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**Derivations in rings with involution**

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# Derivations in Rings with Involution.

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Let $R$ be a prime ring and $d$ be a non-zero derivation of $R$. In [1] B. Felzenszwalb and A. Giambruno proved that if $R$ has no non-zero nil ideals and $d$ is such that $d(x^n) = 0$ for all $x \in R$, where $n = n(x) \geqslant 1$ is an integer, then $R$ is an infinite commutative domain of characteristic $p \neq 0$ and $p \mid n(x)$.

In this paper we shall consider a generalization of this theorem to the case when the ring $R$ is equipped with an involution *. Let now $R$ be a ring with involution * and $S = \{x \in R: x = x^*\}$ the set of symmetric elements of $R$.

If $R$ is a domain of characteristic not 2 or 3 and $d$ a non-zero derivation of $R$ such that $d(s^n) = 0$ for any $s \in S$, where $n = n(s) \geqslant 1$ is an integer, then we shall prove that $R$ is an order in a division algebra at most 4-dimensional over its center, $\operatorname{char} R = p > 0$ and for all $s \in S$ either $p \mid n(s)$ or $d(s) = 0$. This result also generalizes [2, Theorem 2] to the case of non necessarily inner derivations.

More generally, if $R$ is equipped with an involution which is positive definite, we shall show that the derivation becomes inner in the Martindale quotients ring of $R$.

THEOREM 1. Let $R$ be a domain, $\operatorname{char} R \neq 2, 3$ and let $d \neq 0$ be a derivation of $R$ such that $d(s^n) = 0$, $n = n(s) \geqslant 1$, for all $s \in S$. Then $R$ is an order in a division algebra at most 4-dimensional over its center, $\operatorname{char} R = p \neq 0$, and for all $s \in S$ either $p \mid n(s)$ or $d(s) = 0$.

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PROOF. Suppose $S \not\subset Z(R)$ and let $s \neq 0$ in $S$. If $n \geqslant 1$ is such that $d(s^n) = 0$, let $A = C_R(s^n) = \{a \in R: as^n = s^n a\}$ the centralizer of $s^n$ in $R$. $A$ is invariant under $d$ and we may consider $d$ as a derivation on $A$. Now $A$ is a domain stable under * and $Z(A)$ is non-zero because $0 \neq s^n \in Z(A)$.

By localizing $A$ at $Z(A) - \{0\}$ we obtain a domain $Q \supset A$ whose center is a field containing $s^n$ and in particular $s$ is invertible in $Q$. As it is well known, $d$ extends uniquely to a derivation on $Q$ (which we shall also denote by $d$) defined by: $d(az^{-1}) = d(a)\, z^{-1} - ad(z)\, z^{-2}$ for $a \in A$ and $z \in Z(A) - \{0\}$. Under the induced involution the symmetric elements of $Q$ are of the form $t_1 t_2^{-1}$ where $t_1 \in S \cap A$ and $t_2 \in S \cap \big(Z(A) - \{0\}\big)$.

By our basic hypothesis on $d$, there exists $m \geqslant 1$ such that $d\big((t_1 t_2^{-1})^m\big) = 0$ and

$$d\big(s^{-1}(t_1 t_2^{-1})^m s\big) = d\Big(\big(s^{-1}(t_1 t_2^{-1})\, s\big)^m\Big) = 0\ .$$

Now we have:

$$d\big(ss^{-1}(t_1 t_2^{-1})^m s\big) = d\big((t_1 t_2^{-1})^m s\big) = (t_1 t_2^{-1})^m\, d(s)$$

and

$$d\big(ss^{-1}(t_1 t_2^{-1})^m s\big) = d(s)\, s^{-1}(t_1 t_2^{-1})^m s + sd\big(s^{-1}(t_1 t_2^{-1})^m s\big) = d(s)\, s^{-1}(t_1 t_2^{-1})^m s\ .$$

Hence $(t_1 t_2^{-1})^m d(s) = d(s)\, s^{-1}(t_1 t_2^{-1})^m s$ and, by multiplying from the right by $s^{-1}$, we obtain $(t_1 t_2^{-1})^m d(s)\, s^{-1} = d(s)\, s^{-1}(t_1 t_2^{-1})^m$.

Since $t_1 t_2^{-1}$ was an arbitrary symmetric element of $Q$, it follows that $d(s)\, s^{-1} \in H(Q)$, the symmetric hypercenter of $Q$. By [2] $H(Q) = Z(Q)$ and, so, $d(s)\, s^{-1} \in Z(Q)$. Commuting now $d(s)\, s^{-1}$ with $s$, we get $sd(s) = d(s)\, s$ for all $s \in S$. But then, by [3], $R$ is an order in a division algebra at most 4-dimensional over its center.

Also $sd(s) = d(s)\, s$ implies that

$$nd(s)\, s^{n-1} = d(s^n) = 0\ ,$$

hence $nd(s) = 0$. It follows that $\operatorname{char} R = p \neq 0$ and for all $s \in S$ either $p \mid n(s)$ or $d(s) = 0$.

If $S \subset Z(R)$, clearly $R$ is an order in a division algebra at most four dimensional over its center and by the same argument of the last paragraph, the conclusion of the theorem follows also in this case.

Recall that an involution is positive definite if $xx^* = 0$ implies $x = 0$. We now prove:

LEMMA 1. If $R$ is a ring with a positive definite involution, then $d(x)x = 0$ for all $x \in R$ such that $x^2 = 0$.

PROOF. Let $0 \neq x \in R$ be such that $x^2 = 0$. By our basic hypothesis on $d$ we have $d((xx^*)^n) = 0$ for some integer $n \geqslant 1$ and

$$0 = xd((xx^*)^n) = x^2 d(x^*(xx^*)^{n-1}) + xd(x)x^*(xx^*)^{n-1} = xd(x)x^*(xx^*)^{n-1}.$$

Since * is positive definite we get either

$$xd(x)x^*(xx^*)^{n/2} = 0 \qquad \text{or} \qquad xd(x)x^*(xx^*)^{n-1/2} = 0$$

according as $n$ is even or odd. A repeated application of this argument leads to $xd(x)x^* = 0$.

Now, since $x^2 = 0$, we obtain

$$0 = d(x^2x^*) = xd(x)x^* + d(x)xx^* + x^2d(x^*) = d(x)xx^*$$

and since * is positive definite $d(x)x = 0$.

We now make a remark that will be used in the proof of the next theorem:

REMARK. If $R$ is a ring with a positive definite involution *, then $R$ has no nonzero nil right ideals. In fact, let $\varrho \neq 0$ be a nil right ideal of $R$. Then, if $0 \neq x \in \varrho$, $xx^* \in \varrho$ and $(xx^*)^n = 0$ for some integer $n > 1$. By the hypothesis on * we have $(xx^*)^{n/2} = 0$ or $(xx^*)^{n-1/2} = 0$ according as $n$ is even or odd, a repeated application of this argument leads to $xx^* = 0$ and so $x = 0$.

We are finally able to prove the second result of this note. Let us write $Q$ for the Martindale quotients ring of the ring $R$.

THEOREM 2. If $R$ is a ring with a positive definite involution, then $d(x) = 0$ for all $x \in R$ such that $x^2 = 0$. Moreover there exists $\tilde{q} \in Q$ such that $\tilde{q}^2 = 0$ and $d$ is the inner derivation induced by $\tilde{q}$.

PROOF. Let $x, y$ be such that $xy = 0$. If $r \in R$ then $(yrx)^2 = 0$ and, by Lemma 1, $d(yrx)yrx) = yrxd(yrx) = 0$. Therefore, since $xy = 0$, we get

$$yrd(x)yrx = yrxd(y)rx = 0 . \tag{1}$$

Since $d(xy) = 0$ implies $d(x)y = -xd(y)$, then, by setting

$$a = xd(y) - d(x)y = 2xd(y) ,$$

it can be easily seen that (1) implies $(ar)^3 = 0$ for all $r \in R$. But then $aR$ is a nil right ideal of $R$. Since $*$ is positive definite, $R$ has no nil right ideals, forcing $a = 0$. We have proved that $xy = 0$ implies $xd(y) = d(x)y = 0$.

Define

$$A = \{b \in R\colon b^2 = 0\} \text{ and } T = \{r \in R\colon xy = 0 \text{ implies } xry = 0\} .$$

Notice that if $0 \neq t \in A \cap T$ then $t^2 = t^{*2} = 0$ and, since $t \in T$, $t^*tt^* = 0$. Since $*$ is positive definite we get $t = 0$. Thus $A \cap T = (0)$.

Take now $b \in A$ and $x, y \in R$ such that $xy = 0$. Since $x(1 + b)\cdot\cdot(1 - b)y = xy = b^2 = 0$, by what we have proved above it follows that $x(1 + b)d\big((1 - b)y\big) = bd(b) = xd(y) = 0$ and this implies

$$0 = (x + xb)d(y - by) = (x + xb)\big(d(y) - bd(y) - d(b)y\big) = -xd(b)y .$$

We have proved that $d(A) \subset T$. On the other hand $b^2 = 0$ implies $d(b)b = d^2(b)b = 0$ and from this we get $d\big(d(b)b\big) = d^2(b)\,b + d(b)^2 = d(b)^2$. This says that $d(A) \subset A$. Putting all the pieces together we have shown that $d(A) \subset A \cap T = (0)$; thus for all $b \in A$ $d(b) = 0$ and so, for all $r \in R$, $(brb)^2 = 0$ implies $d(brb) = bd(r)b = 0$.

By [4] (Proposition 1.1) $d$ is inner in $Q$, the Martindale quotients ring of $R$, induced by the element $\tilde{q} = \mathrm{cl}\,(RbR, q)$ such that $q(xby) = d(x)by$ for all $x, y \in R$. Because $bd(x)b = 0$ for all $x \in R$, we have $\big(d(x)b\big)^2 = 0$ and so $d\big(d(x)b\big) = 0 = d^2(x)b$.

Thus $q^2(xby) = q\big(d(x)by\big) = d^2(x)by = 0$ for all $x, y \in R$.

Then, since an element $\tilde{f} = \mathrm{cl}\,(I, f) \in Q$ is zero if $f(I) = 0$ ([6], pp. 20-21), the element $\tilde{q}$ is such that $\tilde{q}^2 = \mathrm{cl}\,(RbR, q^2) = 0$. This completes the proof.

REFERENCES

[1] B. Felzenszwalb - A. Giambruno, *A commutativity theorem for rings with derivations*, Pacific J. Math., **102** (1982), pp. 41-45.

[2] A. Giambruno, *On the symmetric hypercenter of a ring*, Canad. J. Math., **36** (1984), pp. 421-435.

[3] P. H. Lee - T. K. Lee, *Derivations centralizing symmetric or skew elements*, Bull. Inst. Math. Acad. Sinica, **14** (1986), pp. 249-256.

[4] J. Bergen - S. Montgomery, *Smash products and outher derivations*, Israel J. Math., **53** (1986), pp. 321-345.

[5] I. N. Herstein, *Noncommutative Rings*, Carus Mathematical Monographs, No. 15, American Math. Soc. and John Wiley and Sons, 1968.

[6] I. N. Herstein, *Rings with Involution*, Univ. Chicago Press Chicago 1976.

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