \times \mathcal{B}(\Omega) \to [0, +\infty]$ *and let* $c_1, c_2$ *with* $0 < c_1 \leq c_2 < +\infty$ *be such that*

(P$'_1$) $$c_1 H^{n-1}(S_u \cap A) \leq \mathcal{F}(u, A) \leq c_2 H^{n-1}(S_u \cap A)$$

$$\forall u \in BV(\Omega, T) \text{ and for every open } A \subset\subset \Omega;$$

(P$'_2$) $\mathcal{F}(u, \cdot)$ *is a positive measure on* $\mathcal{B}(\Omega)$, $\quad \forall u \in BV(\Omega, T)$;

(P$'_3$) $\mathcal{F}(u, A) = \mathcal{F}(v, A)$ *for every open* $A \subset\subset \Omega$ *and*

$$\forall u, v \in BV(\Omega, T) \text{ such that } u(x) = v(x) \quad \forall x \in A.$$

The family of functionals verifying (P$'_1$)-(P$'_3$) will be denoted by $F_{c_1, c_2}(\Omega; T)$. A typical example is the functional:

$$\mathcal{F}(u, B) = \int_{S_u \cap B} \phi(x, tr^{+}(x, u, \nu), tr^{-}(x, u, \nu), \nu)\, dH^{n-1}(x)$$

where $\phi$ is Borel measurable with values in $[c_1, c_2] \subset \mathbf{R}$, $\nu$ is normal to $S_u$ and $tr^{\pm}$ are the traces of $u$ at $x \in S_u$ from both sides along $\nu$.

Functionals of this type have recently been studied by various authors: we mention in particular the paper [12] to which we refer also for the meaning of the symbols used.

We now proceed to state and prove a group of results (Lemmas 4.1 to 4.6) which constitute a general framework for the regularization of «optimal partitions», i.e. partitions $\{U_1, \ldots, U_m\}$ associated with simple $BV$ functions $u$ minimizing functionals of the class $F_{c_1, c_2}(\Omega; T)$, possibly with perturbations.

They are of interest in various settings, e.g. in problems of segmentation of images in Computer Vision Theory, where $u$ is a piecewise constant approximation to a given function representing the intensity of light at points of a given image (see [13], [21], [22], [23], [30], [31]).

Optimal partitions are expected to have smooth boundaries (the interfaces separating different components $U_j$), except possibly for a «small» singular set where three or more components meet (see e.g. [2], [22], [25]). In particular the jump set $S_u$ of a minimizer $u$ is expected to be closed in $\Omega$ (which is not the case, of course, of general functions $u \in BV(\Omega; T)$) and to enjoy good density properties.

A result of this type is obtained at the end of the present section (Theorem 4.7). It relies heavily on «fracture» and «elimination» results, similar to those discussed in the preceding section, based in turn on «decay lemmas», most as in Section 3 again. The use of isoperimetric estimates, both of direct and inverse type is crucial here.

The first result we present is analogous to Lemma 3.2; however, the «decay parameter» here is the average $(n-1)$ dimensional measure of the jump set $S_u$. Later on (Lemma 4.3) we shall state a second «decay lemma», which is an actual generalization of Lemma 3.2 and which is formulated in terms of the average Lebesgue measure of sets in the partition $\{U_j\}$ associated with $u$.

LEMMA 4.1. *For every* $n \geq 2$; *for every* $c_1, c_2, c_3$ *s.t.* $0 < c_1 \leq c_2 < +\infty$, $0 \leq c_3 < +\infty$, *for every* $d \in (0, +\infty)$ *and for every* $\gamma \in (0, 1]$, *there exist two constants* $\beta = \beta(n, c_1, c_2, c_3 d, \gamma) > 0$ *and* $\tau = \tau(n, c_1, c_2, c_3 d, \gamma) \in (0, 1/6)$ *s.t. if* $T \subset \mathbf{R}$, $r > 0$, $0 < s \leq r$ (*or* $r = 0$, $s > 0$), $\mathcal{F} \in F_{c_1, c_2}(A_{r,s}, T)$, $u \in BV(A_{r,s}, T)$ *verify the following hypotheses*:

($H_0$) $T$ *is finite and* $\operatorname{diam} T = d$;

($H_1'$) $\mathcal{F}(u, A_{r,s}) \leq \mathcal{F}(v, A_{r,s}) + c_3 \|u - v\|_{L^{n/(n-1)}(A_{r,s})}$

$$\forall v \in BV(A_{r,s}; T) \ \text{s. t.} \quad \operatorname{support}(u - v) \subset A_{r,s};$$

($H_2'$) $s^{1-n} H^{n-1}(S_u \cap A_{r,s}) < \beta$;

*then for every* $r_0, s_0$ *verifying*

$$r \leq r_0 < r_0 + s_0 \leq r + s, \qquad s_0 \geq \gamma s$$

*there exists* $r_1$ *s.t., setting* $s_1 = \tau s_0$ *then*

$$r_0 + s_0/3 < r_1 < r_1 + s_1 < r_0 + 2s_0/3$$

*and*

$$s_1^{1-n} H^{n-1}(S_u \cap A_{r_1, s_1}) \leq \frac{1}{2} s^{1-n} H^{n-1}(S_u \cap A_{r,s}).$$

The proof of Lemma 4.1 will be obtained as an immediate consequence of the following three lemmas that have also an independent interest. In the first of them we obtain a general isoperimetric inequality, which does not require the minimality of $u$. It is essentially based on the relative isoperimetric inequality established in Section 2.

LEMMA 4.2. *There exist two constants* $\alpha_5, \alpha_6$ *depending only on the dimension* $n$, *such that if* $u \in BV(A_{r,s}, T)$ *with* $r > 0$, $0 < s \leq r$ (*or* $r = 0$, $s > 0$) *and* $T = \{t_1, \ldots, t_m\}$ *verifies*

$$(*) \qquad H^{n-1}(S_u \cap A_{r,s}) < \alpha_5 s^{n-1}$$

*then there exists* $j_0 \in \{1, \ldots, m\}$ *such that*

$$(**) \qquad |A_{r,s} - U_{j_0}|^{(n-1)/n} \leq \alpha_6 H^{n-1}(S_u \cap A_{r,s}).$$

PROOF. In the case $r > 0$, $0 < s \leq r$, set $2\alpha_5 = \min\{\alpha_3, \alpha_4 \omega_n^{(n-1)/n}\}$ where $\alpha_3, \alpha_4$ are the constants of inequality ($D_3$) of Section 2. From $(*)$ and from ($D_3$), keeping in mind (I), we obtain $\forall j = 1, \ldots, m$

$$(\min\{|U_j|, |A_{r,s} - U_j|\})^{(n-1)/n} \leq \alpha_4^{-1} |D\phi_{U_j}|(A_{r,s})$$

hence summing over $j$, we have because of ($D_4$) and (I):

$$(***) \qquad \left(\sum_{j=1}^{m} \min\{|U_j|, |A_{r,s} - U_j|\}\right)^{(n-1)/n} < 2\alpha_4^{-1} H^{n-1}(S_u \cap A_{r,s}).$$

Now if $|U_j| \leq |A_{r,s} - U_j|$ were to hold $\forall j$, then since

$$\left(\sum_j |U_j|\right)^{(n-1)/n} = |A_{r,s}|^{(n-1)/n} \geq \omega_n^{1-1/n} s^{n-1}$$

$(*)$ and $(***)$ would lead to a contradiction, on the account of the choice of $\alpha_5$. It follows that there must exist $j_0 \in \{1, \ldots, m\}$ such that $|U_{j_0}| > |A_{r,s} - U_{j_0}|$; $(***)$ implies therefore

$$2^{1-1/n} |A_{r,s} - U_{j_0}|^{(n-1)/n} \leq 2\alpha_4^{-1} H^{n-1}(S_u \cap A_{r,s})$$

hence $(**)$ with $\alpha_6 = 2^{1/n} \alpha_4^{-1}$.

In the case $r=0$, $s>0$ (i.e. $A_{r,s}\equiv B_s$) the proof is analogous and uses ($D_2$) rather than ($D_3$). Q.E.D.

The second result we proceed to prove is the second version of the decay Lemma already announced.

We note that Lemma 3.2 is a particular case of it, which corresponds to characteristic functions of Caccioppoli sets, i.e. to the choice of $m=2$, $T=\{0,1\}$, $u=\phi_E$ and $t_{j_0}=0$.

LEMMA 4.3. *For every* $n\geq 2$, *for every* $c_1, c_2, c_3$ *such that* $0<c_1\leq c_2<+\infty$, $0\leq c_3<+\infty$, *for every real positive* $d$ *and for every* $\gamma\in(0,1]$ *there exist two constants* $\eta=\eta(n,c_1,c_2,c_3 d,\gamma)>0$ *and* $\sigma=\sigma(n,c_1,c_3 d,\gamma)\in(0,1/6]$ *s.t. if for* $T\subset\mathbf{R}$, $r\geq 0$, $s>0$, $\mathcal{F}\in F_{c_1,c_2}(A_{r,s},T)$, $u\in BV(A_{r,s},T)$ *and* $t_{j_0}\in T$ *the following assumptions hold:*

($H_0$) $\quad T=\{t_1,\dots,t_m\}$ *is finite and* $\operatorname{diam} T=d$;

($H_1''$) $\quad \mathcal{F}(u,A_{r,s})\leq\mathcal{F}(v,A_{r,s})+c_3\|u-v\|_{L^{n/(n-1)}(A_{r,s})} \qquad \forall v\in BV(A_{r,s},T)$

*s.t.* $\operatorname{support}(u-v)\subset A_{r,s}$ *and* $|v(x)-t_{j_0}|\leq|u(x)-t_{j_0}| \qquad \forall x\in A_{r,s}$;

($H_2''$) $\quad |A_{r,s}-U_{j_0}|<\eta s^n$

*then* $\forall r_0, s_0$ *verifying*

$$r\leq r_0<r_0+s_0\leq r+s, \qquad s_0\geq\gamma s$$

*there exists* $r_1$ *such that, if* $s_1=\sigma s_0$, *one has*

$$r_0+s_0/3<r_1<r_1+s_1<r_0+2s_0/3,$$

$$s_1^{-n}|A_{r_1,s_1}-U_{j_0}|\leq\frac{1}{2}s^{-n}|A_{r,s}-U_{j_0}|.$$

We remark explicitly that, in order to simplify the notation, we have denoted in the same way (i.e. $A_{r_1,s_1}$) the two rings in the statement of Lemmas 4.1 and 4.3.

Moreover we recall that $U_j=\{x\in A_{r,s}: u(x)=t_j\}$.

PROOF OF LEMMA 4.3.

*Step* 1. We begin by choosing, for every positive integer $k$, $k^2$ intermediate rings located on the «middle third» of the original ring $A_{r_0,s_0}$, on whose boundaries the sets $U_j$ will have $(n-1)$-dimensional measures that are suitably controlled. To this end, for a fixed $k\in\mathbf{N}$ we set

$\forall h \in \{1, \ldots, k^2\}$ and for $\rho \in (r_0 + s_0/3, r_0 + s_0/3 + s_0/3(k^2+1))$:

$$A_h(\rho) = A_{\rho', s'}$$

where $\rho' = \rho + \dfrac{h-1}{k^2+1}\dfrac{s_0}{3}$, and $s' = \dfrac{s_0}{3(k^2+1)}$.

The finiteness of the perimeter of $U_j$ in $A_{r,s}$ implies

$$H^{n-1}(\partial A_h(\rho) \cap \partial^* U_j) = 0 \qquad \forall j = 1, \ldots, m, \;\; \forall h = 1, \ldots, k^2$$

and for every $\rho$ but for at most a countable number of values. Recalling that $\forall j$

$$H^{n-1}(A_{r,s} - [U_j(0) \cup U_j(1) \cup \partial^* U_j]) = 0$$

outside this exceptional set of values one has

$$\partial A_h(\rho) =_{n-1} [\partial A_h(\rho) \cap U_j(0)] \cup [\partial A_h(\rho) \cap U_j(1)]$$

(where $E =_{n-1} F$ means $H^{n-1}(E \Delta F) = 0$) hence one easily gets

$$(*) \qquad \partial A_h(\rho) =_{n-1} \bigcup_{j=1}^{m} [\partial A_h(\rho) \cap U_j(1)]$$

that is

$$(**) \qquad H^{n-1}[\partial A_h(\rho) - U_{j_0}(1)] = \sum_{j \neq j_0} H^{n-1}[\partial A_h(\rho) \cap U_j(1)]$$

which holds $\forall h = 1, \ldots, k^2$ and $\forall \rho$ outside the exceptional set. Since evidently

$$\int_{r_0 + s_0/3}^{r_0 + s_0/3 + s_0/3(k^2+1)} \left( \sum_{h=1}^{k^2} H^{n-1}[\partial A_h(\rho) - U_{j_0}(1)] \right) d\rho \leq 2|A_{r,s} - U_{j_0}|,$$

for every fixed $k \in \mathbb{N}$ it will be possible to find

$$\rho(k) \in (r_0 + s_0/3, r_0 + s_0/3 + s_0/3(k^2+1))$$

such that

$$(***) \quad H^{n-1}[\partial A_h(\rho(k)) \cap \partial^* U_j] = 0 \qquad \forall j = 1, \ldots, m \text{ and } \forall h = 1, \ldots, k^2,$$

$$(****) \quad \sum_{h=1}^{k^2} H^{n-1}[\partial A_h(\rho(k)) - U_{j_0}(1)] \leq 6\,\frac{k^2+1}{\gamma}\,\frac{|A_{r,s} - U_{j_0}|}{s},$$

since $s_0 \geq \gamma s$ by assumption.

*Step* 2. We now use Lemma 2.1 of Section 2. For $\xi = |A_{r,s} - U_{j_0}| > 0$ (otherwise there is nothing to prove), let us assume that

$$(*) \qquad \min_{1 \leqslant h \leqslant k^2} |A_h(\rho(k)) - U_{j_0}| \geqslant 2^{-k}\xi$$

where the $A_h(\rho(k))$'s are the rings constructed in Step 1. Because of Lemma 2.1, we can find $k$ rings $A_{h_i} \equiv A_{h_i}(\rho(k))$, $i = 1, \ldots, k$ such that

$$(**) \qquad \max_{1 \leqslant i \leqslant k} |A_{h_i} - U_{j_0}| \leqslant 2 \min_{1 \leqslant i \leqslant k} |A_{h_i} - U_{j_0}|\,.$$

*Step* 3. Now we set

$$v(x) = \begin{cases} u(x), & \text{for } x \in A_{r,s} - \bigcup_{i=1}^{k} \overline{A}_{h_i}, \\ t_{j_0}, & \text{in } K \equiv \bigcup_{i=1}^{k} \overline{A}_{h_i}. \end{cases}$$

Obviously, $v \in BV(A_{r,s}; T)$, $v = u$ outside the compact $K \subset A_{r,s}$ and $|v - t_{j_0}| \leqslant |u - t_{j_0}|$. By virtue of assumption ($H_1''$) and of the properties ($P_1'$)-($P_3'$) of the functional $\mathcal{F}$ we get:

$$(*) \qquad 0 \leqslant \mathcal{F}(v, A_{r,s}) - \mathcal{F}(u, A_{r,s}) + c_3 \|u - v\|_{L^{n/(n-1)}(A_{r,s})} \leqslant$$

$$\leqslant \mathcal{F}\left(v, \bigcup_{i=1}^{k} A_{h_i}\right) + \mathcal{F}\left(v, \bigcup_{i=1}^{k} \partial A_{h_i}\right) - \mathcal{F}\left(u, \bigcup_{i=1}^{k} A_{h_i}\right) +$$

$$+ c_3 \left( \int_{A_{r,s}} |u - v|^{n/(n-1)}\, dx \right)^{(n-1)/n} \leqslant$$

$$\leqslant c_2 \sum_{i=1}^{k} H^{n-1}(S_v \cap \partial A_{h_i}) - c_1 \sum_{i=1}^{k} H^{n-1}(S_u \cap A_{h_i}) +$$

$$+ c_3 \left( \sum_{i=1}^{k} \int_{A_{h_i}} |u(x) - t_{j_0}|^{n/(n-1)}\, dx \right)^{(n-1)/n}.$$

Now we estimate separately the terms in the last sum.
First, because of the definition of $v$, we have

$$\operatorname*{aplim}_{y \to x} v(y) = t_{j_0} \qquad \forall x \in \partial A_{h_i} \cap U_{j_0}(1)$$

and hence $S_v \cap \partial A_{h_i} \cap U_{j_0}(1) = \emptyset$ $\forall i = 1, \ldots, k$.

It follows that $\forall i = 1, \ldots, k$:

$$H^{n-1}(S_v \cap \partial A_{h_i}) \leq H^{n-1}[\partial A_{h_i} - U_{j_0}(1)].$$

As for the second sum in (*) we note that $\forall i, j$:

$$H^{n-1}[\partial^*(A_{h_i} \cap U_j)] = H^{n-1}[\partial^* U_j \cap A_{h_i}] + H^{n-1}[\partial A_{h_i} \cap U_j(1)]$$

(see [14] and recall (***) of step 1). By virtue of identity (I) established at the beginning of the present chapter and of (**) of step 1 we have

$$H^{n-1}(S_u \cap A_{h_i}) = \frac{1}{2} \sum_{j=1}^{m} H^{n-1}(A_{h_i} \cap \partial^* U_j) \geq$$

$$\geq \frac{1}{2} \sum_{j \neq j_0} H^{n-1}[\partial^*(A_{h_i} \cap U_j)] - \frac{1}{2} H^{n-1}[\partial A_{h_i} - U_{j_0}(1)].$$

Finally $\forall i = 1, \ldots, k$ the following holds:

$$\int_{A_{h_i}} |u(x) - t_{j_0}|^{n/(n-1)}\, dx =$$

$$= \sum_{j=1}^{m} \int_{A_{h_i} \cap U_j} |u(x) - t_{j_0}|^{n/(n-1)}\, dx \leq d^{n/(n-1)} |A_{h_i} - U_{j_0}|.$$

Going back to (*) we have:

$$0 \leq \left(\frac{c_1}{2} + c_2\right) \sum_{i=1}^{k} H^{n-1}[\partial A_{h_i} - U_{j_0}(1)] - \frac{c_1}{2} \sum_{i,j \neq j_0} H^{n-1}[\partial^*(A_{h_i} \cap U_j)] +$$

$$+ c_3 d \left( \sum_{i=1}^{k} |A_{h_i} - U_{j_0}| \right)^{(n-1)/n} \leq \left(\frac{c_1}{2} + c_2\right) \frac{6(k^2+1)}{\gamma} \frac{|A_{r,s} - U_{j_0}|}{s} +$$

$$- \frac{c_1}{2} n \omega_n^{1/n} \sum_{i,j \neq j_0} |A_{h_i} \cap U_j|^{(n-1)/n} + c_3 d \left( \sum_{i=1}^{k} |A_{h_i} - U_{j_0}| \right)^{(n-1)/n}$$

(because of (****) of step 1 and of the isoperimetric inequality ($D_1$))

$$\leq 6\left(\frac{c_1}{2} + c_2\right) \frac{(k^2+1)}{\gamma} \frac{\xi}{s} - \frac{c_1}{2} n \omega_n^{1/n} \sum_{i=1}^{k} |A_{h_i} - U_{j_0}|^{(n-1)/n} +$$

$$+ c_3 d \left( \sum_{i=1}^{k} |A_{h_i} - U_{j_0}| \right)^{(n-1)/n}$$

(by virtue of inequality ($D_4$))

$$\leqslant 6\left(\frac{c_1}{2}+c_2\right)\frac{(k^2+1)}{\gamma}\,\frac{\xi}{s}-\left\{\frac{c_1 n\omega_n^{1/n}}{2}-\frac{2^{(n-1)/n}c_3 d}{k^{1/n}}\right\}\sum_{i=1}^{k}|A_{h_i}-U_{j_0}|^{(n-1)/n}$$

(on account of inequality ($D_5$) and (**) of step 2)

$$\leqslant 6\left(\frac{c_1}{2}+c_2\right)\frac{k^2+1}{\gamma}\,\frac{\xi}{s}-\left\{\frac{c_1 n\omega_n^{1/n}}{2}-\frac{2^{(n-1)/n}c_3 d}{k^{1/n}}\right\}k2^{-k(n-1)/n}\xi^{(n-1)/n}$$

by virtue of (*) of step 2 and provided $k$ is large enough, in such a way that the quantity in curly brackets be positive; this happens, for instance, if

$$k \geqslant k_0 = \operatorname{int}\left\{\frac{2^{2n-1}}{\omega_n}\left[\frac{c_3 d}{nc_1}\right]^n\right\}+1$$

($\operatorname{int}\{x\}$ = integral part of $x$).

Under such assumption it follows that

$$\frac{\xi^{1/n}}{s} \geqslant \frac{c_1}{c_1+2c_2}\left\{\frac{n\omega_n^{1/n}}{2}-2^{(n-1)/n}k^{-1/n}\frac{c_3 d}{c_1}\right\}\frac{k}{k^2+1}\,\frac{\gamma}{3\cdot 2^{k(n-1)/n}} \equiv$$

$$\equiv \psi^{1/n}(n, c_1, c_2, c_3 d, \gamma, k)\,.$$

*Step* 4. Finally if

$$k_1 = \min\{k \geqslant k_0 \colon 2^k \geqslant 2\cdot 3^n\cdot\gamma^{-n}(k^2+1)^n\} = k_1(n, c_1, c_3 d, \gamma)\,,$$

$$\eta = \psi(n, c_1, c_2, c_3 d, \gamma, k_1)\,,$$

and if $\xi/s^n < \eta$, then there must exist $h \in \{1, \dots, k_1^2)$ such that

$$|A_h(\rho(k_1)) - U_{j_0}| < 2^{-k_1}\xi\,;$$

otherwise, (*) of step 2 would hold for $k = k_1$, and step 3 would then lead to a contradiction.

With such a choice of the constant $\eta$ and under the assumptions of the lemma, we have thus found a ring $A_h(\rho(k_1))$, that we denote by $A_{r_1,s_1}$, with

$$s_1 = \frac{s_0}{3(k_1^2+1)} \equiv \sigma s_0 \quad \text{(see step 1)}$$

such that

$$|A_{r_1,s_1} - U_{j_0}| < 2^{-k_1}|A_{r,s} - U_{j_0}| \leqslant \frac{1}{2}\left(\frac{\gamma s}{3(k_1^2+1)}\right)^n \frac{|A_{r,s} - U_{j_0}|}{s^n} \leqslant$$

$$\leqslant \frac{1}{2} s_1^n \frac{|A_{r,s} - U_{j_0}|}{s^n}$$

because of the choice of $k_1$ and since, by assumption $s_0 \geqslant \gamma s$.
Q.E.D.

Our third result provides, under the same assumptions as in Lemma 4.3, a «reverse» isoperimetric inequality.

LEMMA 4.4. *Under the same assumptions as in Lemma* 4.3 *we have in addition that* $\forall r_0, s_0, \varepsilon$ *verifying*

$$\begin{cases} r \leqslant r_0 < r_0 + s_0 \leqslant r + s\,, \\ s_0 \geqslant \gamma s\,, \\ \varepsilon > 0\,, \end{cases}$$

*there exist* $r_\varepsilon, s_\varepsilon$ *such that*

$$\begin{cases} r_0 + s_0/3 < r_\varepsilon < r_\varepsilon + s_\varepsilon < r_0 + 2s_0/3\,, \\ s_\varepsilon^{1-n} H^{n-1}(S_u \cap A_{r_\varepsilon, s_\varepsilon}) \leqslant \varepsilon s^{1-n} |A_{r,s} - U_{j_0}|^{(n-1)/n}\,. \end{cases}$$

PROOF. By repeated application of Lemma 4.3, $\forall i \in \mathrm{N}$ it is possible to find a ring $A_{r_i,s_i}$ with

$$r_0 + s_0/3 \leqslant r_i < r_i + s_i \leqslant r_0 + 2s_0/3\,, \qquad s_i = \sigma^i s_0$$

and

$$(*) \qquad s_i^{-n}|A_{r_i,s_i} - U_{j_0}| \leqslant 2^{-i} s^{-n} |A_{r,s} - U_{j_0}|\,.$$

If, for every fixed $i \in \mathrm{N}$ and for every $\rho \in (r_i, r_i + s_i/2)$ we define

$$A_i(\rho) \equiv A_{\rho, s_i/2}$$

then, since

$$\int_{r_i}^{r_i + s_i/2} H^{n-1}[\partial A_i(\rho) - U_{j_0}(1)]\, d\rho = |A_{r_i,s_i} - U_{j_0}|$$

it will be possible to determine $\rho_i \in (r_i, r_i + s_i/2)$ such that

$$(**) \qquad H^{n-1}[\partial A_i(\rho_i) - U_{j_0}(1)] \leqslant 2s_i^{-1} |A_{r_i, s_i} - U_{j_0}|$$

(see step 1 in the proof of Lemma 4.3).

Let us set now $A_i \equiv A_i(\rho_i)$ and

$$v = \begin{cases} t_{j_0} & \text{in } \overline{A}_i, \\ u & \text{in } A_{r,s} - \overline{A}_i, \end{cases}$$

and let us use assumption ($H_1''$); by arguing as in step 3 of the proof of Lemma 4.3 we get

$$c_1 H^{n-1}(S_u \cap A_i) \leqslant \mathcal{F}(u, A_i) \leqslant \mathcal{F}(v, \overline{A}_i) + c_3 \left( \int_{A_i} |u - t_{j_0}|^{n/(n-1)} dx \right)^{(n-1)/n} \leqslant$$

$$\leqslant c_2 H^{n-1}(S_v \cap \partial A_i) + c_3 d |A_i - U_{j_0}|^{(n-1)/n} \leqslant$$

$$\leqslant 2c_2 s_i^{-1} |A_{r_i, s_i} - U_{j_0}| + c_3 d |A_{r_i, s_i} - U_{j_0}|^{(n-1)/n}$$

on the account of (**) above. It follows that

$$\left(\frac{s_i}{2}\right)^{1-n} H^{n-1}(S_u \cap A_i) \leqslant$$

$$\leqslant \frac{2^n c_2}{c_1} \frac{|A_{r_i, s_i} - U_{j_0}|}{s_i^n} + \frac{2^{n-1} c_3 d}{c_1} \left[ \frac{|A_{r_i, s_i} - U_{j_0}|}{s_i^n} \right]^{(n-1)/n} \leqslant$$

$$\leqslant \left[ \frac{|A_{r,s} - U_{j_0}|}{s^n} \right]^{(n-1)/n} \left\{ \frac{2^n c_2}{c_1} 2^{-i} \eta^{1/n} + \frac{2^{n-1} c_3 d}{c_1} 2^{(1/n - 1)i} \right\}$$

by virtue of (*) above and of ($H_2''$). The proof is achieved by choosing $A_{r_\varepsilon, s_\varepsilon} = A_i = A_{\rho_i, s_i/2}$, with $i$ suitable large depending on $\varepsilon$.

Notice that in such a way we have $s_\varepsilon = s_i/2 = (\sigma^i s_0)/2$, where $i = i(\varepsilon)$ depends on $\varepsilon$ and on the constants $n, c_1, c_2, c_3 d$, and where $\sigma$ is the constant of Lemma 4.3. Q.E.D.

We are now ready for the:

Proof of Lemma 4.1. Set $\beta = \min\{\alpha_5, \alpha_6^{-1} \eta^{(n-1)/n}\}$ where $\alpha_5$ and $\alpha_6$ are the constants of Lemma 4.2 and $\eta$ that of Lemma 4.3. Under the assumption ($H_2'$), by virtue of Lemma 4.2 we have, for a suitable $j_0 \in \{1, \ldots, m\}$

$$(*) \quad |A_{r,s} - U_{j_0}|^{(n-1)/n} \leq \alpha_6 H^{n-1}(S_u \cap A_{r,s}) < \alpha_6 \beta s^{n-1} < \eta^{(n-1)/n} s^{n-1}$$

and Lemma 4.4 can then be applied. Setting $\bar{\varepsilon} = 1/2\alpha_6$ and calling $r_1, s_1$ the corresponding $r_{\bar{\varepsilon}}, s_{\bar{\varepsilon}}$ provided by Lemma 4.4 (notice that $s_1 = \tau s_0$ with $\tau = \sigma^{i(\bar{\varepsilon})}/2$; see the closing part of the proof of Lemma 4.4), we obtain:

$$s_1^{1-n} H^{n-1}(S_u \cap A_{r_1, s_1}) \leq \frac{1}{2\alpha_6} s^{1-n} |A_{r,s} - U_{j_0}|^{(n-1)/n}$$

and hence the assertion by virtue of (*) above. Q.E.D.

As in Section 3, Lemma 3.1, from the preceding results we can deduce the following «fracture lemma»:

LEMMA 4.5. *Under the assumptions of Lemma* 4.1 *or, indifferently, under those of Lemma* 4.3 *we have that, for every* $r_0, s_0$ *verifying:*

$$r \leq r_0 < r_0 + s_0 \leq r + s, \qquad s_0 \geq \gamma s$$

*there exists* $\bar{r} \in (r_0 + s_0/3, r_0 + 2s_0/3)$ *such that*

$$S_u \cap \partial B_{\bar{r}} = \emptyset .$$

Indeed, under the assumptions of Lemma 4.3 we obtain, proceeding as in the proof of Lemma 3.1:

$$x \in U_{j_0}(1) \qquad \forall x \in \partial B_{\bar{r}} .$$

The following lemma corresponds to the «elimination result» of Section 3 (Lemma 3.4); the «optimal threshold» is now expressed in terms of the product $c_3 d$.

LEMMA 4.6. *Under the assumptions* ($H_0$), ($H_1'$) *of Lemma* 4.1 (*or, indifferently,* ($H_0$), ($H_1''$) *of Lemma* 4.3), *we suppose in addition that* $\bar{r}_1, \ldots, \bar{r}_{k+1}$ *verify:*

$$\begin{cases} r < \bar{r}_1 < \ldots < \bar{r}_{k+1} < r + s , \\ \partial B_{\bar{r}_i} \subset U_{j_0}(1) \quad \forall i = 1, \ldots, k+1 , \end{cases}$$

*(see the comment following Lemma* 4.5*). Setting* $\bar{s}_i = \bar{r}_{i+1} - \bar{r}_i$*, we have*

(i) *if* $c_3 d < n\omega_n^{1/n} c_1$ *then* $|A_{\bar{r}_i, \bar{s}_i} - U_{j_0}| = 0 \quad \forall i = 1, \dots, k$;

(ii) *if* $c_3 d = n\omega_n^{1/n} c_1$ *then there exists at most one index* $i_0 \in \{1, \dots, k\}$ *s.t.* $|A_{\bar{r}_{i_0}, \bar{s}_{i_0}} - U_{j_0}| > 0$*; actually, in such a case* $A_{\bar{r}_{i_0}, \bar{s}_{i_0}}$ *is a n-dimensional ball on which u is constant.*

The proof is a slight modification of that of Lemma 3.4 (see also the proof of Lemma 4.4).

As a straighforward application of the results obtained so far, we derive the following theorem, which is a first important step in the study of the regularization of the jump set of functions taking on a finite number of values and minimizing functionals of the type of the area (see [8], [15]).

THEOREM 4.7. *Let* $\Omega$ *be an open subset of* $\mathbb{R}^n$, $\mathcal{F} \in F_{c_1, c_2}(\Omega; T)$ *with T finite and* $d = \operatorname{diam} T$*. Assume that* $u \in BV(\Omega, T)$ *verifies*

$$\mathcal{F}(u, A) \leq \mathcal{F}(v, A) + c_3 \|u - v\|_{L^{n/(n-1)}(A)}$$

*for every open* $A \subset\subset \Omega$ *and for every* $v \in BV(\Omega; T)$ *with support* $(u - v) \subset A$.

*If in addition we have*

$$c_3 d \leq n\omega_n^{1/n} c_1$$

*then*

(i) $$S_u \cap \Omega = \overline{S}_u \cap \Omega;$$

(ii) $$\liminf_{\rho \to 0} \rho^{1-n} H^{n-1}(S_u \cap B_{x,\rho}) \geq \beta > 0 \quad \forall x \in \overline{S}_u \cap \Omega,$$

*where* $\beta = \beta(n, c_1, c_2, c_3 d)$ *is the constant of Lemma* 4.1 *corresponding to the choice* $\gamma = 1$.

PROOF. For simplicity, let us assume that $0 \in \Omega - S_u$; for a suitable $t_{j_0} \in T$ we then have $0 \in U_{j_0}(1)$; therefore there exists $R > 0$ s.t. $B_R \subset\subset \Omega$, $|B_R - U_{j_0}| < \eta \mathrm{R}^n$ ($\eta$ is the constant of Lemma 4.3, with $\gamma = 1$).

Lemma 4.5 (with $r = 0$, $s = R$, $\gamma = 1$) then yields $\bar{r} \in (R/3, 2R/3)$

such that $\partial B_{\bar{r}} \subset U_{j_0}(1)$; setting

$$v = \begin{cases} t_{j_0} & \text{in } \overline{B}_{\bar{r}}\,, \\ u & \text{in } \Omega - \overline{B}_{\bar{r}}\,, \end{cases}$$

and arguing as in the proof of Lemma 3.4 or 4.6 (using the assumptions of the Theorem and the isoperimetric inequality ($D_1$)) we obtain either that $B_{\bar{r}} - U_{j_0}$ is empty or that it is a ball on which $u$ is constant: since $0 \in U_{j_0}(1)$, we have in any case $0 \notin \overline{S}_u$ hence (i).

Assertion (ii) is obtained in an analogous manner, by the preliminary use of Lemma 4.2 (recall the choice of $\beta$ in the proof of Lemma 4.1). Q.E.D.

*Acknowledgement.* We would like to thank Prof. E. De Giorgi for helpful discussions during the preparation of this work.

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Manoscritto pervenuto in redazione il 13 agosto 1990.