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Some Remarks on Time-Dependent Evolution Systems in the Hyperbolic Case.

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0. Introduction.

In this note we are concerned with some remarks on the regularity of the evolution operator for the time dependent Cauchy problem in the hyperbolic case:

$$\text{(P)} \qquad \begin{cases} u'(t) = A(t)\,u(t) + f(t) & t \in]s;\, T]\,, \\ u(s) = x\,. \end{cases}$$

Existence and regularity of the evolution operator for such a problem has been considered by Kato in a series of papers (see [5], [6], [7], [9]) and also by Da Prato and Iannelli (in [2], [3]) with a different method, Here we set us in the framework of [2], [3] and show that, also in this framework, the reversibility condition on the family $\{A(t)\}_{t\in[0,T]}$, as considered in [5], leads to regularity of the evolution operator.

Next section is devoted to introduce some definitions and to recall some basic results from [2] and [3], then in section 2 we give our results.

1. Here we recall some notations of [2], [3]. Throughout this sections, X, Y are Banach spaces with norms $|\cdot|$, $\|\cdot\|$ respectively and T is a positive real number, the symbol $\|\cdot\|$ will be often used to indicate other norms (i.e. operator's norm) too. Let $\{A(t)\}_{t\in[0,T]}$ be a family of linear operators in X and let $D(t)$ be the domain of $A(t)$.

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1.1. DEFINITION. We say that $\{A(t)\}_{t\in[0,T]}$ is *ω-measurable* in X, where $\omega\in R$, if:

(i) $\varrho(A(t))\supset(\omega,+\infty)$ for $t\in[0,T]$;

(ii) $\forall\lambda>\omega$, $\forall x\in X$, $t\to R(\lambda,A(t)x)$ is measurable in $[0,T]$.

1.2. DEDINITION. The family $\{A(t)\}_{t\in[0,T]}$ is called *(M, ω)-stable* in X, where $M>0$, $\omega\in R$, if:

(i) $\varrho(A(t))\supset(\omega,+\infty)$ for $t\in[0,T]$;

(ii) $\|R(\lambda,A(t_k))R(\lambda,A(t_{k-1}))\dots R(\lambda,A(t_1))\|\leqslant M(\lambda-\omega)^{-k}$, for $\lambda>\omega$ and for every finite sequence $0\leqslant t_1\leqslant t_2\leqslant\dots\leqslant t_k\leqslant T$.

Define the following operator γ_0:

$$\begin{cases} D(\gamma_0):=W^{1,p}([0;T];X)\cap D(A)\,,\\ \gamma_0 u\;:=(u'-Au,u(0))\in L^p([0;T],X)\times X\,,\end{cases}$$

where $p\in[1,+\infty)$ and

$$\begin{cases} D(A):=\{u\in L^p([0;T];X):\\ \qquad\qquad u(t)\in D(t)\text{ a.e. and } t\to A(t)\in L^p([0;T];X)\}\,,\\ Au(t):=A(t)u(t)\,.\end{cases}$$

Let γ be the closure of γ_0. The study of evolution problem (P) is then reduced to

$$\text{(P)}\qquad \gamma u=(f,x)\in L^p([0,T];X)\times X\,.$$

Finally, let γ_n be defined like γ_0 with $A(t)$ replaced by its Yosida approximation $A_n(t)=n^2R(n,A(t))-nI$ (n is a positive integer). Consider also the problem

$$(\mathrm{P}_n)\qquad \gamma_n u_n=(f,x)\,.$$

In [2] the following result is given

1.3. THEOREM. We will make the following assumptions:

(J_0) X is reflexive and Y si densely and continuously imbedded in X;

(J_1) $\{A(t)\}_{t\in[0,T]}$ is ω-measurable and (M, ω)-stable in X;

(J_2) $D(t) \supset Y$, $A(t) \in BL(Y; X)$ amd $\|A(t)\|_{BL(Y,X)} \leqslant \alpha \in R_+$ for every t in $[0, T]$;

(J_3) the family $\{A_Y(t)\}_{t\in[0,T]}$ of the parts of $\{A(t)\}_{t\in[0,T]}$ in Y is ω^*-measurable and (M^*, ω^*)-stable in Y.

Then γ_0 is preclosed in $L^p([0, T]; X)$ and, for $p \in (1, +\infty)$, we have:

(i) if $(f, x) \in L^p([0, T]; X) \times X$ then there exists a unique $u \in D(\gamma)$ satisfying (P); moreover $C(I; X) \supset D(\gamma)$ and $u_n \to u$ uniformly in $[0, T]$;

(ii) if $(f, x) \in L^p([0, T]; Y) \times Y$ then $u \in D(\gamma_0)$; if in addition Y is reflexive then $u \in L^\infty([0, T]; Y)$. □

Obviously, Theorem 1.3 is still valid when the initial-time 0 is replaced by any $s \in [0, T]$; for $x \in X$ let $G(t, s)x$ and $G_n(t, s)x$ be the solutions of (P) and (P_n) with $f = 0$. Then G and G_n are evolution systems in X, $\|G(t, s)\|_{BL(X)} \leqslant M \exp[\omega(t-s)]$ and $G_n(t, s)x \to G(t, s)x$ in X for every $x \in X$, $0 \leqslant s \leqslant t \leqslant T$.

The proof of the next result is extracted from that of Theorem 5.2 in [5].

1.4. Theorem. Let X, Y be Banach spaces such that Y is uniformly convex; moreover let G be evolution system satisfying the following condition with $M^* = 1$:

(C) $Y \supset G(t, s)Y$, $\|G(t, s)\|_{BL(Y)} \leqslant M^* \exp[\omega^*(t-s)]$ for each $0 \leqslant s \leqslant$ $\leqslant t \leqslant T$ and $(t, s) \to G(t, s)$ is Y-weakly continuous.

Then, for fixed t_0, s_0 in $[0, T]$,

(i) $(t, s) \to G(t, s)$ is strongly Y-continuous in (s_0, s_0);

(ii) $s \to G(t_0, s)$ is strongly Y-continuous in $[0, t_0]$;

(iii) $t \to G(t, s_0)$ is strongly right Y-continuous in $[s_0, T]$ and, for $y \in Y$, $t \to G(t, s_0)y$ is Y-continuous in $[s_0, T]$ with the exception of a set σ which is countable at most. □

2. Now we state some propositions that will allow us to apply the results of section 1;

2.1. PROPOSITION. Under the hypothesis of Theorem 1.3, if Y is reflexive, then condition (C) holds.

PROOF. From [3] we know that, if $y \in Y$, $0 \leqslant s \leqslant t \leqslant T$ and $n \geqslant N > \omega^*$ then $G_n(t, s)y \in Y$ and

$$\|G_n(t, s)y\| \leqslant M^* \exp[n\omega^*(t-s)/(n-\omega^*)]\, \|y\| \leqslant \leqslant M^* \exp[N\omega^*(t-s)/(N-\omega^*)]\, \|y\| .$$

Because of $\lim G_n(t, s)y = G(t, s)y$ in X and since Y is reflexive, we see that $G(t, s)y \in Y$ and

$$\|G(t, s)y\| \leqslant M^* \exp[N\omega^*(t-s)/(N-\omega^*)] , \qquad \text{for any } N > \omega^* .$$

The weak continuity can be obtained by the some proof as in Kato's theorem (see [5]). □

Assuming Y uniformly convex and $M^* = 1$, in addition to the hypothesis of Proposition 2.1, and applying Theorem 1.4 we obtain the regularity results (i), (ii) and (iii). To remove the singularities in (iii) we introduce the following

2.2. DEFINITION. Assuming (J_0), we say that $\{A(t)\}_{t\in[0,T]}$ is *reversible* if both the families $\{A(t)\}_{t\in[0,T]}$ and $\{\bar{A}(t) := -A(T-t)\}_{t\in[0,T]}$ satisfy conditions (J_1), (J_2) and (J_3).

Under the hypothesis of reversibility let $\bar{G}$ be the evolution system generated by $\{\bar{A}(t)\}_{t\in[0,T]}$; then we have:

2.3. PROPOSITION. Assume (J_0) and let $\{A(t)\}_{t\in[0,T]}$ be reversible, then:

$$G(t, s)\bar{G}(T-s, T-t) = \bar{G}(T-s, T-t)G(t, s) = Id \qquad \text{for } 0 \leqslant s \leqslant t \leqslant T .$$

PROOF. For $x \in X$ the map $\tau \to \bar{G}(\tau, T-t)G(t, s)x$ is the solution of

$$\begin{cases} u'(\tau) = \bar{A}(\tau)u(\tau) = -A(T-\tau)u(\tau) & \tau \in [T-t; T] , \\ u(T-t) = G(t, s)x , \end{cases}$$

and so $\tau \to \bar{G}(T-\tau, T-t)G(t,s)x$ is solution of

$$\begin{cases} v'(\tau) = A(\tau)v(\tau) & \tau \in [0; t], \\ v(t) = G(t,s)x, \end{cases}$$

and by uniqueness it must be $G(\tau, s)x = v(\tau)$ for each τ in $[s, t]$; in particular we obtain, for $\tau = s$, $x = \bar{G}(T-s, T-t)G(t,s)x$ for $0 \leqslant s \leqslant t \leqslant T$. Exchanging G and $\bar{G}$ we conclude. □

Moreover we have

2.4. Proposition. Let G and $\bar{G}$ satisfy (i), (ii) and (iii) of Theorem 1.4 and assume that the equality

$$\bar{G}(T-s, T-t)G(t,s) = G(t,s)\bar{G}(T-s, T-t) = Id$$

holds; then $(t,s) \to G(t,s)$ is strongly Y-continuous at every (t_0, s_0) with $0 \leqslant s_0 \leqslant t_0 \leqslant T$.

Proof. by (i) of Theorem 1.4 we can suppose $t_0 > s_0$ and $t \geqslant a \geqslant s$ for fixed a with $t_0 > a > s_0$, so that $G(t,s) = G(t,a)G(a,s)$. It remain to prove that $\sigma = \Phi$; actually we have that, if $t < t_0$ and $y \in Y$ then

$$G(t,a)y = \bar{G}(T-t, T-t_0)G(t_0,t)G(t,a)y = \\ = \bar{G}(T-t, T-t_0)G(t_0,a)y \xrightarrow{Y} G(t_0,a)y,$$

when $t \to t_0$. □

2.5. Proposition. Assume (J_0) and let $\{A(t)\}_{t\in[0,T]}$ be reversible with $M^* = 1$ and Y uniformly convex.

Then $(t,s) \to G(t,s)$ is strongly Y-continuous on $0 \leqslant s \leqslant t \leqslant T$.

Proof. For Propositions 2.3 and 2.1, we can apply Proposition 2.4. □

We conclude with

2.6. Proposition. Assume the conditions (J_0)-(J_3), with $M^* = 1$ and Y uniformly convex and let $y \in Y$. Then there exists a zero measure set $N \subset [0, T]$ such that, for $t \in (0, T]$,

(i) for each s in $[0, t)\setminus N$ there exists $D_s^+ G(t, s)y = -G(t, s)\cdot A(s)y$ and for each s in $(0, t]\setminus N$ there exists $D_s^- G(t, s)y = -G(t, s)A(s)y$,

(ii) if $t \to A(t)$ is a continuous $BL(Y, X)$-valued map then $N = \Phi$

(where the derivatives are in strong sense in X).

PROOF. Because of ω-measurability and since $A_n(t)y \to A(t)y$ in X, $t \to A(t)y$ is measurable in X; from this, in addition to $|A(t)y| \leqslant \alpha \|y\|$ we see that the map $t \to A(t)y$ is in $L^1([0, T], X)$. Hence, if we define $N := [0, T]\setminus\{\text{Lebesgue's points}\}$, we know from measure theory that the measure of N is zero.

Now, $t \to G(t, s)y$ is in $D(\gamma_0)$, so for any $h > 0$,

$$\left|\frac{G(t, s+h)y - G(t, s)y}{h} + G(t, s)A(s)y\right| =$$

$$= \left|G(t, s+h)\left[G(s+h, s)A(s)y - \frac{1}{h}\int_s^{s+h} A(\tau)G(\tau, s)y\, d\tau\right]\right| \leqslant$$

$$\leqslant M \exp[\omega(t-s-h)]\left\{|G(s+h, s)A(s)y - A(s)y| + \right.$$

$$\left. + \frac{1}{h}\int_s^{s+h}|A(s)y - A(\tau)y|\, d\tau + \frac{1}{h}\int_s^{s+h}|A(\tau)\big(y - G(\tau, s)y\big)|\, d\tau\right\}$$

and by (i) in Theorem 1.4 we obtain the first part of (i). The second part can be proved analogously. Finally (ii) is a consequence of (i) and of the definition of N. □

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