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Fixed Points for Automorphisms in Cartan Domains of Type IV.

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ABSTRACT - In this paper we study the set of fixed points for holomorphic automorphisms of a Cartan domain of type four, $\mathcal{D}_n$. We give a direct proof of the fact that each holomorphic automorphism f of $\mathcal{D}_n$ extends to a continuous function $\tilde{f}$ on $\overline{\mathcal{D}}_n$, the closure of $\mathcal{D}_n$, in itself. Using this result we give a classification of the set of fixed points of $\tilde{f}$, the continuous extension of f, in $\overline{\mathcal{D}}_n$ in the case in which f has no fixed points in $\mathcal{D}_n$: in almost all cases this set has the following structure: it contains p isolated points and the intersection of r affine complex lines with $\overline{\mathcal{D}}_n$, moreover $p+2r \leqslant 4$.

0. Introduction.

In this note we shall investigate the structure of the set of fixed points for holomorphic automorphisms of Cartan domains of type four. A Cartan domain of type four $\mathcal{D}$ is a bounded symmetric homogeneous domain defined by

$$\mathcal{D} = \{z \in \boldsymbol{C}^n : |z| < 1 \quad \text{and} \quad 1 - 2|z|^2 + |{}^t zz|^2 > 0\},$$

and can be expressed as the open unit ball for the norm p, where $p^2(z) = |z|^2 + \sqrt{|z|^4 - |{}^t zz|^2}$, see [Harris 1]. The Shilov boundary of $\mathcal{D}$ is $\mathcal{L} = \{e^{i\theta} x \colon x \in S^{n-1} \subset \boldsymbol{R}^n\}$.

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The group of automorphism of $\mathcal{D}$ has the following representation. Let

$$G = \left\{ g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL(n+2, \boldsymbol{R}) | A \in GL(n, \boldsymbol{R}), \right.$$

$$B \in M(n, 2, \boldsymbol{R}), \quad C \in M(2, n, \boldsymbol{R}),$$

$$\left. D \in GL(2, \boldsymbol{R}) \colon \det D > 0, {}^t g \begin{bmatrix} I_n & 0 \\ 0 & -I_2 \end{bmatrix} g = \begin{bmatrix} I_n & 0 \\ 0 & -I_2 \end{bmatrix} \right\}.$$

In the first section we prove that, given $g \in G$ as above,

$$(0.1) \qquad d(z) = (1 i) \left(Cz + D \begin{bmatrix} \frac{1}{2}({}^t zz + 1) \\ \frac{i}{2}({}^t zz - 1) \end{bmatrix} \right) \neq 0, \quad \text{for all } z \in \overline{\mathcal{D}},$$

where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$.

Then, for all g in G, the holomorphic C^n-valued function

$$(0.2) \qquad \Psi_g(z) = \left(Az + B \begin{bmatrix} \frac{1}{2}({}^t zz + 1) \\ \frac{i}{2}({}^t zz - 1) \end{bmatrix} \right) \cdot \left((1 i) \left(Cz + D \begin{bmatrix} \frac{1}{2}({}^t zz + 1) \\ \frac{i}{2}({}^t zz - 1) \end{bmatrix} \right) \right)^{-1}$$

is well defined on $\overline{\mathcal{D}}$. We show that $g \mapsto \Psi_g$ is a surjective homomorphism of G onto $\operatorname{Aut} \mathcal{D}$, whose kernel is $\pm I_{n+2}$.

A proof can be found joining [Hau 1] and [Satake 1] (see also [Hirzebruch 1]); as the notations in these two papers are quite different, here we give a direct and complete proof.

Moreover (0.1) gives a direct proof of the known fact that every $f \in \operatorname{Aut} \mathcal{D}$ can be extended to a holomorphic—hence continuous—function in a neighborhood of $\overline{\mathcal{D}}$.

In the second section we investigate the case in which $f \in \operatorname{Aut} \mathcal{D}$ has a fixed point in $\mathcal{D}$. Setting $\operatorname{fix} f = \{z \in \mathcal{D} \colon f(z) = z\}$, it is known that $\operatorname{fix} f$ (if not empty) is connected. It is actually arcwise holomorphically connected, in the sense that for all x, y in $\operatorname{fix} f$ there exists a holomorphic map φ from Δ to $\mathcal{D}$ which is a complex geodesic for the Kobayashi metric such that $x, y \in \varphi(\Delta) \subset \operatorname{fix} f$. Then it is natural to ask whether there is more than one complex geodesic having this property. We show that this is true iff x and y satisfy a condition on complex extreme points.

In the third section we consider the case in which $f \in \mathrm{Aut}\,\mathcal{D}$ has no fixed points in $\mathcal{D}$. Denoting by the same symbol f the continuous extension of f to $\overline{\mathcal{D}}$ and setting $\mathrm{Fix} f = \{z \in \overline{\mathcal{D}}: f(z) = z\}$, Brouwer's fixed point theorem ensures that $\mathrm{Fix} f \neq \emptyset$. We shall show that for «almost all» $f \in \mathrm{Aut}\,\mathcal{D}$ (in a sense that shall be made more precise later) such that $\mathrm{fix} f = \emptyset$, the set $\mathrm{Fix} f$ contains $p \geq 0$ points and $r \geq 0$ intersections of affine lines with $\overline{\mathcal{D}}$, with $p + 2r \leq 4$.

1. Extension of automorphisms to continuous maps on $\overline{\mathcal{D}}$.

According to a general result of W. Kaup and H. Upmeier (see [Kaup-Upmeier 1]), every holomorphic automorphism of a ball in a Banach space can be extended to a continuous function on the closure of the ball. A direct proof of this fact will be given here.

We begin by briefly describe the «projective representation» due to Satake.

Let S be a quadratic form on a real vector space V of dimension $n+2$ with signature $(n, 2)$ and let h_S be the hermitian form on the complexification V_C of V extending S, that is $h_S(x, y) = S(x, \overline{y})$.

PROPOSITION 1.1. There exists a bijection of the set of all real, oriented two-planes V_- in V, such that $S_{|V_-} < 0$ onto the set of all complex lines W in V_C such that $S_{|W} = 0$ and $h_{S|W} < 0$ which identifies V_{-C} with $W \oplus \overline{W}$ and is such that $ix \wedge x$ (where x is in $W - \{0\}$) is positive for the orientation of V_-.

For a proof see [Satake 1].

The set $M = \{W$ is a complex line in V_C such that $S_{|W} = 0\}$ is a quadric hypersurface in $\boldsymbol{P}(V_C)$, the complex projective space.

Let $\mathcal{D}^*$ be the open set in M defined by $h_{S_{|W}} < 0$.

By Proposition 1.1 $\mathcal{D}^*$ has two connected components. We prove that one of these components is $\mathcal{D}$. For $x \in V$, $\langle x \rangle_C$ is the complex line generated by x.

Choosing a base $e_1 \ldots e_{n+2}$ in V such that $S = \begin{bmatrix} I_n & 0 \\ 0 & -I_2 \end{bmatrix}$, if $W = \left\langle \sum_{j=1}^{n+2} z_j e_j \right\rangle_C$ is contained in $\mathcal{D}^*$, then we have

$$\sum_{j=1}^{n} z_j^2 - z_{n+1}^2 - z_{n+2}^2 = 0 \quad \text{and} \quad \sum_{j=1}^{n} |z_j|^2 - |z_{n+1}|^2 - |z_{n+2}|^2 < 0 ,$$

and this implies that z_{n+1} and z_{n+2} are linearly independent on $\boldsymbol{R}$, whence $\operatorname{Im}\left(\frac{z_{n+2}}{z_{n+1}}\right) \neq 0$.

Let $\mathcal{D}_1$ be the connected component of $\mathcal{D}^*$ containing $W^0 = \langle e_{n+1} - - ie_{n+2} \rangle_C$ i.e. the component where $\operatorname{Im}\left(\frac{z_{n+2}}{z_{n+1}}\right) < 0$. Thus we can normalize setting $z_{n+1} + iz_{n+2} = 1$; $\left(\text{because } z_{n+1} + iz_{n+2} = 0 \text{ implies } \operatorname{Im}\left(\frac{z_{n+2}}{z_{n+1}}\right) = 1 > 0\right)$. From now on we set $w = {}^t zz$.

As a consequence of the normalization we find

$$w = z_{n+1} - iz_{n+2} = \sum_{j=1}^{n} z_j^2 \quad \text{and} \quad 1 + |w|^2 = 2(|z_{n+1}|^2 + |z_{n+2}|^2),$$

therefore $|w| < 1$ $\left(-w \text{ is the Cayley transform of } -\frac{z_{n+2}}{z_{n+1}}\right)$ and

$$\sum_{j=1}^{n} |z_j|^2 < \frac{1 + |w|^2}{2} < 1 \quad \text{and} \quad \operatorname{Im}\left(\frac{z_{n+2}}{z_{n+1}}\right) < 0,$$

showing that $\mathcal{D}_1$, the component containing W_0, is biholomorphic to $\mathcal{D}$.

Now we want to prove that every automorphism of $\mathcal{D}$ can be extended to a continuous function on a neighborhood of $\overline{\mathcal{D}}$. First of all we establish (0.1). This implies that Ψ_g is holomorphic on a neighborhood of $\overline{\mathcal{D}}$ if $g \in G$. Then we show that Ψ is a surjective homomorphism of G into $\operatorname{Aut} \mathcal{D}$.

Notice that every element $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in G leaves S and h_S invariant and maps $\mathcal{D}_1$ in $\mathcal{D}_1$. In fact the definition of $\mathcal{D}^*$ and the invariance of S and h_S imply that g maps $\mathcal{D}^*$ onto itself. As $\mathcal{D}^*$ has two connected components, one of which is $\mathcal{D}_1$, $\mathcal{D}_1 \cap g\mathcal{D}_1 \neq \emptyset$ gives $g\mathcal{D}_1 = \mathcal{D}_1$. So we are left to prove that $\mathcal{D}_1 \cap g\mathcal{D}_1 \neq \emptyset$. Then we compute the image of W_0 which is the complex line spanned by

$$\frac{1}{2}\begin{bmatrix} B\begin{bmatrix} 1 \\ -i \end{bmatrix} \\ D\begin{bmatrix} 1 \\ -i \end{bmatrix} \end{bmatrix}$$

and, setting $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we obtain

$$\operatorname{Im} \frac{c-id}{a-ib} = \frac{1}{a^2+b^2} \operatorname{Im}(c-id)(a+ib) = -\frac{1}{a^2+b^2}(ad-bc) < 0 .$$

Hence $g\mathcal{D}_1 = \mathcal{D}_1$.

If

$$d(z) = (1i)\left(Cz + D \begin{bmatrix} \frac{1}{2}(w+1) \\ \frac{i}{2}(w-1) \end{bmatrix} \right) \neq 0$$

we can define $\Psi_g(z)$; then it is enough to show that this term is different from 0 on $\overline{\mathcal{D}}$.

For $z \in \overline{\mathcal{D}}$ let

$$q = \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ \dfrac{w+1}{2} \\ \dfrac{i(w-1)}{2} \end{bmatrix} .$$

The above discussion on the projective representation shows that q has the following properties: $S(q, q) = 0$ and $h_S(q, q) \leq 0$.

We denote by $z_1' \ldots z_{n+2}'$ the coordinates of gq, i.e.

$$z' = gq = \begin{bmatrix} z_1' \\ \vdots \\ z_n' \\ z_{n+1}' \\ z_{n+2}' \end{bmatrix} ;$$

then we must show that $z_{n+1}' + iz_{n+2}' \neq 0$, so we can define Ψ_g on $\overline{\mathcal{D}}$.

It is obvious that $d(z) = z_{n+1}' + iz_{n+2}' \neq 0$ on $\mathcal{D}$: if $z_{n+1}' + iz_{n+2}' = 0$ then $gq = {}^t(z_1' \ldots z_n' \, z_{n+1}' \, z_{n+2}')$ would not be in $\mathcal{D}_1$, while we have shown that every element of G maps $\mathcal{D}_1$ in $\mathcal{D}_1$.

Now suppose that $z \in \mathcal{D}$. If $z'_{n+1} + iz'_{n+2} = 0$ then we have two cases: either 1) $z'_{n+1} = 0$ or 2) $z'_{n+1} \neq 0$.

In the first case $z'_{n+2} = 0$: as g preserves h_S and since $h_S(q, q) \leqslant 0$ we have $h_S(z', z') = h_S(q, q) \leqslant 0$, then $\sum_{j=1}^{n} |z'_j|^2 \leqslant |z'_{n+1}| + |z'_{n+2}|^2 = 0$, so $z'_j = 0$ for all $j = 1, \dots, n+2$; as z' is in $\boldsymbol{PC}^{n+1}$ this is impossible.

In the second case $z'_{n+1} \neq 0$. Let

$$z'(t) = g \begin{bmatrix} tz_1 \\ \vdots \\ tz_n \\ \dfrac{t^2 w + 1}{2} \\ \dfrac{i(t^2 w - 1)}{2} \end{bmatrix}.$$

It is easily seen that $S(z'(t), z'(t)) = 0$ and $h_S(z'(t), z'(t)) \leqslant 0$, $\forall t \in [0, 1]$. $z'(t)$ is a continuous function of t. If $t \in [1/2, 1)$ then $z'(t)$ is in $\mathcal{D}_1$ because ${}^t(tz_1 \dots tz_n)$ is in $\mathcal{D}$.

Let us define $\rho(t) = \operatorname{Im}\left(\dfrac{z'_{n+2}(t)}{z'_{n+1}(t)}\right)$: this is a continuous negative function on $[1/2, 1)$; moreover $z'_{n+1}(1) = z'_{n+1} \neq 0$, so ρ is continuous on $[1/2, 1]$ and $\rho(t) \leqslant 0$ on this interval; then it is not possible that $z'_{n+1} + iz'_{n+2} = 0$ because this implies $\rho(1) = 1$.

Thus we have established the following

PROPOSITION 1.2. For every z in $\overline{\mathcal{D}}$ and $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$, then $d(z) \neq 0$.

Hence Ψ_g is an element of $\operatorname{Hol}(\mathcal{D}, \boldsymbol{C}^n)$ for all $g \in G$ and every element in $\Psi(G)$ can be extended to a neighborhood of $\overline{\mathcal{D}}$.

Actually, we have shown that $\Psi(G) \subset \operatorname{Hol}(\mathcal{D}, \mathcal{D})$. A direct computation shows that Ψ is an homomorphism, so we have that $\Psi(G) \subset \operatorname{Aut} \mathcal{D}$.

Since the proof that $\operatorname{Ker} \Psi = \pm I_{n+2}$ is straightforward, we are left to prove that Ψ is surjective. To do this we show that $\Psi(G)$ is transitive on $\mathcal{D}$ and that the isotropy group of the origin is contained in $\Psi(G)$.

If $z_0 \in \mathcal{D}$, we exhibit and element g_{z_0} in G such that $\Psi_{g_{z_0}}(z_0) = 0$. Setting $w_0 = {}^t z_0 z_0$ and defining

$$X_0 = 2(z_0 \overline{z_0}) \begin{bmatrix} w_0 + 1 & \overline{w_0} + 1 \\ i(w_0 - 1) & -i(\overline{w_0} - 1) \end{bmatrix}^{-1},$$

a simple computation gives that X_0 is in $M(n, 2, \boldsymbol{R})$,

$$X_0 \begin{bmatrix} \frac{1}{2}(w_0 + 1) \\ \frac{i}{2}(w_0 - 1) \end{bmatrix} = z_0,$$

and $I_2 - {}^t X_0 X_0 > 0$ (which also implies $I_n - X_0^t X_0 > 0$).

Hence there exists $A \in \mathrm{Gl}(n, \boldsymbol{R})$ such that $A(I_n - X_0^t X_0)\,{}^t A = I_n$. Defining

$$D = \frac{1}{2} \frac{1}{(1 - 2|z_0|^2 + |w_0|^2)^{1/2}} \begin{bmatrix} -i(w_0 - \overline{w_0}) & w_0 + \overline{w_0} + 2 \\ w_0 + \overline{w_0} - 2 & -i(w_0 \overline{w_0}) \end{bmatrix}$$

it is easily seen that $\det D > 0$ and $D(I - {}^t X_0 X_0)\,{}^t D = I_2$.

Then

$$g_{z_0} = \begin{bmatrix} A & -AX_0 \\ -D^t X_0 & D \end{bmatrix}$$

is in G and $\Psi_{g_{z_0}}(z_0) = 0$ (in fact

$$Az_0 - AX_0 \begin{bmatrix} \frac{1}{2}(w_0 + 1) \\ \frac{i}{2}(w_0 - 1) \end{bmatrix} = A\left(z_0 - X_0 \begin{bmatrix} \frac{1}{2}(w_0 + 1) \\ \frac{i}{2}(w_0 - 1) \end{bmatrix} \right) = 0).$$

Now we must show that the isotropy of the origin, $(\mathrm{Aut}\,\mathcal{D})_0$ consists of the elements $z \mapsto e^{i\theta} Az$, where $\theta \in \boldsymbol{R}$ and $A \in O(n)$.

Let $f \in (\mathrm{Aut}\,\mathcal{D})_0$: as $\mathcal{D}$ is a bounded circular domain and $0 \in \mathcal{D}$, then f is the restriction of a linear automorphism Q of $\boldsymbol{C}^n$ by Cartan's lemma, (see [Vesentini 6]).

If $z \in \boldsymbol{C}^n$ we define

$$\lambda_1(z) = \left(|z|^2 + \sqrt{|z|^4 - |{}^t zz|^2}\right)^{1/2},$$

$$\lambda_2(z) = \left(|z|^2 + \sqrt{|z|^4 - |{}^t zz|^2}\right)^{1/2};$$

according to [Abate 1] we call λ_1 and λ_2 the «modules». Notice that λ_1 is the norm p. The Kobayashi distance on $\mathcal{D}$ is given by $k_{\mathcal{D}}(0, z) = \omega(0, p(z))$, where ω is the Poincaré distance on the unit disk Δ. As f is an automorphism of $\mathcal{D}$, then it preserves k and from $f(0) = 0$ we obtain that $\lambda_1(z) = \lambda_1(f(z))$ for all $z \in \mathcal{D}$; the fact that Q is linear and that $\mathcal{D}$ is a n open neighborhood of the origin in $\boldsymbol{C}^n$ gives $\lambda_1(z) = \lambda_1(Qz)$ for all $z \in \boldsymbol{C}^n$.

LEMMA 1.3. For all $z \in \boldsymbol{C}^n$ there exist $\theta \in \boldsymbol{R}$, $A \in O(n)$ such that $e^{i\theta} Az = {}^t(a, ib, 0, \ldots, 0)$, where $a, b \in \boldsymbol{R}^+$.

For a proof see [Hirzebruch 1].

A straightforward computation gives

$$a = \frac{\lambda_1(z) + \lambda_2(z)}{2}, \qquad b = \frac{\lambda_1(z) - \lambda_2(z)}{2}.$$

If $\theta \in \boldsymbol{R}$ and $A \in O(n)$, we call $z \mapsto e^{i\theta} Az$ a orthogonal automorphism of $\mathcal{D}$. Obviously the orthogonal automorphisms preserve the modules.

As an easy consequence of Lemma 1.3 we have that $\lambda_1(z) = \lambda_2(z) = 1$ implies $\lambda_1(f(z)) = \lambda_2(f(z))$. In fact by Lemma 1.3 we can suppose that $z = {}^t(1\,0 \ldots 0)$ and

$$f(z) = {}^t\!\left(\frac{i(1 + \lambda_2(f(z)))}{2}, \frac{i(1 - \lambda_2(f(z)))}{2}, 0, \ldots, 0\right).$$

It is easily seen that, if $\lambda_2(f(z)) \neq 1$, $f(z)$ is not a complex estreme point for $\overline{\mathcal{D}}$, while z is, and this a contraddiction, because f is a linear automorphism of $\mathcal{D}$.

Let $e_1, \ldots, e_n$ be the standard base of $\boldsymbol{C}^n$ and set $v_j = Q(e_j)$, $j = 1, \ldots, n$.

Let $t \in \boldsymbol{R}$ and note that

$$\lambda_1\left((e_j + te_h)/\sqrt{1 + t^2}\right) = \lambda_2\left((e_j + te_h)/\sqrt{1 + t^2}\right) = 1$$

if $h \neq j$; we have

$$\lambda_1\left((v_j + tv_h)/\sqrt{1 + t^2}\right) = \lambda_2\left((v_j + tv_h)/\sqrt{1 + t^2}\right) = 1,$$

i.e.

$$|v_j + tv_h| = 1 + t^2 \quad \text{and} \quad |{}^t(v_j + tv_h)(v_j + tv_h)| = 1 + t^2:$$

hence

(1.1) $|v_j| = 1, \quad |{}^t v_j v_j| = 1, \quad {}^t v_j v_h = 0, \quad \text{and} \quad \operatorname{Re}(v_j, v_h) = 0, \quad \text{if } j \neq h.$

Moreover $\lambda_1^2(e_j + ie_h) = \lambda_1^2(v_j + iv_h)$, that is, using (1.1),

$$4 = 1 + 1 + 2\operatorname{Re} i(v_j, v_h) + \sqrt{(1 + 1 + 2\operatorname{Re} i(v_j, v_h))^2 - (1 + i^2)^2}.$$

Then we obtain $\operatorname{Re}(i(v_j, v_h)) = 0$ if $j \neq h$, hence $(v_j, v_h) = 0$.

Hence we have proved that Q is a unitary matrix, so that $\lambda_1(z) = \lambda_1(Q(z))$ gives

$$|{}^t zz| = |{}^t z\, {}^t QQz| \quad \text{for all} \quad z \in \boldsymbol{C}^n.$$

Define $K = {}^t QQ$ and consider last equation for $z = e_h + \lambda e_j$, we obtain

$$|\lambda^2 + 1| = |{}^t(\lambda e_j + e_h)(\lambda e_j + e_h)| =$$
$$= |{}^t(\lambda e_j + e_h) K(\lambda e_j + e_h)| = |\lambda^2 k_{jj} + 2\lambda k_{jh} + k_{hh}|;$$

then we have $k_{jh} = 0$ if $h \neq j$ and $k_{jj} = k_{hh}$ with $|k_{jj}| = 1$; this ensures that there exists $\eta \in \boldsymbol{R}$ such that $K = e^{i\eta} I_n$.

From this we have immediately that there exists $A \in O(n)$ and $\theta \in \boldsymbol{R}$ such that $Q = e^{i\theta} Az$, hence $f(z) = e^{i\theta} Az$ for all $z \in \mathcal{D}$.

Then we have proved the following

PROPOSITION 1.4. The map $\Psi: G \to \operatorname{Aut} \mathcal{D}$ is a surjective homomorphism whose kernel is $\pm I_{n+2}$.

In view of this result Proposition 1.2 yields the following:

THEOREM 1.5. Every automorphism of $\mathcal{D}$ has a holomorphic extension in a neighborhood of $\overline{\mathcal{D}}$.

2. Fixed points in $\mathcal{D}$.

Let $f \in \operatorname{Aut} \mathcal{D}$ be such that $\operatorname{fix} f \neq \emptyset$. There is no restriction in assuming $0 \in \operatorname{fix} f$, so that there are $A \in O(n)$ and $\theta \in \boldsymbol{R}$ such that $f(z) = e^{i\theta} Az$ for all $z \in \mathcal{D}$.

Hence the set $\operatorname{fix} f$ is the intersection of $\mathcal{D}$ with a complex vector space; then it is convex and a fortiori connected. Thus the set of fixed points of an element in $\operatorname{Aut} \mathcal{D}$ is either connected or empty.

Throughout the following, the space $\operatorname{Hol}(\mathcal{D}, \boldsymbol{C}^n)$ of all holomorphic maps of $\mathcal{D}$ in $\boldsymbol{C}^n$ will always be endowed with the topology of uniform convergence on compact sets of $\mathcal{D}$. By Montel's theorem, every sequence in $\mathrm{Hol}(\mathcal{D}, \mathcal{D})$ contains a convergent subsequence.

In the following we shall consider the iterates of an automorphism of $\mathcal{D}$. In the case of the euclidean ball $\Delta_n = \{z \in \boldsymbol{C}^n : |z| < 1\}$ we have the following theorem due to Hervé:

THEOREM 2.1. Let $f \in \mathrm{Aut}\, \Delta_n - \{id\}$, then

a) if f has a fixed point in Δ_n the sequence $\{f^n\}$ does not converge and all coverging subsequences converge to an automorphism of $\mathcal{D}$;

b) if f has no fixed points in Δ_n then $\{f^n\}$ converges uniformly on compact sets of Δ_n to a constant function, mapping Δ_n to a point in $\partial\Delta$.

For a proof see [Hervé 1].

In the case of $\mathcal{D}$ a weaker result holds, which turns out to be the best possible in this direction.

THEOREM 2.2. Let $f \in \mathrm{Aut}\, \mathcal{D} - \{id\}$, then

a) if f has a fixed point in $\mathcal{D}$, the sequence $\{f^n\}$ does not converge and all converging subsequences converge to an automorphisms of $\mathcal{D}$;

b) if f has no fixed points in $\mathcal{D}$; then $\{f^n\}$ does not necessarily converge. If a subsequence of $\{f^n\}$ converges to a limit function h such that $h(\mathcal{D}) \cap \mathcal{L} \neq \emptyset$, then h is constant. Any converging subsequence converges to holomorphic maps from $\mathcal{D}$ into $\partial\mathcal{D}$.

REMARK.. Before proving the theorem we give an example showing that it is possible that f has no fixed points in $\mathcal{D}$ and the sequence $\{f^n\}$ does not converge.

The domain $\mathcal{D}_2$ is biholomorphic to $\Delta \times \Delta$ via the map

$$\varepsilon(z_1 z_2) = (z_1 + iz_2 z_1 - iz_2). \tag{2.1}$$

Let $h: \Delta \times \Delta \to \Delta \times \Delta$ defined by

$$h\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \dfrac{z_1 \cosh \alpha + \sinh \alpha}{z_1 \sinh \alpha + \cosh \alpha} \\ e^{i\theta} z_2 \end{bmatrix},$$

where θ, $\alpha \in \boldsymbol{R}$. Then $\mathrm{fix}\, h = \emptyset$.

Obviously, if $\theta \neq 2k\pi$, $(k \in \boldsymbol{Z})$, then $\{h^n\}$ does not converge, but there is a converging subsequence whose limit function is $z \mapsto \begin{bmatrix} 1 \\ e^{i\mu} z_2 \end{bmatrix}$, that maps $\Delta \times \Delta$ in $\{1\} \times \Delta$.

PROOF.. Since $\mathcal{D}$ is a ball (hence a taut domain), the limit function of a convergent subsequence in $\operatorname{Aut}\mathcal{D}$ is an element of $\operatorname{Hol}(\mathcal{D}, \mathcal{D})$ or a holomorphic map of $\mathcal{D}$ into $\partial\mathcal{D}$.

Applying this result to convergent subsequences of $\{f^n\}$, say $\{f^{n_j}\}$, and again to a convergent subsequences of $\{f^{-n_j}\}$, we obtain that the limit function h is either an automorphism of $\mathcal{D}$ or is such that maps $\mathcal{D}$ into $\partial\mathcal{D}$.

If f has a fixed point in $\mathcal{D}$ this is a fixed point for all iterates and therefore h is an element in $\operatorname{Aut}\mathcal{D}$. Moreover, if f^{n_k} converges to h, then f^{n_k+1} converges to $hf \neq h$, and therefore f^n does not converge.

If f has a fixed points in $\mathcal{D}$ three cases are possible: i) the limit function h is in $\operatorname{Aut}\mathcal{D}$, ii) $h(\mathcal{D}) \cap \mathcal{L} \neq \emptyset$, iii) $h(\mathcal{D}) \subset \partial\mathcal{D} - \mathcal{L}$.

Since $\mathcal{D}$ is convex the first case can not occur, according to a result of M. Abate; in fact for convex domains «f has no fixed points in $\mathcal{D}$» is equivalent to «f is compactly divergent». A proof of this theorem can be found in [Abate 2], together with a detailed exposition of the general theory of iterates.

In the second case, let $z_0 \in \mathcal{D}$ be such that $h(z_0) \in \mathcal{L} = \{e^{i\theta}x\colon x \in S^{n-1}\}$ (see p. 161) and let $\varphi\colon \mathcal{D} \to \boldsymbol{C}$, be defined by $\varphi(z) = {}^t\overline{h}(z_0)\, h(z)$; this is a holomorphic map with $\varphi(\mathcal{D}) \subset \overline{\Delta}$ and $\varphi(z_0) = 1$. By the maximum principle φ is constant, then $h(z) = h(z_0)$ for all $z \in \mathcal{D}$.

In the third case we have $h(\mathcal{D}) \subset \partial\mathcal{D} - \mathcal{L}$ and nothing more can be said in general on the behaviour of h. ∎

We recall a few facts concerning the notion of complex geodesic, that is often an important tool in the investigation of fixed points of automorphisms.

If V is a bounded convex domain in $\boldsymbol{C}^n$, the Kobayashi and the Carathéodory pseudodistances coincide and they induce on V the natural metric topology, hence V is a complete domain with respect to these distances, and V is taut. If V is the unit ball in a Banach space with respect to a continuous norm p we have $k_V(0, z) = \omega(0, p(z))$, where ω is the Poincaré distance on the disk Δ.

DEF. 1. A complex geodesic for the Kobayashi metric is a map $\varphi\colon \Delta \to V$ that is an isometry for the Kobayashi distance.

We recall here the following theorems, due to [Vesentini 2,3] and [Vigué 2].

THEOREM 2.3. Let $\xi \in \Delta$ and $\varphi \in \mathrm{Hol}(\Delta, V)$. If

1) $\kappa_V(\varphi(\xi), \dot{\varphi}(\xi)) = \kappa_\Delta(\xi, 1)$ or

2) there is $\nu \in \Delta - \xi$ such that $k_V(\varphi(\nu), \varphi(\xi)) = k_\Delta(\nu, \xi)$, then φ is a complex geodesic.

THEOREM 2.4. Two complex geodesics ψ and φ have the same image if and only if there is an automorphism l of Δ such that $\varphi \circ l = \psi$.

THEOREM 2.5. If V is a bounded convex domain in C^n then for every pair $x, y \in V$ there is a complex geodesic μ such that $x, y \in \mu(\Delta)$.

We start with the following

LEMMA 2.6. Let $f \in \mathrm{Hol}(V, V)$ and x, y in fix$f$. Let $\varkappa$ be a complex geodesic such that $x, y \in \mu(\Delta)$; if $f(\mu(\Delta)) = \mu(\Delta)$ then $\mu(\Delta) \subset \mathrm{fix} f$.

PROOF.. By Schwarz's lemma, if a holomorphic map of Δ into Δ has two fixed points then it is the identity map.

By Theorem 2.3 $f \circ \mu$ is still a complex geodesic whose range coincides with that of μ. Then, by Theorem 2.4, there exists $l \in \mathrm{Aut}\,\Delta$ such that $f \circ \mu = \mu \circ l$.

Let a and b be points of Δ such that $\mu(a) = x$ and $\mu(b) = y$.

Then $\mu(l(a)) = f(\mu(a)) = f(x) = x$, and $\mu(l(b)) = f(\mu(b)) = f(y) = y$. Since the isometry μ is one-to-one $l(a) = a$ and $l(b) = b$, and so $l = id$.

This implies that $f \circ \mu = \mu$, i.e. $\mu(\Delta)$ is contained in the set of fixed points of f. ■

The set of fixed points of $f \in \mathrm{Hol}(V, V)$ is connexted if V is a bounded convex domain in C^n, as a consequence of a result of J.-P. Vigué whereby for every pair of fixed points x and y of $f \in \mathrm{Hol}(V, V)$ there is a complex geodesic whose range contains x and y and is contained in the set fix$f$ (see [Vigué 1,2]). This result extends the one we found directly in the case in which f is in Aut$\mathcal{D}$.

It is interesting to ask whether there is more than one complex geodesic whose image contains two fixed points and is contained in the set fix f. There is no restriction in choosing 0 as one of the fixed points. By Schwarz's lemma it is obvious that, if x is a fixed point different from 0, the linear map $\varphi(z) = (z/p(x))\,x$ is a complex geodesic such that $0,\ x \in \varphi(\Delta)$ and $\varphi(\Delta)$ is contained in fix$f$; from now on we call this map the linear geodesic.

DEF. 2. Let μ a complex geodesic whose range contains 0 and x. We say that μ is normalized if $\mu(0) = 0$ and $\mu(p(x)) = x$. We note that a normalization as in the above definition is always possible: in fact if ψ is a complex geodesic whose range contains 0 and x we can always find $\alpha \in \operatorname{Aut}\Delta$ such that $\psi \circ \alpha$ is normalized (because Δ is homogeneous and $(\operatorname{Aut}\Delta)_0 \sim S^1$); viceversa two normalized complex geodesic whose ranges coincide, coincide too.

From now on we set $y = (1/p(x))\,x$. If y is contained in the Shilov boundary the unique normalized complex geodesic μ whose range contains 0 and x is given by the linear one (for the proof see [Vesentini 3]).

Then, if $y = (1/p(x))\,x$ is a point in the Shilov boundary, there exists only one normalized geodesic whose range contains 0 and x. Hence we turn our attention to the case in which y is not a point in the Shilov boundary.

In the first section we stated

LEMMA 1.3. For all $z \in \boldsymbol{C}^n$ there exist $\theta \in \boldsymbol{R}$, $A \in O(n)$ such that $e^{i\theta}Az = {}^t(a,\ ib,\ 0, \ldots, 0)$ where $a, b \in \boldsymbol{R}^+$ and

$$a = \frac{\lambda_1(z) + \lambda_2(z)}{2}, \qquad b = \frac{\lambda_1(z) - \lambda_2(z)}{2}.$$

If $z \in \partial\mathcal{D}$, then $\lambda_1(z) = 1$, hence $a \in [0, 1]$ and $b = 1 - a$; moreover z is in the Shilov boundary if and only if $a = 0, 1$.

We set $\Delta(r) = \{z \in \boldsymbol{C}\colon |z| < r\}$.

PROPOSITION 2.7. Let $y \in \partial\mathcal{D} - \mathcal{L}$. If $A \in O(n)$ and $\theta \in \boldsymbol{R}$ are such that $e^{i\theta}Ay = {}^t(a, i(1-a), 0, \ldots, 0)$, where $a \in (0, 1)$, then $y + \Delta(r)z \subset \overline{\mathcal{D}}$ with $r > 0$ if and only if $z = e^{-i\theta}A^{-1}\,{}^t(1, -i, 0, \ldots, 0)$.

PROOF. It is enough to establish the proposition for

$$y = {}^t(a, i(1-a), 0, \ldots, 0).$$

Let $z = {}^t(z_1, \ldots, z_n)$ be such that $y + \Delta(r)z \subset \overline{\mathcal{D}}$.

It is easily seen that $p(u_1, \ldots, u_n) \geq p(u_1, \ldots, u_{n-1}, 0)$, for all $(u_1, \ldots, u_n)$ in $\boldsymbol{C}^n$ and that the equality holfs if and only if $u_n = 0$.

Hence $y + \Delta(r)z \subset \overline{\mathcal{D}}$ implies that $y + \Delta(r)\,{}^t(z_1, z_2, 0, \ldots, 0) \subset \overline{\mathcal{D}}$.

Passing to $\Delta \times \Delta$ via the biholomorphism (2.1) we obtain that ${}^t(z_1, z_2) = \alpha(1, -i)$, for some $\alpha \in \boldsymbol{C}$.

As $p(y + c\,{}^t(1, -i, 0, \ldots, 0)) = 1$, if $|c| < r$ we have that $z_3 = \ldots z_n = 0$, and this proves the proposition. ■

PROPOSITION 2.8. Let $f \in \operatorname{Aut} \mathcal{D}$. Suppose 0, $x \in \operatorname{fix} f$ and $y = (1/p(x))\,x$ is not on the Shilov boundary. If the point $z \neq 0$ such that

$$y + tz \in \partial\mathcal{D} \qquad \forall t \in \Delta(r) \tag{2.2}$$

is in $\operatorname{fix} f$, then there is a normalized complex geodesic different from the linear one, joining x and 0, whose range is contained in $\operatorname{fix} f$. Otherwise the unique normalized geodesic whose range contain 0 and x and is contained in $\operatorname{fix} f$ is the linear one.

We first open a paranthesis and consider convex circular domains: let V be a bounded convex circular neighborhood of 0 in $\boldsymbol{C}^n$ and let $y \in \partial V$.

The family $\mathcal{P} = \{P$ convex circular subset of $\boldsymbol{C}^n$ such that $y + P \subset \overline{V}\}$ has a maximal element $P(y)$.

We indicate by p the Minkowski norm associated to V. Then we have the following

THEOREM 2.9. Let $f \in \operatorname{Aut} V$ such that 0, $x \in \operatorname{fix} f$. Then $\operatorname{fix} f \cap P\left(\frac{1}{p(x)}x\right) \neq \{0\}$ if and only if there is a complex geodesic different from the linear one, whose range contains x and 0 and is contained in $\operatorname{fix} f$.

PROOF. As V is a bounded circular domain such that $0 \in V$, then f is linear by Cartan's lemma. In [Gentili 1], it is shown that, for all $h: \Delta \to P\left(\frac{1}{p(x)}x\right)$ holomorphic and such that $h(0) = h(p(x))$, the map $\xi \mapsto \frac{\xi}{p(x)}x + h(\xi)$ is a complex geodesic whose range contains 0 and x. Viceversa, if φ is a normalized complex geodesic whose range contains 0 and x, then there exists a holomorphic map $h: \Delta \to \bigcup_{t>1} tP\left(\frac{1}{p(x)}x\right)$ such that $h(0) = h(p(x)) = 0$, $h \neq 0$ and $\varphi(\xi) = \frac{\xi}{p(x)}x + h(\xi)$ $\forall \xi \in \Delta$.

If $P\left(\frac{1}{p(x)}x\right) \cap \operatorname{fix} f \neq \{0\}$, choose $w \in P\left(\frac{1}{p(x)}x\right) \cap \operatorname{fix} f - \{0\}$, and $\tau: \Delta \to \Delta$ holomorphic such that $\tau(0) = \tau(p(x)) = 0$, $\tau \neq 0$: the map

$\varphi\colon \xi \mapsto \dfrac{\xi}{p(x)} x + \tau(\xi) w$ is a complex geodesic whose range contains 0 and x.

Moreover

$$f\left(\frac{\xi}{p(x)} x + \tau(\xi) w\right) = f\left(\frac{\xi}{p(x)} x\right) + f(\tau(\xi) w) = \frac{\xi}{p(x)} x + \tau(\xi) w ,$$

hence $\varphi(\Delta) \subset \operatorname{fix} f$ and φ is a linear geodesic, that proves the first assertion.

Viceversa, if there is a complex geodesic φ different from the linear one, with the properties $0, x \in \varphi(\Delta) \subset \operatorname{fix} f$, we find a holomorphic map $h\colon \Delta \to \bigcup_{t>1} tP\left(\dfrac{1}{p(x)} x\right)$ such that $h(0) = h(p(x)) = 0$, $h \neq 0$ and $\varphi(\xi) = \dfrac{\xi}{p(x)} x + h(\xi)$.

Then $f(\varphi(j)) = \varphi(\xi)$ for all $\xi \in \Delta$ implies that

$$\operatorname{fix} f \cap \bigcup_{t>1} tP\left(\frac{1}{p(x)} x\right) \neq \{0\}.$$

As $\operatorname{fix} f$ is convex, $\operatorname{fix} f \cap P\left(\dfrac{1}{p(x)} x\right) \neq \{0\}$. ■

Then Proposition 2.8 becomes an easy corollary of Theorem 2.9.

3. Fixed points on the boundary $\partial\mathcal{D}$.

Since $\mathcal{D}$ is the open unit ball of $\boldsymbol{C}^n$ for the norm p defined by $p^2(z) = |z|^2 + \sqrt{|z|^4 - |{}^t zz|^2}$, then $\mathcal{D}$ is homeomorphic to Δ_n and the homeomorphism can be extended to $\overline{\mathcal{D}}$. Because of the Brouwer theorem and of the results of § 1, we have

THEOREM 3.1. Let $f \in \operatorname{Aut} \mathcal{D}$ be such that $\operatorname{fix} f = \emptyset$. Then the unique holomorphic extention of f to a neighborhood of $\overline{\mathcal{D}}$ has at least a fixed point in $\partial\mathcal{D}$.

Now we state a classification of elements in $\operatorname{Aut} \mathcal{D}$ which have no fixed points in $\mathcal{D}$.

THEOREM 3.2. Let $f \in \operatorname{Aut} \mathcal{D}$ be such that $\operatorname{fix} f = \emptyset$ and let $g \in G$ be such that $\Psi_g = f$. If both 1 and -1 are eigenvectors of g whose geometric multiplicity does not exceed 2, then the set of fixed points of f in $\overline{\mathcal{D}}$

is given by p isolated points and by the intersections of r complex affine lines with $\overline{\mathcal{D}}$. If neither 1 or -1 are eigenvalues for g then $0<p+2r\leq 4$.

We begin with some preliminary observations about the statement.

Set $\operatorname{Fix} f=\{z\in\overline{\mathcal{D}}\colon f(z)=z\}$.

REMARK 1. We have proved that, if $f\in\operatorname{Aut}\mathcal{D}$ (or even $f\in\operatorname{Hol}(\mathcal{D},\mathcal{D})$) and $\operatorname{fix} f\neq\emptyset$, then the set $\operatorname{fix} f$ is connected. This is not necessarily true for $\operatorname{Fix} f$ if $f\in\operatorname{Aut}\mathcal{D}$. Let $g\in G$ be expressed by

$$g=\begin{bmatrix}\cosh\alpha & 0 & 0 & \sinh\alpha & 0\\ 0 & \cosh\alpha & 0 & 0 & -\sinh\alpha\\ 0 & 0 & I_{n-2} & 0 & 0\\ \sinh\alpha & 0 & 0 & \cosh\alpha & 0\\ 0 & \sinh\alpha & 0 & 0 & \cosh\alpha\end{bmatrix},$$

where $\alpha\in\boldsymbol{R}$. The fixed points of Ψ_g are the solutions of the system

$$(3.1)\quad z_1\cosh\alpha+\frac{w+1}{2}\sinh\alpha= = z_1\left((z_1-iz_2)\sinh\alpha+\frac{(w+1)-(w-1)}{2}\cosh\alpha\right),$$

$$(3.2)\quad z_2\cosh\alpha-1\frac{w-1}{2}\sinh\alpha= = z_2\left((z_1-iz_2)\sinh\alpha+\frac{(w+1)-(w-1)}{2}\cosh\alpha\right),$$

$$(3.3)\quad \begin{bmatrix}z_3\\ \vdots\\ z_n\end{bmatrix}=\begin{bmatrix}z_3\\ \vdots\\ z_n\end{bmatrix}\left((z_1-iz_2)\sinh\alpha+\frac{(w+1)-(w-1)}{2}\cosh\alpha\right)$$

where, as before $w=\sum_{j=1}^{n} z_j^2$.

Equations (3.1) and (3.2) imply that, if $\sinh\alpha\neq 0$, $\frac{1}{2}(w+1)= = z_1(z_1-iz_2)$ and $\frac{1}{2i}(w-1)=z_2(z_1-iz_2)$.

Then $2=2(z_1-iz_2)^2$ and therefore $z_1-iz_2=\pm 1$.

If $\alpha \neq 0$, then $\pm \sinh \alpha + \cosh \alpha \neq \pm 1$, using this we see that, if $\alpha \neq 0$, $z_3 = \ldots = z_n = 0$.

The set of fixed points of Ψ_g in $\overline{\mathcal{D}}$ is

$$\{z \in \overline{\mathcal{D}} : z_1 - iz_2 = \pm 1, \quad z_3 = \ldots = z_n = 0\}:$$

we have two components ($z_1 - iz_2 = 1$ and $z_1 - iz_2 = -1$) which are the intersections of $\overline{\mathcal{D}}$ with two parallel complex affine lines.

REMARK 2. If z is a fixed point of $f = \Psi_g$ in $\overline{\mathcal{D}}$, setting as before $w = {}^t zz = \sum_{j=1}^{n} z_j^2$, $z_{n+1} = \dfrac{w+1}{2}$, $z_{n+2} = \dfrac{i(w-1)}{2}$, then

$$u = {}^t(z_1, \ldots, z_{n+2}) \tag{3.4}$$

has the following properties:

$$S(u,u) = 0, \qquad h_S(u,u) \leq 0, \qquad \operatorname{Im}\left(\frac{z_{n+2}}{z_{n+1}}\right) < 0 . \tag{3.5}$$

Viceversa, every eigenvector of g satisfying (3.5) gives a fixed point of f in $\overline{\mathcal{D}}$.

This method imitates the one used by Hayden and Suffridge in the case of the open unit ball in a complex Hilbert spaces, see [Hayden-Suffridge 1], leading to the following

THEOREM 3.3. *Let B be the unit ball of a complex Hilbert space, an let $F \in \operatorname{Aut} B$, unit ball of and Hilbert space. If F has no fixed points in B, then its (unique) continuous extention to $\overline{B}$ has at least one and at most two fixed points in ∂B.*

DEF.. We say that $x = {}^t(x_1, \ldots, x_{n+2}) \in C^{n+2}$ with $|x_{n+1}| + |x_{n+2}| > 0$, is normalized if $x_{n+1} + ix_{n+2} = 1$.

Notice that, for every point in $\mathcal{D}$, the representation (3.4) yields a normalized vector which satisfies (3.5), viceversa every normalized vector in C^{n+2} which satisfies condition (3.5) corresponds to a point in $\overline{\mathcal{D}}$.

REMARK 3. The condition on the eigenvalues 1 and -1 is essential, as shown by the following example, in which the set of fixed points in contained in the Shilov boundary and contains a manifold of real dimension $k-2$ where $k = 2, \ldots, n$.

Let

$$g = \begin{bmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha & 0 \\ 0 & I_{k-1} & 0 & 0 & 0 \\ 0 & 0 & -I_{n-k} & 0 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\alpha \in \boldsymbol{R}$.

The eigenvalues of g are: $\cosh\alpha - \sinh\alpha$, $\cosh\alpha + \sinh\alpha$, 1 with multiplicity k and -1 with multiplicity $n-k$. The eigenvectors associated to $\cosh\alpha - \sinh\alpha$ and $\cosh\alpha + \sinh\alpha$ are ${}^t(1, 0, \dots 0, -1)$ and ${}^t(1, 0, \dots 0, 1)$. A base for the eigenvectors associated to 1 is $e_2, \dots, e_k, e_{n+2}$, while for -1 we can choose $e_{k+1} \dots e_n$.

We now look for the fixed points of Ψ_g in $\overline{\mathcal{D}}$ coming from eigenvectors associated to 1 and -1: first of all the normalization condition excludes all eigenvectors associated to -1 and implies that, if $v = {}^t(0, v_2, \dots, v_k, 0, \dots, 0, v_{n+2})$ is a normalized eigenvector associated to 1, then $v_{n+2} = -i$. This implies $\sum_{j=1}^{n} v_j^2 = -1$. Since $\sum_{j=1}^{n} |v_j|^2 \leq 1$, then $z = {}^t(0, v_2, \dots, v_k, 0, \dots)$ is of the form $z = e^{i\theta}x$, where $x \in \{0\} \times S^{k-2} \times \times \{0\}$ and $e^{2i\theta} = -1$. Thus the set of fixed points contains iS^{k-2}. As both $\cosh\alpha - \sinh\alpha$ and $\cosh\alpha + \sinh\alpha$ correspond to a point, the set of fixed points in $\overline{\mathcal{D}}$ consists of two isolated points and a sphere S^{k-2}.

Proof (of the theorem).. Coming now to the proof of the theorem we shall denote by S and h_S not only the quadratic and hermitian forms, but also the scalar products they induce. Let $\mathcal{F} = \{x \in \boldsymbol{C}^{n+2}: h_S(x, x) = 0\}$; if x is an eigenvector of g with eigenvalue ξ and $|\xi| \neq 1$, then x must be in $\mathcal{F}$ because g preserves h_S.

If y is another eigenvector with eigenvalue σ and $\bar{\xi}\sigma \neq 1$ then $h_S(x, y) = 0$ for the same reason.

As h_S has Witt index 2, i.e. the dimension of a maximal complex subspace on which h_S vanishes identically is 2, for every eigenvalue whose modulus is different from 1 there are no more than two linearly independent eigenvectors.

Moreover if x and y are two eigenvectors with eigenvalues ξ and σ respectively and if both of them are not contained in the unit circle, then $x, y \in \mathcal{F}$. If $\xi\bar{\sigma} \neq 1$, then $h_S(x, y) = 0$, this implies that there are no more than two eigenvalues which are not conjugated under the involution $\lambda \mapsto \bar{\lambda}^{-1}$ (because eigenvectors associated to different eigenvalues are linearly independent).

Let us consider now the quadratic form S: if x, y, ξ and σ are as above, we must have, as g preserves S, either $\xi^2 = 1$ or $x \in \mathcal{S}$, where $\mathcal{S} = \{u \in \boldsymbol{C}^{n+2} | S(u, u) = 0\}$.

Moreover if $\xi\sigma \neq 1$, then $S(x, y) = 0$.

Hence we can divide the distinct eigenvalues in three sets, which we list together with bases of corresponding eigenvectors:

1 with a base of eigenvectors $p_1^1, \dots p_{k_1}^1$,
-1 with a base of eigenvectors $p_1^2, \dots p_{k_2}^2$,
$e^{i\theta_1}$ with a base of eigenvectors $q_1^1, \dots q_{r_1}^1$, with $\theta_1 \neq 0$ (mod. π),
$\vdots$
$e^{i\theta_s}$ with a base of eigenvectors $q_1^s, \dots q_{r_s}^s$, with $\theta_s \neq 0$ (mod. π),
l_1 with a base of eigenvectors $u_1^1, \dots u_{t_1}^1$,
$\vdots$
l_a with a base of eigenvectors $u_1^a, \dots u_{t_1}^a$,

where $|l_j| \neq 1$. (It is possible that some of these are not present).

By the previous observations there are no more than 4 eigenvalues whose modulus are different from 1, and thus $a \leq 4$. Moreover we can suppose that l_1 is conjugated to l_2 and l_3 to l_4 (if they exist). We can say that t_j is 0, 1, 2 and, if we admit rearrangements, we can think that $t_1 \geq t_2$, t_3 and t_4; if t_1 is 2 then t_3 and t_4 must be 0 because h_s has Witt index 2.

What we saw before implies that S is identically 0 on the vector space of eigenvectors associated to any one of the eigenvalues in the second or in the third set. To examine fixed points for the transformation Ψ_g we need the following

LEMMA 3.4. Let x and y in $\mathcal{F}$ be normalized and $\sum_{k=1}^{n} |x_k|^2 \leq 1$, $\sum_{k=1}^{n} |y_k|^2 \leq 1$. Let us suppose that the vector space spanned by x and y is contained in $\mathcal{S}$, that is $S(x, x) = S(x, y) = S(y, y) = 0$. If the complex affine line joining $\tilde{x} = {}^t(x_1, \dots, x_n)$ and $\tilde{y} = {}^t(y_1, \dots, y_n)$ in $\boldsymbol{C}^n$ does not intersect $\mathcal{D}$, then $h_S(x, y) = 0$.

PROOF.. If both $\tilde{x}$ and $\tilde{y}$ are contained in the Shilov boundary, replacing if necessary $\tilde{x}$ and $\tilde{y}$ by $A\tilde{x}$ and $A\tilde{y}$, for a suitable $A \in O(n)$, we can suppose that $\tilde{x} = e_1$ and $\tilde{y} = e^{i\beta}\,{}^t(\cos\mu, \sin\mu, 0, \dots, 0)$ where $\mu \in \boldsymbol{R}$.

Then we can apply the linear biholomorphism $\varphi = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$ between $\Delta \times \Delta$ and $\mathcal{D}_2$. If the affine line joining $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{i\beta} \begin{bmatrix} \cos\mu \\ \sin\mu \end{bmatrix}$ does not intersect $\Delta \times \Delta$, then $\beta \pm \mu = 0$.

$$\beta + \mu = 0 \text{ gives } y = e^{i\mu}\,{}^t(\cos\mu, \sin\mu, 0, \ldots, 0, \cos\mu, -\sin\mu),$$

$$\beta - \mu = 0 \text{ gives } y = e^{i\mu}\,{}^t(\cos\mu, -\sin\mu, 0, \ldots, 0, \cos\mu, -\sin\mu),$$

so $h_S(x, y) = 0$ in both cases.

If at least one of the two vectors, say x, is not in the Shilov boundary, then $\sum_{k=1}^{n} |x_k|^2 < 1$. Every point on the complex affine line defined by x and y is normalized and in a neighborhood U of x we still have $\sum_{k=1}^{n} |x_k + t(y_k - x_k)|^2 < 1$ and $S(x + t(y - x),\ x + t(y - x)) = 0$.

If the affine line joining $\tilde{x}$ and $\tilde{y}$ does not intersect $\mathcal{D}$, we must have $h_S(x + t(y - x),\ x + t(y - x)) \geqslant 0$ in the neighborhood U.

As x amd y are in $\mathcal{F}$ we have $h_S(x + t(y - x),\ x + t(y - x)) = 2\,\mathrm{Re}\,(\bar{t}h_S(x, y)) + |t|^2 \mathrm{Re}\, h_S(x, y) \geqslant 0$: the fact that this is not negative in a neighborhood of $t = 0$ yields $h_S(x, y) = 0$. ∎

We now examine the three sets of eigenvalues.

We start with a trivial remark whereby the vector space spanned by two eigenvectors u and v associated to different eigenvalues contains no eigenvector which is not collinear to u or v; hence we are mainly interessed in the case of eigenvalues with geometric multiplicity greater than 1.

We have seen that, if the third set of eigenvalues contains an eigenvalue λ with geometric multiplicity 2, this set contains only λ and $\bar{\lambda}^{-1}$ and $\bar{\lambda}^{-1}$ has multiplicity less than two. Hence the third set yields at most four isolated fixed points or the set consisting of the intersections of $\overline{\mathcal{D}}$ with one or two complex affine lines.

Consider now the second set: as before we are mainly interessed in the case of geometric multiplicity greater than 1.

Let $v_1, \ldots, v_j$ be a base of eigenvectors associated to $e^{i\theta}$. We only consider affine combinations representing points in $\overline{\mathcal{D}}$, i.e. normalized and satisfying conditions (3.5).

We can suppose that v_1 satisfies these properties: if it is the unique vector in the vector space spanned by $v_1, \ldots, v_k$ which satisfies (3.5) then we change eigenvalue; if this is not the case we choose v_2 satisfying (3.5): on the vector space spanned by v_1 and v_2 S vanishes identically (because $e^{2i\theta} \neq 1$). By Lemma 4 we have $h_S(v_1, v_2) = 0$.

Hence, if w is a normalized vector verifying (3.5) and is not an affine combination of v_1 and v_2, the form h_S restricted to the vector space spanned by v_1, v_2 and w is identically 0. But this is not possible because h_S has Witt index 2.

Every eigenvalue of the second set yields an isolated fixed point or the intersection of $\overline{\mathcal{D}}$ with a complex affine line; in the last case the geometric multiplicity of the eigenvalue must be two.

The bounds we posed on the geometric multiplicity of eigenvalues 1 and -1 ensure that each of them yields the intersection of $\overline{\mathcal{D}}$ with complex affine line or an isolated fixed point, so we have proved the first part of our assertion.

Now we suppose that $u^1, \ldots, u^4$ are linearly independent eigenvectors associated to $\mu_1, \ldots, \mu_4$ which are eigenvalues different from ± 1 and that each u_j corresponds to a fixed point of f in $\partial\mathcal{D}$.

We have seen that $S(u^j, u^j) = 0$, $h_S(u^j, u^j) = 0$ and $\sum_{k=1}^{n} |u_k^j|^2 \leq 1$, $j = 1, \ldots, 4$.

We know that $h_S(u^a, u^b) = 0$, if $\mu_a \bar{\mu}_b \neq 1$, and $S(u^a, u^b) = 0$, if $\mu_a \mu_b \neq 1$.

Moreover we have proved that if u and v are eigenvectors associated to the eigenvalue $\mu \neq \pm 1$ and correspond to fixed points of f in $\partial\mathcal{D}$, then $h_S(v, u) = 0$ (if $|\mu| \neq 1$ it is obvious, if $|\mu| = 1$ see Lemma 4).

Let u be an eigenvector corresponding to a fixed point of f in $\partial\mathcal{D}$, and let μ be the associated eigenvalue of u. Moreover we suppose that u does not belong to the complex affine lines spanned by any pair of u^j.

The conditions (3.5) are $S(u, u) = 0$ and $h_S(u, u) = 0$. If we choose u normalized, then $\sum_{k=1}^{n} |u_k|^2 \leq 1$.

We now show that the fact that u does not belong to a complex affine line spanned by some u^k and u^j yields a contradiction.

We have two possible cases.

1) $\mu_1 = \mu_2 \neq \mu_3 = \mu_4$. Since we saw before that there is no eigenvalue different from ± 1 with more than 2 linearly independent eigenvectors, then $\mu \neq \mu_1, \mu_3$.

Moreover by Lemma 4 $h_S(u^1, u^2) = h_S(u^3, u^4) = 0$.

If $\mu\bar{\mu}_1 \neq 1$, then $h_S(u, u^j) = 0$, $j = 1, 2$, hence u, u^1 and u^2 are three linearly independent eigenvectors and they span a vector space of complex dimension three which is totally isotropic for h_S. However this is not possible because h_S has index 2; thus $\mu\bar{\mu}_1 = 1$.

With the same method we prove that $\bar{\mu}\mu_3 = 1$, and that implies $\mu\bar{\mu}_1 = 1 = \mu\bar{\mu}_3$, whence $\mu_1 = \mu_3$ but this is a contraddiction.

2) We are left to consider the case in which there are at least three different μ_j, which we call μ_1, μ_2, μ_3. If μ_4 coincides with one of them, say μ_3, we get $h_S(u^3, u^4) = 0$ by Lemma 4. As $\mu_1 \neq \mu_2$, it is not possible that $\bar{\mu}_1 \mu_3 = 1$ and $\bar{\mu}_2 \mu_3 = 1$, so one of them is different from 1, so

the three vectors u^1, u^3 and u^4 (if $\mu_1\bar{\mu}_3 \neq 1$) or u^2, u^3 and u^4 (if $\mu_2\bar{\mu}_3 \neq 1$) span a vector space of dimension three which is totally isotropic for h_S and this is impossible.

Then the μ_j are all distinct, and thus we can find at least three of them, say μ_1, μ_2, μ_3 such that $\mu\bar{\mu}_j \neq 1$. As they are all different, we can find two of them for which $\mu_j\bar{\mu}_k \neq 1$, with $j \neq k$. Hence u, u^k and u^j span a vector space of complex dimension three on which h_S is identically 0, and this is a contradiction.

So we have proved that there are at most four linearly independent eigenvectors associated to eigenvalues different from 1 and -1; this implies the bound $p + 2r \leqslant 4$, because a point corresponds to an eigenvector, while the intersection of a complex affine line with $\bar{\mathcal{D}}$ to a vector space of dimension 2 in C^{n+2}. ■

REMARK 4. The key-point in which we use the fact that f has no inner fixed points is Lemma 4: in fact in the proof of the theorem we never used the assumptions that f has no fixed points in $\mathcal{D}$ except in the proof of the lemma.

Notice that, in the case in which f has fixed points in $\mathcal{D}$, Lemma 4 is not true. Consider, for example,

$$g = \begin{bmatrix} \operatorname{rot}\theta_1 & 0 & \dots & (0) & 0 \\ 0 & \ddots & 0 & (0) & 0 \\ 0 & \dots & \operatorname{rot}\theta_m & (0) & 0 \\ (0) & (\dots) & (0) & (1) & (0) \\ 0 & 0 & 0 & (0) & \operatorname{rot}\theta \end{bmatrix},$$

where parenthesis indicate elements that exists iff n is odd.

Let us choose $\theta = \theta_1 = \theta_2$. Then $u = e_1 + e_{n+1}$, and $v = e_2 + e_{n+1}$ are two normalized eigenvectors corresponding to two fixed points on the Shilov boundary with $h_S(u, v) = -1$; while it is evident $\Psi_g(0) = 0$.

BIBLIOGRAPHY

[Abate 1] M. ABATE, *Automorphism groups of the classical domains*, Rend. Sci. Acc. Lin. (8), **76** (1985), pp. 127-131.

[Abate 2] M. ABATE, *Iteration theory of holomorphic maps on taut* manifolds, Mediterranean Press, Cosenza, 1989.

[Eisenman 1] D. EISENMAN, *Holomorphic mappings into tight manifolds*, Bull. Am. Math. Soc., **76** (1970), pp. 46-48.

[Gentili 1] G. GENTILI, *On complex geodesics of balanced convex domains*, Ann. Mat. Pura e Appl. (4), **144** (1986), pp. 113-130.

[Harris 1] L. A. HARRIS, *Bounded symmetric homogeneous domains in infinite dimensional holomorphy*, in *Proceedings on infinite dimensional holomorphy, Kentucky*, Lect. Notes in Math., **364** (Springer-Verlag, 1973), pp. 13-40.

[Hayden-Suffridge 1] T. K. HAYDEN - T. J. SUFFRIDGE, *Biholomorphic maps in Hilbert spaces have a fixed point*, Pac. J. Math., **38** (1971), pp. 419-422.

[Hervé 1] M. HERVÉ, *Quelques propriétés des applications analytiques d'une boule à m dimensions dans elle-même*, J. Math. Pures et Appl. (9), **42** (1963), pp. 117-147.

[Hirzebruch 1] U. HIRZEBRUCH, *Halbräume und ihre holomorphen Automorphismen*, Math. Ann., **153** (1964), pp. 395-417.

[Hua 1] L. K. HUA, *Harmonic analysis of functions of several variables in the classical domains*, Trans. Am. Math. Soc., R.I., 1963.

[Kaup-Upmeier 1] W. KAUP - H. UPMEIER, *Banach spaces with biholomorphically equivalent unit balls are isomorphic*, Proc. Amer. Math. Soc., **58** (1976), pp. 129-133.

[Kiernan 1] P. KIERNAN, *On the relation between taut, tight and hyperbolic manifolds*, Bull. Am. Math. Soc., **76** (1970), pp. 49-51.

[Kobayashi 1] S. KOBAYASHI, *Hyperbolic manifolds and holomorphic mappings*, Dekker, N.Y., 1970.

[Kobayashi 2] S. KOBAYASHI, *Intrinsic distances, measures and geometric function theory*, Bull. Am. Math. Soc., **82** (1976), pp. 1-82.

[Satake 1] I. SATAKE, *Algebraic structures of symmetric domains*, Princeton University Press, 1980.

[Vesentini 1] E. VESENTINI, *Variation on a theme of Carathéodory*, Ann. Scuola Norm. Sup. (9), **7** (1979), pp. 39-68.

[Vesentini 2] E. VESENTINI, *Complex geodesics*, Compos. Math., **44** (1981), pp. 375-394.

[Vesentini 3] E. VESENTINI, *Complex geodesics and holomorphic maps*, Symp. Math., **26** (1982), pp. 211-231.

[Vesentini 4] E. VESENTINI, *Semigroups of holomorphic isometries*, Adv. in Math., **65** (1987), pp. 272-306.

[Vesentini 5] E. VESENTINI, *Semigroups on Krein spaces*, to appear.

[Vesentini 6] E. VESENTINI, *Capitoli scelti della teoria delle funzioni olomorfe*, U.M.I. Gubbio, 1980.

[Vigué 1] P. VIGUÉ, *Géodesiques complexes et points fixes d'applications holomorphes*, Adv. in Math., **61** (1984), pp. 241-247.

[Vigué 2] P. VIGUÉ, *Points fixes d'applications holomorphes dans un domaine borné convexe de* C^n, Trans. Am. Math. Soc., **289** (1985), pp. 345-353.

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