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On oscillation, continuation, and asymptotic expansions of solutions of linear differential equations

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On Oscillation, Continuation, and Asymptotic Expansions of Solutions of Linear Differential Equations.

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1. Introduction.

There is a classical result due mainly to E. Hille (see [13; p. 345] or [22; p. 282]) which states that for any second-order linear differential equation

$$w'' + P(z)w' + Q(z)w = 0, \tag{1.1}$$

where $P(z)$ and $Q(z)$ are polynomials, there exist finitely many rays, $\arg z = \varphi_j$, for $j = 1, \ldots, m$, (which can be explicitely calculated from the equation), with the property that for any $\varepsilon > 0$, all but finitely many zeros of any solution $f \not\equiv 0$ must lie in the union of the sectors $|\arg z - \varphi_j| < \varepsilon$ for $j = 1, \ldots, m$.

In [1], [6], and [7], an investigation was carried out to determine the corresponding situation for higher-order equations,

$$w^{(n)} + R_{n-1}(z)w^{(n-1)} + \ldots + R_0(z)w = 0. \tag{1.2}$$

It was first shown in [6] that when the coefficients $R_j(z)$ are polynomials, the situation for $n > 2$ can be far different than that for $n = 2$, since equations of order $n > 2$ can have the following property (which we call the *global oscillation property*): For any ray, $\arg z = \varphi$, and any $\varepsilon > 0$, there is a solution $f \not\equiv 0$ having infinitely many zeros in the sector, $|\arg z - \varphi| < \varepsilon$. The examples having the global oscillation property

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which were constructed in [6], are,

$$w^{(n)} + z^2 w'' + zw' + w = 0, \qquad \text{for any } n > 2. \tag{1.3}$$

In [7], a rather complete answer was given to the question of determining the situation concerning the possible location of the zeros of solutions of (1.2) when $n \geqslant 2$ and the $R_j(z)$ are any rational functions, by proving the following result [7; Theorem 1]:

THEOREM 1. Given the equation (1.2), where $n \geqslant 1$ and $R_0(z), \dots, R_{n-1}(z)$ are any rational functions. Then, one of the following holds:

(*A*) For any θ in $(-\pi, \pi)$ and any $\varepsilon > 0$, there exist positive constants δ and K, with $\delta < \min\{\varepsilon, \theta + \pi, \pi - \theta\}$, and a solution $f \not\equiv 0$ of (1.2) such that f is analytic and has infinitely many zeros $z_1, z_2, \dots$, with $\lim_{m \to \infty} |z_m| = +\infty$, on the region defined by $|\operatorname{Arg} z - \theta| < \delta$ and $|z| > K$.

(*B*) There exist a positive integer λ and real numbers $\sigma_1, \dots, \sigma_\lambda$ lying in $(-\pi, \pi]$ such that for any $\varepsilon > 0$ and any solution $f \not\equiv 0$ of (1.2) which is meromorphic on the plane, all but finitely many zeros of f lie in the union for $k = 1, \dots, \lambda$, of the sectors, $|\arg z - \sigma_k| < \varepsilon$.

It was also shown in [7] that there is an effective method for deciding which property (*A*) or (*B*) in Theorem 1 holds for a given equation (1.2), and in the case where Property (*B*) holds, our methods produced the set $\{\sigma_1, \dots, \sigma_\lambda\}$. (These results from [7] are stated in §§ 4, 6 below for the reader's convenience.) However (see § 11), it is not difficult to construct examples where the set $\{\sigma_1, \dots, \sigma_\lambda\}$ produced by our method in [7] contains extraneous elements σ_j, in the sense that in some ε-sector, $|\arg z - \sigma_j| < \varepsilon$, no solution $f \not\equiv 0$ of (1.2) has infinitely many zeros. One of the main results of the present paper (Theorem B in § 7), sets forth a simple method for deciding which (if any) of the numbers σ_j are extraneous in the above sense.

In order to explain the method in Theorem B, it is necessary to discuss how the set $\{\sigma_1, \dots, \sigma_\lambda\}$ is produced. It was shown in [7] that for any equation (1.2) whose coefficients $R_j(z)$ are rational functions, there exist functions $W_1, \dots, W_n, M_1, \dots, M_n$ (which can all be explicitely calculated from the equation) such that: (*a*) Each $W_j(z)$ is either identically zero or is an analytic function which possesses an asymptotic expansion as $z \to \infty$ in the slit plane, in terms of decreasing powers of z; (*b*) Each $M_j(z)$ is a function of the form $z^{\alpha_j} (\operatorname{Log} z)^{k_j}$, for some complex number α_j and some nonnegative integer k_j; (*c*) In sectorial regions S of the plane, the equation possesses a fundamental set of solutions $f_1, \dots, f_n$, of the form,

$$(1.4) \qquad f_j(z) = \psi_j(z) \exp \int W_j(z),$$

where for each j, the function $\psi_j(z)$ is analytic in S and satisfies $\psi_j(z)/M_j(z) \to 1$ as $z \to \infty$ in S. (In the present paper, the set of functions,

$$(1.5) \qquad \left\{M_1 \exp \int W_1, M_2 \exp \int W_2, \ldots, M_n \exp \int W_n\right\},$$

will be called the *asymptotic set* for (1.2), and any corresponding fundamental set $\{f_1, \ldots, f_n\}$ satisfying (1.4) will be called a *basic fundamental set* in S for (1.2).) Roughly speaking, the rays $\arg z = \sigma_j$ in Part (B) of Theorem 1 which are produced by the method in [7] consist mainly of two types: (i) The boundary rays of the sectors S (where the representation (1.4) is valid), and (ii) all rays where the asymptotic behavior (as $z \to \infty$) of some ratio of distinct elements in the asymptotic set changes from «large» to «small». Our result in Theorem B shows that all rays of Type (ii) are non-extraneous, and only those rays of Type (i) which are also of Type (ii) are non-extraneous. This is accomplished by first proving a «continuation» result (Theorem A in § 7) which produces from two basic fundamental sets in adjoining sectors, a third basic fundamental set in the union of the two sectors (including the dividing ray). The proof makes extensive use of the Phragmen-Lindelöf principle.

Our final result (Theorem C in § 13) concerns the precise determination of the asymptotic behavior of the elements f_j in a basic fundamental set. As mentioned earlier, the asymptotic expansions in the slit plane of the functions W_j in (1.4) can be explicitely calculated by the methods in [7]. The function $\psi_j(z)$ appearing in the basic fundamental set (1.4) obviously satisfies the linear differential equation in u obtained from (1.2) by the change of dependent variable $y = (\exp \int W_j) u$. When this latter equation is divided by $\exp \int W_j$, we obtain a linear differential equation in u whose coefficients all possess asymptotic expansions in terms of decreasing powers of z, as $z \to \infty$ in the slit plane. In Theorem C, we show that one can choose the $\psi_j(z)$ to have a generalized asymptotic expansion (in known sectors and to any number of terms desired) in terms of asymptotically decreasing functions of the form $z^a (\operatorname{Log} z)^b$, which can all be calculated in advance from the original equation (1.2). Thus, for these choices of the functions $\psi_j(z)$, the asymptotic behavior of the functions $f_j(z)$ in the basic fundamental set (1.4), is known precisely.

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2. Concepts from the Strodt theory [16].

(*a*) [16; § 94]: *The neighborhood system* $F(a, b)$. Let $-\pi \leqslant a < b \leqslant \pi$. For each nonnegative real-valued function g on $(0, (b - a)/2)$, let $V(g)$ be the union (over all $\delta \in (0, (b-a)/2)$) of all sectors, $a + \delta < \operatorname{Arg}(z - h(\delta)) < b - \delta$, where $h(\delta) = g(\delta) \exp[i(a+b)/2]$. The set of all $V(g)$ (for all choices of g) is denoted $F(a, b)$, and is a filter base which converges to ∞. Each $V(g)$ is a simply-connected region (see [16; § 93]), and we require the following simple fact which is proved in [6; § 2]:

LEMMA 2.1. Let V be an element of $F(a, b)$, and let $\varepsilon > 0$ be arbitrary. Then there is a constant $R_0(\varepsilon) > 0$ such that V contains the set, $a + \varepsilon \leqslant \operatorname{Arg} z \leqslant b - \varepsilon$, $|z| \geqslant R_0(\varepsilon)$.

(*b*) [16; § 13]. *The relation of asymptotic equivalence.* If $f(z)$ is an analytic function on some element of $F(a, b)$, then $f(z)$ is called *admissible* in $F(a, b)$. If c is a complex number, then the statement $f \to c$ in $F(a, b)$ means (as is customary) that for any $\varepsilon > 0$, there exists an element V of $F(a, b)$ such that $|f(z) - c| < \varepsilon$ for all $z \in V$. The statement $f \ll 1$ in $F(a, b)$, means that in addition to $f \to 0$, all the functions $\theta_j^k f \to 0$ in $F(a, b)$, where θ_j denotes the operator $\theta_j f = z(\operatorname{Log} z)\ldots(\operatorname{Log}_{j-1} z) f'(z)$, and where (for $k \geqslant 0$), θ_j^k is the k-th iterate of θ_j. The statements $f_1 \ll f_2$ and $f_1 \sim f_2$ in $F(a, b)$ mean respectively $f_1/f_2 \ll 1$ and $f_1 - f_2 \ll f_2$. (As usual, z^α and $\operatorname{Log} z$ will denote the principal branches of these functions on $|\operatorname{Arg} z| < \pi$.) We will write $f_1 \approx f_2$ to mean $f_1 \sim c f_2$ for some nonzero constant c. (We remark that this strong relation of asymptotic equivalence is designed to ensure that if $f \ll 1$ in $F(a, b)$, then $\theta_j f \ll 1$ in $F(a, b)$ for all $j \geqslant 1$. (See [16; § 28]).) If $f \sim c z^{-1+d}$ in $F(a, b)$, where $c \neq 0$ and $d \geqslant 0$, then the indicial function of f is the function,

$$IF(f, \phi) = \operatorname{Cos}(d\phi + \arg c) \quad \text{for } a < \phi < b. \tag{2.1}$$

If g is any admissible function in $F(a, b)$, we will denote by $\int g$, a primitive of g in an element of $F(a, b)$. We will require the following two results, (see [7; § 2]:

LEMMA 2.2. Let $f \sim c z^{-1+d}$ in $F(a, b)$, where $c \neq 0$ and $d > 0$. If (a_1, b_1) is any subinterval of (a, b) on which $IF(f, \phi) < 0$ (respectively, $IF(f, \phi) > 0$), then for all real α, $\exp\int f \ll z^\alpha$ (respectively, $\exp\int f \gg z^\alpha$) in $F(a_1, b_1)$.

LEMMA 2.3. Let $\alpha = a + bi$ be a complex number. Then for any $\varepsilon > 0$, we have $z^{a-\varepsilon} \ll z^{\alpha}$ and $z^{\alpha} \ll z^{a+\varepsilon}$ in $F(-\pi, \pi)$.

(*c*) [18; p. 244]. *Logarithmic Fields.* A function of the form cz^{α}, for complex $c \neq 0$ and real α, is called a logarithmic monomial of rank zero. The set of all logarithmic monomials of rank zero will be denoted Φ_0. A *logarithmic differential field of rank zero over* $F(a, b)$ is a set Γ of functions, each defined and admissible in $F(a, b)$, with the following properties: (i) Γ is a differential field (where, as usual, we identify two elements of Γ if they agree on an element of $F(a, b)$); (ii) Γ contains Φ_0; (iii) for every element f in Γ except zero, there exists M in Φ_0 such that $f \sim M$ over $F(a, b)$. (The simplest example of such a field is the set of rational combinations of the elements of Φ_0.) If $f \sim cz^{\alpha}$ over $F(a, b)$, we will denote α by $\delta_0(f)$. If $f \equiv 0$, we will set $\delta_0(f) = -\infty$.

Now let $G(z, v) = \sum_{j=0}^{n} f_j(z) v^j$ be a polynomial in v of degree $n \geqslant 1$ whose coefficients belong to a logarithmic differential field of rank zero in $F(a, b)$. A logarithmic monomial $M = cz^{\alpha}$ of rank zero is called a *critical monomial* of G if there exists an admissible function $h \sim M$ in $F(a, b)$ for which $G(z, h(z))$ is not $\sim G(z, M(z))$ in $F(a, b)$. The multiplicity of M is the smallest positive integer j such that M is not a critical monomial of $\partial^j G/\partial v^j$. There is an algorithm (see [5; § 26]) which produces the sequence (counting multiplicity) of critical monomials of $G(z, v)$. (By [5; § 29], the sequence has $n - d$ members, where d is the smallest $k \geqslant 0$ for which $f_k \not\equiv 0$.) The algorithm is based on a Newton polygon method (e.g., [12; p. 105]). (One simply finds the values of α which have the following properties: When $v = cz^{\alpha}$ is inserted into the individual terms in G, at least two such terms have the same δ_0, and this value of δ_0 is at least as large as the other terms produce. The constant c is then determined by requiring that the terms with the largest δ_0 cancel.) The crititical monomials of G give the first terms of the asymptotic expansions of the roots of G. This is shown by the following fact:

LEMMA 2.4. Let $G(z, v) = \sum_{j=0}^{n} f_j v^j$ be a polynomial in v of degree $n \geqslant 1$, whose coefficients $f_0, \ldots, f_n$ are elements of a logarithmic differential field of rank zero over $F(a, b)$. Then

(*a*) There exists an extension logarithmic differential field of rank zero over $F(a, b)$, in which $G(z, v)$ factors completely.

(*b*) If M is a simple critical monomial of $G(z, v)$, then there exists a unique admissible function $h(z)$ in $F(a, b)$ having the following two properties: (i) $h \sim M$ in $F(a, b)$, and (ii) $G(z, h(z)) \equiv 0$. In addition, the function $h(z)$ belongs to a logarithmic differential field of rank zero over $F(a, b)$.

PROOF. Part (*a*) is proved in [18; Theorem II, p. 244]. Part (*b*) follows easily from [18; §§ 24, 26] and from Part (*a*).

When $f_0 \not\equiv 0$ in $G(z, v)$, the polynomial $G(z, v)$ possesses one or more special critical monomials, $M = cz^\alpha$, called *principal monomials* (see [16; § 67]) which arise as follows: When $v = cz^\alpha$ is inserted into the individual terms of $G(z, v)$, the power $\delta_0(f_0)$ is at least as large as the δ_0 produced by the other terms. The principal monomials are the critical monomials which are of minimal rate of growth in $F(-\pi, \pi)$ (see [16; § 67]). We will require the following facts which are proved in [5; §§ 3, 31(*c*)]:

LEMMA 2.5. *Let M be a simple critical monomial of a polynomial $G(z, v)$ whose coefficients belong to a logarithmic differential field of rank zero over $F(a, b)$, and assume $G(z, M(z)) \not\equiv 0$. Let $G_1(z, w) = G(z, M(z) + w)$. Then $G_1(z, w)$ possesses a unique principal monomial M_1. In addition, M_1 is simple, and $M_1 \ll M$ in $F(-\pi, \pi)$.*

3. Preliminaries.

Given an equation (1.2) where the $R_j(z)$ are functions which belong to a logarithmic differential field of rank zero over $F(a, b)$, we first rewrite the equation in terms of the operator θ which is defined by $\theta w = zw'$. (It is easy to prove by induction that for each $m = 1, 2, \ldots$,

$$w^{(m)} = z^{-m}\left(\sum_{j=1}^{m} b_{jm}\,\theta^j w\right), \tag{3.1}$$

where θ^j is the j-th iterate of the operator θ, and where the b_{jm} are integers with $b_{mm} = 1$. In fact, as polynomials in x,

$$\sum_{j=1}^{n} b_{jn}\, x^j = x(x-1)\ldots(x-(n-1)).\,) \tag{3.2}$$

When written in terms of θ, let (1.2) have the form

$$\sum_{j=0}^{n} B_j(z)\,\theta^j w = 0. \tag{3.3}$$

(Of course, the $B_j(z)$ belong to the same field as the $R_j(z)$.) By dividing equation (3.3) through by z^d where d is the maximum of $\delta_0(B_j)$ for $j = 0, \ldots, n$, we may assume that for each j, we have either $B_j \ll 1$ or $B_j \approx 1$

in $F(a, b)$, and there exists an integer $p \geqslant 0$ such that $B_j \ll 1$ for $j > p$, while B_p is $\sim$ to a nonzero constant (denoted $B_p(\infty)$). The integer p is called the *critical degree* of the equation (1.2). The equation,

$$F^*(\alpha) = \sum_{j=0}^{n} B_j(\infty)\, \alpha^j = 0, \tag{3.4}$$

is called the *critical equation* of (1.2). Clearly, $F^*(\alpha)$ is a polynomial in α, of degree p, having constant coefficients. Let the distinct roots of $F^*(\alpha)$ be $\alpha_0, \ldots, \alpha_r$, with α_k having multiplicity m_k. (Thus, $\sum m_k = p$.) Let $M_1, \ldots, M_p$ be the p distinct functions of the form $z^{\alpha_k}(\operatorname{Log} z)^j$ for $0 \leqslant k \leqslant r$, and integers j satisfying $0 \leqslant j \leqslant m_k - 1$. We call the set $\{M_1, \ldots, M_p\}$, the *logarithmic set* for (1.2).

When (1.2) is written in the form (3.3), we form the algebraic polynomial in v,

$$H(z, v) = \sum_{j=0}^{n} z^j B_j(z)\, v^j, \tag{3.5}$$

which we will call the *full factorization polyomial* for (1.2). Clearly, the coefficients of $H(z, v)$ belong to the same logarithmic differential field as do the coefficients of (1.2). If p is the critical degree of (1.2), it is shown in [14; Lemma 6.1], that $H(z, v)$ possesses precisely $n - p$ critical monomials $N_1, \ldots, N_{n-p}$, (counting multiplicity) satisfying $\delta_0(N_j) > -1$. We will call the set $\{N_1, \ldots, N_{n-p}\}$, the *exponential set* for (1.1). If T_j is the set of zeros on (a, b) of the function $IF(N_j, \varphi)$ (see (2.1)), then the union of the sets T_j for $j = 1, \ldots, n-p$, will be called the *transition set* for (1.2) on (a, b).

4. A result from [7].

THEOREM 2. Let $n \geqslant 1$, and let $R_0, R_1, \ldots, R_{n-1}$ belong to a logarithmic differential field of rank zero over $F(a, b)$. Let $\Lambda(w)$ be the n-th order linear differential operator,

$$\Lambda(w) = w^{(n)} + R_{n-1}(z)\, w^{(n-1)} + \ldots + R_0(z)\, w. \tag{4.1}$$

Let p be the critical degree of $\Lambda(w) = 0$, and let $\{M_1, \ldots, M_p\}$ be the logarithmic set for this equation. Let $r_1 < r_2 < \ldots r_t$ be the transition set for $\Lambda(w) = 0$, and set $r_0 = a$ and $r_{t+1} = b$. (If the transition set is empty, set $t = 0$.) Then, in each of $F(r_0, r_1)$, $F(r_1, r_2), \ldots, F(r_t, r_{t+1})$ separately, the following conclusion holds: For each j, with $1 \leqslant j \leqslant p$, there exists an admissible solution $\varphi_j(z)$ of $\Lambda(w) = 0$, satisfying $\varphi_j \sim M_j$.

REMARK. In view of Theorem 2, we made the following definition in [7]:

DEFINITION 4.1. Under the hypothesis and notation of Theorem 2, if $\{\psi_1, \ldots, \psi_p\}$ is a set of admissible functions in some $F(c, d)$, such that ψ_j is a solution of $\Lambda(w) = 0$ and satisfies $\psi_j \sim M_j$ in $F(c, d)$ for $j = 1, \ldots, p$, then we will call $\{\psi_1, \ldots, \psi_p\}$ a *complete logarithmic set of solutions* of $\Lambda(w) = 0$ in $F(c, d)$. (Thus Theorem 2 asserts the existence of complete logarithmic sets of solutions in each of $F(r_0, r_1), \ldots, F(r_t, r_{t+1})$ separately.)

5. Concepts and notation from [14] and [7].

Let,

$$\Omega(w) = \sum_{j=0}^{n} B_j(z)\, \theta^j w, \tag{5.1}$$

be an n-th order linear differential operator whose coefficients $B_0, \ldots, B_n$ belong to a logarithmic differential field $\mathcal{K}$ of rank zero over $F(a, b)$ and assume $B_n \not\equiv 0$. (As in § 3, $\theta w = zw'$.) Let W belong to an extension logarithmic differential field $\mathcal{K}_1$ of rank zero over $F(a, b)$, and assume $W \gg z^{-1}$ in $F(a, b)$. Set $h = \exp\int W$, and let $\Lambda(v)$ be the operator defined by $\Lambda(v) = \Omega(hv)/h$. Then $\Lambda(v)$ has coefficients belonging to $\mathcal{K}_1$, and we denote,

$$\Lambda(v) = \sum_{j=0}^{n} B_j[W]\, \theta^j v. \tag{5.2}$$

Let $H(u)$ and $K(u)$ denote respectively, the full factorization polynomials for $\Omega(w)$ and $\Lambda(v)$, so that,

$$H(u) = \sum_{j=0}^{n} z^j B_j u^j \quad \text{and} \quad K(u) = \sum_{j=0}^{n} z^j B_j[W]\, u^j. \tag{5.3}$$

In [14; § 10], the following concept is introduced: W is said to have transform type (m, q) with respect to H (briefly, $\operatorname{trt}(W, H) = (m, q)$) if Λ has critical degree m, and if q is the minimum multiplicity of all critical monomials M of $K(u)$ which satisfy $z^{-1} \ll M \ll W$ in $F(a, b)$. (If there are no such M, then we set $q = 0$.) The following results are proved in [14; § 10]:

LEMMA 5.1. With the above notation, assume $W \sim N$ in $F(a, b)$ where N is a critical monomial of $H(u)$ of multiplicity d, satisfying

$N \gg z^{-1}$, and assume that $\operatorname{trt}(W, H) = (m, q)$. Then:

(*a*) $K(u)$ has precisely $d - m$ critical monomials L satisfying $z^{-1} \ll L \ll W$, counting multiplicity.

(*b*) We have $m + q \leqslant d$.

(*c*) If $q = 0$, then $m = d$.

(*d*) If $(m, q) = (0, d)$, and we set,

$$G(u) = \sum_{k=d-1}^{n} \begin{bmatrix} k \\ d-1 \end{bmatrix} B_k(z)\, z^{k-(d-1)} (W + u)^{k-(d-1)}, \tag{5.4}$$

then $G(u)$ possesses a unique principal monomial V. In addition, V has the following properties: (i) V is a simple critical monomial of G; (ii) $V \ll W$; (iii) There is a unique function g satisfying $g \sim V$ in $F(a, b)$ and $G(g) \equiv 0$; (iv) If $U = W + g$, then $U \sim W$ in $F(a, b)$, and $\operatorname{trt}(U, H) = (m_1, q_1)$ where $q_1 < d$.

(REMARK. Conclusions (*a*)-(*c*) are proved in [14; Lemma 10.3]. The conclusion (*d*) follows from [14; Lemmas 10.5, 8.5] and from Lemma 2.4 above.)

In view of Parts (*b*) and (*d*) of Lemma 5.1, we introduced the following notations in [7]:

DEFINITION 5.2. With the above notation, let N be a critical monomial of $H(u)$ of multiplicity d, satisfying $N \gg z^{-1}$, and let $\operatorname{trt}(N, H) = (m, q)$. By Part (*b*), we have $q \leqslant d$. If $q < d$, set $N^* = N$. If $q = d$, then by Part (*b*), we have $m = 0$. We set $N^* = U$, where U is the function in Lemma 5.1, Part (*d*) which is constructed by taking W equal to N. Hence, in all cases, we have,

$$N^* \sim N \quad \text{and} \quad \operatorname{trt}(N^*, H) = (m_1, q_1) \quad \text{where} \quad q_1 < d. \tag{5.5}$$

(REMARK. The *-operation to form N^* depends upon the polynomial $H(u)$, and we will indicate this, where necessary, by saying that it is relative to $H(u)$.)

DEFINITION 5.3. Let $\Omega(w)$ and $H(u)$ be as in (5.1) and (5.3), and let $N \gg z^{-1}$ be a critical monomial of $H(u)$ of multiplicity d. A finite sequence $(V_0, V_1, \ldots, V_r)$, where r is a nonnegative integer, and where the V_j are elements of an extension logarithmic differential field of $\mathcal{K}$, of rank zero over $F(a, b)$, will be called an *N-sequence for* Ω if and only if the following conditions are satisfied: (i) $V_0 = N^*$; (ii) If $r \geqslant 1$, then there is a critical monomial M_1 of

(5.6) $$K_1(u) = \sum_{j=0}^{n} z^j B_j[V_0] u^j,$$

satisfying $z^{-1} \ll M_1 \ll V_0$, such that $V_1 = M_1^*$ (where the *-operation is relative to K_1), and in general, for $1 \leq k \leq r$, there is a critical monomial M_k of

(5.7) $$K_k(u) = \sum_{j=0}^{n} z^j B_j[V_0 + V_1 + \ldots + V_{k-1}] u^j$$

satisfying,

(5.8) $$z^{-1} \ll M_k \ll V_{k-1} \quad \text{and} \quad V_k = M_k^*,$$

where the *-operation in (5.8) is relative to K_k. The set of all N-sequences for Ω will be denoted $\mathcal{D}(N, \Omega)$. If $V^\# = (V_0, \ldots, V_r)$ is an N-sequence for Ω, let $\Lambda_0 = \Omega$, $K_0 = H$, and for $1 \leq k \leq r+1$, set

(5.9) $$\Lambda_k(v) = \sum_{j=0}^{n} B_j[V_0 + V_1 + \ldots + V_{k-1}] \theta^k v.$$

The equation, $\Lambda_{r+1}(v) = 0$, will be called the *terminal* equation for $V^\#$, and its critical degree will be called the *terminal index* for $V^\#$, and will be denoted $t(V^\#)$. We will say that $V^\#$ is *active* if $t(V^\#) > 0$, and we denote the set of all active N-sequences for Ω by $\mathcal{D}_1(N, \Omega)$.

6. Further results from [7].

LEMMA 6.1. Let $\Omega(w) = \sum_{j=0}^{n} B_j \theta^j w$ where the B_j belong to a logarithmic differential field of rank zero over $F(a, b)$, and assume $B_n \not\equiv 0$. Let $N \gg z^{-1}$ be a critical monomial of multiplicity d of the full factorization polynomial for Ω. Then,

(6.1) $$\sum \{t(V^\#) \colon V^\# \in \mathcal{D}(N, \Omega)\} = d,$$

and

(6.2) $$\sum \{t(V^\#) \colon V^\# \in \mathcal{D}_1(N, \Omega)\} = d.$$

THEOREM 3. Given the equation (1.2) where $n \geq 1$ and where the functions $R_0(z), \ldots, R_{n-1}(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$. When written in terms of the operator θ (where

$\theta w = zw'$) let (1.2) have the form $\Omega(w) = 0$, where $\Omega(w) = \sum_{j=0}^{n} B_j(z)\,\theta^j w$. Let p be the critical degree of (1.2), and let $N_1, \ldots, N_s$ be the distinct elements (if any) of the exponential set (see § 3) for (1.2). Let E_1 denote the transition set for (1.2) on (a, b). For each $k = 1, \ldots, s$, and each active N_k-sequence, $V^{\#}$, for Ω, let $E(V^{\#})$ denote the transition set for the terminal equation for $V^{\#}$ (see § 5) on (a, b). Let $E = \{r_1, \ldots, r_q\}$, where $r_1 < r_2 < \ldots < r_q$, denote the union of E_1 and all the sets $E(V^{\#})$ as $V^{\#}$ ranges over the sets $\mathcal{O}_1(N_k, \Omega)$ for $k = 1, \ldots, s$. Let (c, d) denote any of the intervals (a, r_1), $(r_1, r_2), \ldots, (r_{q-1}, r_q)$, (r_q, b), (where we take $(c, d) = (a, b)$ if E is empty). Then the following hold:

(*A*) The equation (1.2) possesses a complete logarithmic set of solutions $\{\varphi_1, \ldots, \varphi_p\}$ in $F(c, d)$.

(*B*) If $k \in \{1, \ldots, s\}$, and $V^{\#} = (V_0, \ldots, V_r)$ is an element of $\mathcal{O}_1(N_k, \Omega)$, then the equation (1.2) possesses ($t(V^{\#})$ admissible solutions, $h_1, \ldots, h_{t(V^{\#})}$, in $F(c, d)$ of the form

$$h_j(z) = \psi_j(z)(\exp\textstyle\int(V_0 + \ldots + V_r), \tag{6.3}$$

where $\{\psi_1, \ldots, \psi_{t(V^{\#})}\}$ is a complete logarithmic set of solutions of the terminal equation for $V^{\#}$ in $F(c, d)$.

(*C*) The total number of solutions represented in Parts (*A*) and (*B*) is precisely n, and these n solutions form a fundamental set of solutions for (1.2) in some element of $F(c, d)$.

In view of Theorem 3, we make the following definitions:

Definition 6.2. Assume the hypothesis and notation of Theorem 3. Let $\{M_1, \ldots, M_p\}$ be the logarithmic set for (1.2). If $k \in \{1, \ldots, s\}$ and $V^{\#} = (V_0, \ldots, V_r)$ is an element of $\mathcal{O}_1(N_k, \Omega)$, let $\{P_1, \ldots, P_{t(V^{\#})}\}$ be the logarithmic set for the terminal equation for $V^{\#}$. The set of n functions consisting of $M_1, \ldots, M_p$ and the functions,

$$P_j(z)(\exp\textstyle\int(V_0 + \ldots + V_r)), \quad \text{for} \quad 1 \leq j \leq t(V^{\#}), \tag{6.4}$$

as k ranges over $\{1, \ldots, s\}$ and $V^{\#}$ ranges over $\mathcal{O}_1(N_k, \Omega)$, will be called the *asymptotic set* for (1.2). (We note that the functions M_j and P_j are all functions of the form $z^{\alpha}(\operatorname{Log} z)^{\beta}$ so that the asymptotic set $\{H_1, \ldots, H_n\}$ consists of functions which are admissible in $F(a, b)$. Of course, they need not be solutions of (1.2).) If (c, d) is a subset of (a, b), and if $\{f_1, \ldots, f_n\}$ is a fundamental set of solutions of (1.2), each admissible in $F(c, d)$ and satisfying $f_j/H_j \to 1$ in $F(c, d)$ for $1 \leq j \leq n$ (where $\{H_1, \ldots, H_n\}$ is the asymptotic set for (1.2)), then we will call $\{f_1, \ldots, f_n\}$

a *basic fundamental set* for (1.2) in $F(c, d)$. Thus, Theorem 3 asserts the existence of a basic fundamental set for (1.2) in each of the neighbourhood systems $F(a, r_1)$, $F(r_1, r_2), \ldots, F(r_q, b)$, separately.

DEFINITION 6.3. Assume the hypothesis and notation of Theorem 3. For each $q = 1, \ldots, s$, and any $V^{\#} = (V_0, \ldots, V_r)$ belonging to $\mathcal{D}_1(N_q, \Omega)$, where $r \geqslant 1$, let $K_k(u)$ be given by (5.7) for $1 \leqslant k \leqslant r$. For each $k = 1, \ldots, r$, let $\mathcal{F}_k(V^{\#})$ denote the set of critical monomials M of $K_k(u)$ satisfying $z^{-1} \ll M \ll V_{k-1}$. We define $\mathcal{E}_k(V^{\#})$ to be the union of the sets of zeros on (a, b) of all the functions, $IF(M, \varphi)$ for $M \in \mathcal{F}_k(V^{\#})$, and $IF(M - M_1, \varphi)$ for distinct elements M and M_1 in $\mathcal{F}_k(V^{\#})$. Finally, we let A denote the union of the sets of zeros on (a, b) of all the functions $IF(N_j - N_k, \varphi)$ where $1 \leqslant k < j \leqslant s$. With the set E as defined in the statement of Theorem 3, we now define the *oscillation set* on (a, b) of equation (1.2) to be the union of the sets E, A, and all the sets $\mathcal{E}_k(V^{\#})$ as $V^{\#} = (V_0, \ldots, V_r)$ ranges over $\mathcal{D}_1(N_q, \Omega)$ for all $q = 1, \ldots, s$, and k ranges over $\{1, \ldots, r\}$. (The oscillation set is clearly finite.)

THEOREM 4. Given the equation (1.2), where $n \geqslant 1$, and where the functions $R_0(z), \ldots, R_{n-1}(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$. Let $N_1, \ldots, N_s$ be the distinct elements (if any) of the exponential set for (1.2), and let (1.2) have the form $\Omega(w) = 0$, where $\Omega(w) = \sum_{j=0}^{n} B_j \theta^j w$, when (1.2) is written in terms of $\theta w = zw'$. Then:

(*A*) Assume that (1.2) satisfies at least one of the following two conditions: (i) The critical equation for (1.1) possesses two distinct roots having the same real part; (ii) For some $k, 1 \leqslant k \leqslant s$, there is an element $V^{\#}$ in $\mathcal{D}_1(N_k, \Omega)$ such that the terminal equation for $V^{\#}$ has the property that its critical equation possesses two distinct roots having the same real part. Then (1.2) has the following property: For any ϕ in (a, b) and any $\varepsilon > 0$, there exist positive constants δ and K, and a solution $f \not\equiv 0$ of (1.2) such that,

$$\delta < \min\{\varphi - a, b - \varphi, \varepsilon\}, \tag{6.5}$$

and such that f is analytic and has infinitely many zeros $z_1, z_2, \ldots$, with $\lim_{m \to \infty} |z_m| = +\infty$, on the region defined by,

$$|\operatorname{Arg} z - \varphi| < \delta \quad \text{and} \quad |z| > K. \tag{6.6}$$

(*B*) Assume that (1.2) satisfies neither of the conditions (i) and (ii) in Part (*A*). Let the oscillation set for (1.2) on (a, b) consist of the points $r_1 < r_2 < \ldots < r_q$, and set $r_0 = a$ and $r_{q+1} = b$. (If the oscillation set is empty, then set $q=0$.) Then, for each $j, 0 \leqslant j \leqslant q$, the following three conclusions hold:

(*a*) If $\{H_1, \ldots, H_n\}$ is the asymptotic set for (1.2), then there is a permutation $\{q_1, \ldots, q_n\}$ of $\{1, \ldots, n\}$ (depending on j) such that

$$H_{q_1} \gg H_{q_2} \gg \ldots \gg H_{q_n} \quad \text{in} \quad F(r_j, r_{j+1}). \tag{6.7}$$

(*b*) A basic fundamental set for (1.2) exists in $F(r_j, r_{j+1})$.

(*c*) If $f \not\equiv 0$ is a solution of (1.2) which is admissible in $F(a, b)$, then there is an element of $F(r_j, r_{j+1})$ on which f has no zeros.

REMARK. Part (*A*) and Section (*c*) of Part (*B*) constitute the original Theorem 4 proved in [7]. However, the other two parts are established in the proof of Theorem 4. In view of Theorem 4, we make the following definition:

DEFINITION 6.4. Assume the hypothesis and notation of Part (*B*) of Theorem 4, and let $\{q_1, \ldots, q_n\}$ be as in (6.7). Then we will call the n-tuple $(H_{q_1}, H_{q_2}, \ldots, H_{q_n})$, *the ordered asymptotic system* for (1.2) in $F(r_j, r_{j+1})$

REMARK. In the case considered in Theorem 1 of § 1, namely where the coefficients $R_j(z)$ of (1.2) are rational functions, then in Theorem 4, we can take $a = -\pi$ and $b = \pi$. It is shown in [7], that the set $\{\sigma_1, \ldots, \sigma_\lambda\}$ in Part (*B*) of Theorem 1, can be taken to be $\{r_1, \ldots, r_{q+1}\}$, where $r_1 < r_2 < \ldots < r_q$ are the points of the oscillation set for (1.2) on $(-\pi, \pi)$, and where $r_{q+1} = \pi$. The next section is devoted to determining which (if any) of these points is extraneous in the sense described in § 1.

7. Statement of main results of present paper.

THEOREM A (Continuation). Assume the hypothesis and notation of Part (*B*) of Theorem 4. Then for any j with $1 \leqslant j \leqslant q$, there exists a basic fundamental set for (1.2) in $F(r_{j-1}, r_{j+1})$.

THEOREM B (Oscillation). Assume the hypothesis and notation of Part (*B*) of Theorem 4. Let $j \in \{1, \ldots, q\}$, and let $(H_{q_1}, \ldots, H_{q_n})$ and $(H_{t_1}, \ldots, H_{t_n})$ be respectively, the ordered asymptotic systems for (1.2) in $F(r_j, r_{j+1})$ and in $F(r_{j-1}, r_j)$. Then:

(*a*) If $(H_{q_1}, \ldots, H_{q_n}) = (H_{t_1}, \ldots, H_{t_n})$, then there exists $\delta > 0$ with the property that for any solution $f \not\equiv 0$ of (1.2) which is admissible in $F(a, b)$, there is a constant $R = R(f) > 0$ such that f has no zeros on the set,

$$|\operatorname{Arg} z - r_j| < \delta, \qquad |z| > R. \tag{7.1}$$

(*b*) If $(H_{q_1}, \ldots, H_{q_n}) \neq (H_{t_1}, \ldots, H_{t_n})$, then there exists a solution $f \not\equiv 0$ of (1.2) such that for any $\delta > 0$, f possesses infinitely many zeros $z_1, z_2, \ldots$ satisfying $|z_m| \to +\infty$ as $m \to \infty$, and lying in $|\operatorname{Arg} z - - r_j| < \delta$.

The proofs of these results will be given in §§ 9, 10.

8. Preliminary results for Theorem A and B.

We will require two preliminary results. The first is a combination of several Phragmen-Lindelöf principles whose proofs can be found in [19; pp. 176-180].

LEMMA 8.1. Let $f(z)$ be analytic and of finite order of growth in a closed sectorial region of the form $\alpha \leqslant \arg z \leqslant \beta$, $|z| \geqslant K$. Then, there exists $\delta > 0$ such that for any real numbers c and d, with $\alpha \leqslant c < d \leqslant \beta$ and $d - c < \delta$, for which the limits,

$$L_1 = \lim_{r \to +\infty} f(re^{ic}) \quad \text{and} \quad L_2 = \lim_{r \to +\infty} f(re^{id}), \tag{8.1}$$

exist and are finite, the following conclusions hold: $L_1 = L_2$, and $f(z) \to L_1$ as $z \to \infty$ in $c \leqslant \arg z \leqslant d$.

LEMMA 8.2. Given the equation (1.2) where the coefficients $R_j(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$, and let $f \not\equiv 0$ be a solution of (1.2) which is admissible in $F(a, b)$. Then, for any real numbers c and d, with $a < c < d < b$, there exists $K > 0$ such that f is analytic and of finite order in the closed sectorial region defined by, $c \leqslant \arg z \leqslant d$, $|z| \geqslant K$.

PROOF OF 8.2. The proof of the lemma parallels very closely the proof of the corresponding result for analytic solutions of first-order algebraic differential equations in sectors, which was given in [2]. For this reason, we will simply sketch the proof. As in [2], the first step is to prove a local version of the result, namely:

LEMMA 8.3. Assume the hypothesis and notation of Lemma 8.2. Then for any $\lambda \in (a, b)$, there exist positive real numbers $\delta(\lambda)$ and $K(\lambda)$

such that f is analytic and of finite order on the set,

$$\lambda - \delta(\lambda) \leqslant \arg z \leqslant \lambda + \delta(\lambda) \qquad |z| \geqslant K(\lambda). \tag{8.2}$$

Once we establish Lemma 8.3, the original result Lemma 8.2 easily follows by a compactness argument for the interval $[c, d]$. To prove Lemma 8.3, we use Lemma 2.1, the definition of a logarithmic differential field of rank zero, and a little geometry, to assert that if $\lambda \in (a, b)$, there is a set of the form (8.2) (where $\delta(\lambda)$ is of the form $\pi/(4m)$ for some integer $m \geqslant 2$), which is contained in a sector,

$$\lambda - 2\delta(\lambda) < \arg(z - Re^{i\lambda}) < \lambda + 2\delta(\lambda), \tag{8.3}$$

for some $R > 0$, and such that on (8.3), the solution f and all coefficients $R_j(z)$ of (1.2) are analytic, and we have $|R_j(z)| \leqslant |z|^\beta$ for some $\beta > 0$ and each $j = 0, 1, \ldots, n-1$. We now follow [2], and map the sector (8.3) conformally onto the unit disk $|t| < 1$ by the sequence of mappings,

$$u = e^{-i\lambda}(z - Re^{i\lambda}), \qquad \zeta = u^m, \qquad t = (\zeta - 1)/(\zeta + 1). \tag{8.4}$$

The original solution $f(z)$ becomes an analytic function $g(t)$ on $|t| < 1$, and a routine calculation shows that $g(t)$ satisfies a linear differential equation,

$$g^{(n)}(t) + \sum_{j=0}^{n-1} E_j(t) g^{(j)}(t) = 0, \tag{8.5}$$

where the coefficients $E_j(t)$ are analytic on the unit disk and all satisfy an estimate of the form $|E_j(t)| \leqslant K_1(1 - |t|)^{-\alpha}$ for constants $K_1 > 0$ and $\alpha > 0$. As in [2; p. 149], we invoke the Valiron-Wiman theory [20; Theorem II, p. 299] (where instead of requiring the Valiron-Wiman condition for just the first derivative as in [2; Formula (24)], we require the full strength, namely

$$g^{(j)}(t) = (1 + \varepsilon_j(t))(n(r)/t)^j g(t), \tag{8.6}$$

for $j = 0, 1, \ldots, n-1$), and we show as in [2] that $g(t)$ must be of finite order of growth in $|t| < 1$. We now retransform back to $f(z)$ using [2; Lemmas D-H], and we show as in [2] that $f(z)$ is of finite order on the sector (8.3). This establishes Lemma 8.3 and thus also Lemma 8.2.

9. Proof of Theorem A.

Let $(H_{q_1}, \ldots, H_{q_n})$ be the ordered asymptotic system for (1.2) in $F(r_j, r_{j+1})$ so that (6.7) holds, and let $(H_{t_1}, \ldots, H_{t_n})$ be the ordered asymptotic system for (1.2) in $F(r_{j-1}, r_j)$ so that,

$$H_{t_1} \gg H_{t_2} \gg \dots \gg H_{t_n} \quad \text{in} \quad F(r_{j-1}, r_j). \tag{9.1}$$

By Theorem 4, a basic fundamental set $\{f_1, \dots, f_n\}$ for (1.2) exists in $F(r_{j-1}, r_j)$, and a basic fundamental set $\{g_1, \dots, g_n\}$ for (1.2) exists in $F(r_j, r_{j+1})$. Thus, for each k, we have

$$f_k/H_k \to 1 \text{ in } F(r_{j-1}, r_j), \qquad \text{and} \qquad g_k/H_k \to 1 \text{ in } F(r_j, r_{j+1}). \tag{9.2}$$

We begin with g_{t_1} which is a solution of (1.2) admissible in $F(r_j, r_{j+1})$. By basic existence theory (e.g. [21; Th. 2.2, p. 3]), the solution g_{t_1} has an extension G_{t_1} which is admissible in $F(a, b)$, and thus for some constants $c_1, \dots, c_n$, we have

$$G_{t_1} = c_1 f_{t_1} + \dots + c_n f_{t_n} \quad \text{in} \quad F(r_{j-1}, r_j). \tag{9.3}$$

In view of (9.1) and (9.2) we see that $G_{t_1}/H_{t_1} \to c_1$ in $F(r_{j-1}, r_j)$, while in $F(r_j, r_{j+1})$ we have $G_{t_1}/H_{t_1} \to 1$ by (9.2) since $G_{t_1} = g_{t_1}$. In view of Lemma 8.2, clearly G_{t_1} (and thus also G_{t_1}/H_{t_1}) is analytic and of finite order in some sector $|\arg z - r_j| \leq \varepsilon$ for sufficiently large $|z|$, and thus by Lemma 8.1, we have $c_1 = 1$, and

$$G_{t_1}/H_{t_1} \to 1 \quad \text{as} \quad z \to \infty \quad \text{in } |\arg z - r_j| \leq \varepsilon, \tag{9.4}$$

for some sufficiently small $\varepsilon > 0$. It follows from [16; § 97] that,

$$G_{t_1}/H_{t_1} \to 1 \quad \text{in} \quad F(r_{j-1}, r_{j+1}). \tag{9.5}$$

In view of (9.1) and (9.5), it is clear that the set $\{G_{t_1}, f_{t_2}, \dots, f_{t_n}\}$ is a fundamental set for (1.2) in $F(r_{j-1}, r_j)$.

We now proceed by induction to construct for each $k \in \{1, \dots, n\}$, solutions $G_{t_1}, \dots, G_{t_k}$ of (1.2) in $F(r_{j-1}, r_{j+1})$ with the following properties:

$$G_{t_i}/H_{t_i} \to 1 \quad \text{in} \quad F(r_{j-1}, r_{j+1}), \qquad \text{for } i \leq k, \tag{9.6}$$

and $\{G_{t_1}, \dots, G_{t_k}, f_{t_{k+1}}, \dots, f_{t_n}\}$ is a fundamental set for (1.2) in $F(r_{j-1}, r_j)$. We have established this fact for $k = 1$, and we now assume it for some $k < n$. We consider the solution $g_{t_{k+1}}$ which, as before, has an extension $D_{t_{k+1}}$ which is admissible in $F(a, b)$. Thus, there are constants $c_{t_1}, \dots, c_{t_n}$ such that in $F(r_{j-1}, r_j)$,

$$D_{t_{k+1}} = \sum_{i=1}^{k} c_{t_i} G_{t_i} + \sum_{m=k+1}^{n} c_{t_m} f_{t_m}. \tag{9.7}$$

In view of the total ordering (6.7) of the set $\{H_1, \dots, H_n\}$ in $F(r_j r_{j+1})$, we let Γ denote the subset of $\{t_1, \ldots, t_k\}$ consisting of those t_i for which $H_{t_{k+1}} \ll H_{t_i}$ in $F(r_j, r_{j+1})$, and let Δ be the set $\{t_1, \ldots, t_k\} - \Gamma$. For ease of notation, we denote the elements of Γ by $t_\alpha, t_\beta, \ldots$, where we assume that $\alpha < \beta < \ldots$. We now rewrite (9.7) in $F(r_{j-1}, r_j)$ as,

$$(9.8) \qquad D_{t_{k+1}} - \sum_{t_i \in \Delta} c_{t_i} G_{t_i} = c_{t_\alpha} G_{t_\alpha} + c_{t_\beta} G_{t_\beta} + \ldots + \sum_{m=k+1}^{n} c_{t_m} f_{t_m}.$$

Let G denote the left side of (9.8), and consider G/H_{t_α}. In view of (9.1), (9.2), and (9.6), we see from (9.8) that $G/H_{t_\alpha} \to c_{t_\alpha}$ in $F(r_{j-1}, r_j)$. On the other hand, in $F(r_j, r_{j+1})$, we have $D_{t_{k+1}} = g_{t_{k+1}}$ so that $D_{t_{k+1}}/H_{t_\alpha} \to 0$ by (9.2) and the definition of the set Γ. In addition, if $t_i \in \Delta$ then $H_{t_i} \ll H_{t_{k+1}}$ in $F(r_j, r_{j+1})$ by definition of Δ, so that by (9.6) we have $G_{t_i}/H_{t_\alpha} \to 0$ since $t_\alpha \in \Gamma$. Thus, by definition of G, we have $G/H_{t_\alpha} \to 0$ in $F(r_j, r_{j+1})$. We can conclude that $c_{t_\alpha} = 0$ by lemma 8.1. We now repeat the above argument using H_{t_β} instead of H_{t_α}, and we conclude that $c_{t_\beta} = 0$. Similarly, we conclude that all $c_{t_i} = 0$ for $t_i \in \Gamma$. Thus, again from (9.1), (9.2), and (9.8), we see that $G/H_{t_{k+1}} \to c_{t_{k+1}}$ in $F(r_{j-1}, r_j)$, but now by the definition of G and Δ, we also have $G/H_{t_{k+1}} \to 1$ in $F(r_j, r_{j+1})$. Thus from Lemma 8.1, we can conclude that $c_{t_{k+1}} = 1$, and that $G/H_{t_{k+1}} \to 1$ as $z \to \infty$ in some sector $|\arg z - r_j| \leq \varepsilon$. It follows from [16; § 97] that $G/H_{t_{k+1}} \to 1$ in $F(r_{j-1}, r_{j+1})$. The form of G shows that G is a solution of (1.2) and so if we denote G by $G_{t_{k+1}}$, then (9.6) holds for $i = k+1$ also. In view of (9.6) and (9.1), it easily follows that $\{G_{t_1}, \ldots, G_{t_{k+1}}, f_{t_{k+2}}, \ldots, f_{t_n}\}$ is a fundamental set for (1.2) in $F(r_{j-1}, r_j)$. Thus, we have established our desired statement (9.6) by induction, and so for $k = n$, the solutions $G_{t_1}, \ldots, G_{t_n}$ form a basic fundamental set for (1.2) in $F(r_{j-1}, r_{j+1})$. This proves Theorem A.

10. Proof of Theorem B.

We are given that (6.7) and (9.1) hold. Under the assumption of Part (*a*), we thus have that (6.7) holds on both $F(r_j, r_{j+1})$ and $F(r_{j-1}, r_j)$. We examine each ratio $H_{q_{k+1}}/H_{q_k}$ for $k = 1, \ldots, n-1$. It is easy to see (e.g. see the proof of [7; Theorem 3]) that this ratio is either a function of the form $z^\alpha (\operatorname{Log} z)^\beta$ (where α is complex, β is an integer, and where either the real part of α is not zero or $\alpha = 0$ while $\beta \neq 0$), or the ratio is a function of the form,

$$(10.1) \qquad z^\alpha (\operatorname{Log} z)^\beta \exp \int V$$

where $V \sim c z^{-1+d}$ in $F(a, b)$, for some $c \neq 0$ and $d > 0$, and some constants α and β. In the first case, Lemma 2.3 shows that since

$H_{q_{k+1}}/H_{q_k} \ll 1$ in $F(r_j, r_{j+1})$, the same relation holds in $F(a, b)$. In the second case (i.e. where the ratio has the form (10.1)), we see from Lemma 2.2 that the indicial function (see (2.1)) of V must remain negative on (r_{j-1}, r_{j+1}) in order that $H_{q_{k+1}}/H_{q_k} \ll 1$ on both $F(r_{j-1}, r_j)$ and $F(r_j, r_{j+1})$. Thus by Lemmas 2.2 and 2.3, we have $H_{q_{k+1}} \ll H_{q_k}$ on $F(r_{j-1}, r_{j+1})$, and thus we see that under the assumption of Part (*a*), the asymptotic relation (6.7) is valid on all of $F(r_{j-1}, r_{j+1})$. Let $\{G_1, \ldots, G_n\}$ be the basic fundamental set for (1.2) in $F(r_{j-1}, r_{j+1})$, whose existence is guaranteed by Theorem A. If $f \not\equiv 0$ is any solution of (1.2) which is admissible in $F(a, b)$, then there are constants $c_{q_1}, \ldots, c_{q_n}$ such that

$$f = c_{q_1} G_{q_1} + \ldots + c_{q_n} G_{q_n} \quad \text{in} \quad F(r_{j-1}, r_{j+1}). \tag{10.2}$$

Since $G_k/H_k \to 1$ in $F(r_{j-1}, r_{j+1})$ for each k, and since (6.7) holds on $F(r_{j-1}, r_{j+1})$, we see that on some element of $F(r_{j-1}, r_{j+1})$, we have $f = c_{q_k} H_{q_k}(1 + E)$, where $E \to 0$ in $F(r_{j-1}, r_{j+1})$ and where k is the smallest index i for which $c_{q_i} \neq 0$. In view of the form of H_{q_k}, we see that on some element of $F(r_{j-1}, r_{j+1})$, the solution f can have no zeros. The conclusion of Part (*a*), now follows immediately from Lemma 2.1.

Under the hypothesis of Part (*b*), clearly by (6.7) and (9.1), there must be two elements H_i and H_k of the asymptotic set for which

$$H_i \ll H_k \quad \text{in} \quad F(r_{j-1}, r_j), \quad \text{and} \quad H_i \gg H_k \quad \text{in} \quad F(r_j, r_{j+1}). \tag{10.3}$$

By the discussion in Part (*a*) concerning the possible forms of the ratio H_i/H_k, clearly the ratio H_i/H_k must have the form (10.1) in order for the asymptotic behavior to change, and in addition, the indicial function of V must have a zero at r_j or else the behavior would not change by Lemma 2.2. As in Part (*a*), we let $\{G_1, \ldots, G_n\}$ be the basic fundamental set for (1.2) in $F(r_{j-1}, r_{j+1})$, so that we have for the indices i and k in (10.3),

$$G_i/H_i \to 1 \quad \text{and} \quad G_k/H_k \to 1 \quad \text{in} \quad F(r_{j-1}, r_{j+1}). \tag{10.4}$$

Let $G = c_i G_i + c_k G_k$ for any non-zero constants c_i and c_k. Then G is an admissible solution of (1.2) in $F(r_{j-1}, r_{j+1})$, and we assert that G satisfies the conclusion of Part (*b*).

To prove this, we observe first that since H_i/H_k has the form (10.1), it follows from (10.4) that the equation $G(z) = 0$ is of the form,

$$z^\alpha (\operatorname{Log} z)^\beta (1 + E_1) \exp \int V = s_1 \tag{10.5}$$

where $E_1 \to 0$ in $F(r_{j-1}, r_{j+1})$ and where $s_1 = -c_k/c_i$. It is easy to see that

(10.5) can be written in the form,

$$\exp((c/d)\, z^d + E_2(z)) = s_1, \tag{10.6}$$

where $E_2/z^d \to 0$ in $F(r_{j-1}, r_{j+1})$, and since the indicial function for V must have a zero at r_j, we have

$$\cos(dr_j + \arg c) = 0. \tag{10.7}$$

The equation (10.6) is precisely the same equation as Equation (122) in [4], where Formulas (158) and (159) of [4] hold. In addition, (10.7) is precisely the same condition as Condition (160) in [4], and so exactly as in [4; pp. 94-95], we show using Rouché's theorem that for any $\delta > 0$, the solution $G(z)$ possesses the required sequence of zeros. This proves Part (*b*) of Theorem B and concludes the proof.

11. Remarks.

(*A*) In the case where the coefficients of (1.2) are rational functions, it follows from the last Remark in § 6 that Theorem B permits us to test all the points $\sigma_1, \dots, \sigma_\lambda$ which appear in Part (*B*) of Theorem 1, and which belong to $(-\pi, \pi)$, for the property of being extraneous in the sense described in § 1. However, the value $\sigma_j = \pi$ cannot be tested this way since it does not belong to the oscillation set. However, it is easy to test π by simply making the change of independent variable $\zeta = -z$ in (1.2). This converts $\arg z = \pi$ to $\arg \zeta = 0$, and one simply tests the transformed equation to see if the value 0 beongs to the oscillation set, and, if so, whether or not it is extraneous.

(*B*) In this section, we give a simple example of an equation (1.2) having polynomial coefficients (so we take the neighborhood system to be $F(-\pi, \pi)$) which has the property that its oscillation set on $(-\pi, \pi)$ has extraneous elements. We consider the equation,

$$w'' - (2z+1)\, w' + (2z+2)\, w = 0. \tag{11.1}$$

We rewrite the equation in the form (3.3), and we compute the polynomial (3.5). We find that the critical degree is zero and the exponential set is $\{1, 2z\}$. Thus the transition set for (11.1) on $(-\pi, \pi)$ consists of $\pm \pi/4$, $\pm \pi/2$, and $\pm 3\pi/4$, and thus all of these points belong to the oscillation set of (11.1) on $(-\pi, \pi)$. Using the method developed in §§ 5, 6, we find that the asymptotic set for (11.1) is $\{z^{-2} e^{z^2}, ze^z\}$. Using Lemma 2.2, we find that $z^{-2} e^{z^2} \ll ze^z$ in $F(\pi/4, 3\pi/4)$ and also in $F(-3\pi/4, -\pi/4)$, while the reverse inequality holds in $F(-\pi, -3\pi/4)$, $F(-\pi/4, \pi/4)$ and $F(3\pi/4, \pi)$. Thus by Part (*a*) of Theorem B, both $\pm \pi/2$ are extraneous, while Part (b) of Theorem B shows that $\pm\pi/4$, $\pm 3\pi/4$ are all non-extraneous.

(C) If one knows the asymptotic set $\{H_1, \ldots, H_n\}$ for an equation (1.2) satisfying the hypothesis of Part (B) of Theorem 4, it is a simple matter to determine the oscillation properties around any given ray $\arg z = \theta_0$, where $\theta_0 \in (a, b)$. We use Lemma 2.2 and 2.3 to decide whether the asumptotic set can be ordered (as in (6.7)) in $F(\theta_0 - \varepsilon, \theta_0 + + \varepsilon)$ for some $\varepsilon > 0$. If the answer is in the affirmative, then we can assert that there is a $\delta > 0$ such that any solution $f \not\equiv 0$ has no zeros on a set of the form, $|\arg z - \theta_0| < \delta$. $|z| > R$ for some $R = R(f) > 0$. If the answer is in the negative, then we can assert that there is a solution $f \not\equiv 0$ such that for any $\delta > 0$, f possesses infinitely many zeros in $|\arg z - \theta_0| < \delta$ which tend to ∞. (The affirmative case follows from Part (a) of Theorem B if θ_0 belongs to the oscillation set, and from the last conclusion of Theorem 4 if θ_0 does not belong to the oscillation set. In the negative case, Section (a) of Part (B) of Theorem 4 shows that θ_0 would have to be in the oscillation set, say $\theta_0 = r_j$, and the proof of Part (a) of Theorem B shows that the hypothesis of Part (a) of Theorem B cannot be satisfied. Thus the hypothesis of Part (b) of Theorem B must hold and the conclusion then follows from Part (b).)

12. Preliminaries for Theorem C.

The concept of «principal monomial» discussed in § 2 for algebraic polynomials, also exists for differential polymomials (see [16; § 67]). In the present paper, we will require this concept for equations of the form $\Omega(w) = \varphi$, where $\Omega(w)$ is a linear differential polynomial of the form (5.1) whose coefficients belong to a logarithmic differential field $\mathcal{K}$ of rank zero over $F(a, b)$, and where $\varphi \not\equiv 0$ is an element of the field generated by the field $\mathcal{K}$ and all functions of the form,

$$(12.1) \qquad N(z) = Kz^{\alpha_0} (\operatorname{Log} z)^{\alpha_1} (\operatorname{Log}\operatorname{Log} z)^{\alpha_1} \ldots (\operatorname{Log}_q z)^{\alpha_q},$$

where q is a nonnegative integer, K is a nonzero complex constant, and where the α_j are real numbers. The algorithm in [16; § 67] produces for any such equation $\Omega(w) = \varphi$, a unique function $M(z)$ of the form (12.1) with the following two properties in $F(a, b)$: (i) $\Omega(M) \sim \varphi$, and (ii) $\Omega(f) \ll \varphi$ if $f \ll M$. The function $M(z)$ is called the *principal monomial* of the differential polynomial $\Omega(w) - \varphi$. We remark that the algorithm in [16; § 67] was carried out in [3; § 3], and an explicit formula for the principal monomial was developed there.

DEFINITION 12.1. Given an equation (1.2) whose coefficients belong to a logarithmic field of rank zero over $F(a, b)$, and let it have the form (3.3) when written in terms of θ, where the $B_j(z)$ satisfy the conditions in § 3. Let $\Omega(w) = \sum_{j=0}^{n} B_j \theta^j w$, and let $M = z^{\alpha}(\operatorname{Log} z)^q$ be an element of the logarithmic set for (1.2). From the conditions in § 3, each B_j is of the form $a_j + h_j$ where a_j is a constant and each $h_j \ll 1$ in $F(a, b)$. It is proved in [7; § 6] that $\sum_{j=0}^{n} a_j \theta^j M \equiv 0$. It is easy to check by induction that for each j, the function $\theta^j M$ is of the form $z^{\alpha} Q_j(\operatorname{Log} z)$, where $Q_j(u)$ is a polynomial in u, with constant coefficients, of degree at most q. It is also easy to check that under the change of variable $w = z^{\alpha} v$, the function $\theta^j w$ becomes $z^{\alpha}(\theta + \alpha)^j(v)$, where $(\theta + \alpha)^j$ is the j-th iterate of the operator $\theta + \alpha$. In view of these facts, we have,

$$z^{-\alpha}\Omega(M + z^{\alpha} v) = \Lambda(v) - \varphi_0(z), \tag{12.2}$$

where $\Lambda(v) = \sum_{j=0}^{n} B_j(\theta + \alpha)^j(v)$, and where,

$$\varphi_0(z) = -\sum_{j=0}^{n} h_j Q_j(\operatorname{Log} z). \tag{12.3}$$

We observe that the operator $\Lambda(v)$ has coefficients in the same logarithmic field that contains the $B_j(z)$. Thus, from the discussion in the first part of this section, if $\varphi_0 \not\equiv 0$ then $\Lambda(v) - \varphi_0$ possesses a unique principal monomial P_1. (The formula in [3; § 3] shows that $P_1 \ll z^{-\varepsilon}$ for some $\varepsilon > 0$, so that

$$z^{\alpha} P_1 \ll M \quad \text{in} \quad F(a, b).\,) \tag{12.4}$$

We now define a sequence of functions $(P_1, P_2, \ldots)$ as follows: If $\varphi_0 \equiv 0$, set $P_1, P_2, \ldots$ all identically equal to zero. If $\varphi_0 \not\equiv 0$, let P_1 be as above, and make the change of dependent variable $v = P_1 + u$ in the operator on the right side of (12.2). We obtain the differential polynomial,

$$\Lambda(u) - \varphi_1(z) \quad \text{where} \quad \varphi_1 = \varphi_0 - \Lambda(P_1). \tag{12.5}$$

If $\varphi_1 \equiv 0$, define $P_2, P_3, \ldots$ all to be identically zero. If $\varphi_1 \not\equiv 0$, let P_2 denote the principal monomial of $\Lambda(u) - \varphi_1(z)$ in (12.5). (It is proved in [16; § 75] that $P_2 \ll P_1$ in $F(a, b)$.) We now continue, and make the change of variable $u = P_2 + y$ in (12.5) which yields the differential polynomial,

$$\Lambda(y) - \varphi_2(z) \quad \text{where} \quad \varphi_2 = \varphi_1 - \Lambda(P_2). \tag{12.6}$$

As before, if $\varphi_2 \equiv 0$ we set $P_3, P_4, \ldots$ all equal to zero, while if $\varphi_2 \not\equiv 0$, we let P_3 denote the principal monomial of $\Lambda(y) - \varphi_2$. Continuing this way, we obtain a sequence of functions $(P_1, P_2, \ldots)$ with the properties that

$$(12.7) \qquad P_{j+1} \ll P_j \quad \text{if } P_j \not\equiv 0, \qquad \text{while} \qquad P_{j+1} \equiv 0 \quad \text{if } P_j \equiv 0.$$

For a given positive integer q we will call the sequence $(P_1, \ldots, P_q)$, the *principal q-tuple* for (1.2) relative to M. (We emphasize that for any q, the principal q-tuple can be easily calculated using [3; § 3].)

13. Theorem C.

Given an equation (1.2) where the coefficients belong to a logarithmic field of rank zero over $F(a, b)$. Let $r_1, \ldots, r_t$ be the transition set for (1.2) on (a, b). Set $r_0 = a$ and $r_{t+1} = b$. (If the transition set is empty, set $t = 0$.) Let M be an element of the logarithmic set for (1.2). Let q be any positive integer, and let $(P_1, \ldots, P_q)$ be the principal q-tuple for (1.2) relative to M. Then, in each of $F(r_0, r_1)$, $F(r_1, r_2), \ldots, F(r_t, r_{t+1})$ separately, the following conclusion holds: The equation (1.2) possesses an admissible solution of the form,

$$(13.1) \qquad w = M + z^{\alpha}(P_1 + P_2 + \ldots + P_q + \psi_q),$$

where $\psi_q \ll P_q$ if $P_q \not\equiv 0$, while $\psi_q \equiv 0$ if $P_q \equiv 0$.

14. Proof of Theorem C.

We first consider the case where $P_q \equiv 0$, and we let k denote the minimum positive integer i for which $P_i \equiv 0$. If $i = 1$, then $\varphi_0 \equiv 0$ in (12.2) and $P_j \equiv 0$ for $j \geqslant 1$. From (12.2), we see that $w = M$ solves (1.2), and is a solution of the form (13.1). If $i = 2$, then $\varphi_1 \equiv 0$ and $P_j \equiv 0$ for $j \geqslant 2$. It follows from (12.2) and (12.5) that $w = M + z^{\alpha} P_1$ solves (1.2) and is the desired solution. In general, if $i \geqslant 2$, then $v = P_1 + \ldots + P_{i-1}$ solves $\Lambda(v) = \varphi_0$, and the desired conclusion follows from (12.2).

We now assume $P_q \not\equiv 0$. From Definition 12.1, we know that for each j, $1 \leqslant j \leqslant q$, the function P_j is the principal monomial of $\Lambda(u) - \varphi_{j-1}$, where φ_0 is given by (12.3), and

$$(14.1) \qquad \varphi_{j-1} = \varphi_{j-2} - \Lambda(P_{j-1}).$$

Thus P_q is the principal monomial of the polynomial,

$$(14.2) \qquad \sum_{j=0}^{n} (a_j + h_j)(\theta + \alpha)^j(u) - \varphi_{q-1}\,,$$

where, as in Definition 12.1, we have written $B_j = a_j + h_j$ where a_j is a constant, and $\delta_0(h_j) < 0$. From [3; Lemma 5, p. 729], it follows that there exists $u^* \sim P_q$ in $F(a, b)$ such that,

$$(14.3) \qquad \sum_{j=0}^{n} a_j (\theta + \alpha)^j (u^*) \equiv \varphi_{q-1}\,.$$

Under the change of dependent variable $u = u^* + v$, the differential polynomial (14.2) becomes,

$$(14.4) \qquad \Lambda(v) + G \quad \text{where} \quad G = \sum_{j=0}^{n} h_j (\theta + \alpha)^j (u^*).$$

Let $\delta_0(P_q) = \sigma$. Then $\delta_0(G) < \sigma - \varepsilon$ for some $\varepsilon > 0$. Set $\sigma_1 = \sigma - \varepsilon/12$, and consider the differential equation,

$$(14.5) \qquad \Omega(y) = -z^{\alpha+\sigma_1-\varepsilon/2} H, \quad \text{where} \quad H = z^{-\sigma_1+(\varepsilon/2)} G,$$

and where $\Omega(y) = \sum_{j=0}^{n} B_j \theta^j y$ as in (12.2). Then $H \ll z^{-\varepsilon/3}$ in $F(a, b)$, and so the differential equation on the left side of (14.5) is exactly the type of equation treated in [7; Equation (6.8)], and it is proved in [7; § 6] that in each $F(r_k, r_{k+1})$ separately, this equation possesses a solution $y^* \ll z^{\alpha+\sigma_1}$. But clearly the operator $\Lambda(v)$ is simply $z^{-\alpha}\Omega(z^\alpha v)$, and hence the function $v^* = z^{-\alpha} y^*$ solves $\Lambda(v) = -G$ (in view of the definition of H in (14.5)). Thus the function $u_0 = u^* + v^*$ is a root of the differential polynomial in (14.2), namely we have $\Lambda(u_0) \equiv \varphi_{q-1}$. In view of the recurrence relation (14.1), we see that,

$$(14.6) \qquad \Lambda(u_0 + \ldots + P_{q-1}) \equiv \varphi_0 \quad \text{in} \quad F(r_k, r_{k+1}),$$

and thus from (12.2), we see that $\Omega(w) = 0$ possesses the solution,

$$(14.7) \qquad w = M + z^\alpha (P_1 + \ldots + P_{q-1} + u_0).$$

Since $u^* \sim P_q$ while $v^* \ll z^{\sigma-(\varepsilon/12)}$ (so that $v^* \ll P_q$) we see that $u_0 \sim P_q$ and so $u_0 = P_q + \psi_q$ where $\psi_q \ll P_q$. Thus (14.7) is the desired solution and the proof is complete.

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