RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

HERMANN HEINEKEN

Nilpotent groups of class two that can appear as central quotient groups

Rendiconti del Seminario Matematico della Università di Padova, tome 84 (1990), p. 241-248

<http://www.numdam.org/item?id=RSMUP_1990__84__241_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 84 (1990)

Nilpotent Groups of Class Two that Can Appear as Central Quotient Groups.

HERMANN HEINEKEN (*)

In this note we will be concerned with the following question: Suppose $C_p \times C_p = G' \subseteq Z(G)$. What can be said about $G/Z(G)$ if G is isomorphic to some central quotient group $H/Z(H)$ of a group H? The answer to the corresponding question for $|G'| = p$ is wellknown for a long time; it is $|G/Z(G)| = p^2$ (see for instance Beyl and Tappe [1; p. 233]).

The proof of the answer (Proposition 3) makes use of our knowledge about vector spaces with two alternating bilinear forms. The bounds obtained are strict for odd primes p; this is shown in the second section. In the third section we give an example of a group G such that

$$G_3 = G^p = 1\,, \qquad |G'| = |Z(G)| = p^n\,, \qquad |G/G'| = p^{2n+\binom{n}{2}}$$

and G is a central quotient. This shows at least quadratic growth for the upper bound of the rank of $G/Z(G)$ with growing rank of G'.

1. The bounds for $|G/Z(G)|$.

In what follows we will have to deal with vector spaces V with two alternating bilinear forms f_1, f_2 which are of a comparatively transparent structure: There are two linear combinations

$$g_1 = af_1 + bf_2 \quad \text{and} \quad g_2 = cf_1 + df_2$$

(*) Indirizzo dell'A.: Universität Würzburg, Federal Republic of Germany.

and a basis $x_1, \ldots, x_m$ of V such that

$$g_1(x_{2t-1}, x_{2t}) = g_2(x_{2t}, x_{2t+1}) = 1 ,$$

$$g_1(x_{2t}, x_{2t+1}) = g_2(x_{2t-1}, x_{2t}) = 0 \quad \text{for all } t \leqslant \frac{m}{2} ,$$

$$g_1(x_i, x_j) = g_2(x_i, x_j) = 0 \quad \text{for } |i-j| > 1 .$$

For brevity we will call such a vector space a string with respect to g_1 and g_2. If $\dim V$ is odd, V is also a string with respect to any two different linear combinations of f_1, f_2; on the other hand if $\dim V$ is even, g_1 is fixed and g_2 can be changed to any other linear combination different from g_1. The reference to the respective bilinear forms will mostly be unnecessary and is then omitted.

A direct sum is called an orthogonal sum, if in addition the summands are orthogonal to each other with respect to all bilinear forms considered.

PROPOSITION 1. If two alternating forms f_1, f_2 are defined on a finite dimensional vector space V, then V is the orthogonal sum

$$V = R \oplus X_1 \oplus X_2 \oplus \ldots \oplus X_t ,$$

where every linear combination $rf_1 + sf_2 \not\equiv 0$ is nondegenerate on R and $X_1, \ldots, X_t$ are strings. Any two such decompositions of V are of the same form.

PROOF. Following Scharlau [2], we can compare the finite dimensional vector space V possessing two alternating forms with a Kronecker module consisting of two spaces and two endomorphisms mapping the first space into the second: for the first space take a subspace which is maximal with respect to both forms reducing to the zero form on it, for the second take the dual of the respective quotient space. Scharlau proves [2; 3.*e*, Theorem p. 14] that the decomposition of such a vector space into an unrefineable orthogonal sum is unique up to isomorphism. For R we take the sum of such summands with all linear combinations $rf_1 + rf_2$ non-degenerate; the remaining summands correspond to Kronecker modules M_2, L_n, L_n^* as described in [2; p. 16], and these are strings in our sense.

PROPOSITION 2. Denote by V a finite-dimensional vector space with two alternating forms f_1 and f_2 such that every lineare combination $rf_1 + rf_2$ is non-degenerate on V. Then

(i) $\dim V$ is even, and at least 4.

(ii) If W is a subspace of V of codimension 1, then $W = T \oplus S$, where every linear combination $rf_1 + rf_2$ is non-degenerate on T while S is a string of odd dimension.

PROOF. Since f_1 is nondegenerate on V, $\dim V$ is even, for $\dim V = 2$ f_1 and f_2 are linearly dependent. On the other hand, $\dim W = \dim V - 1$ is odd, and every linear combination $rf_1 + sf_2$ is degenerate on W, so W is an orthogonal sum with at least one string, and one string of odd dimension. Since every linear combination $rf_1 + sf_2$ is nondegenerate on V, $(rf_1 + sf_2)(x, W) = 0$ has as space of solution a subspace of dimension $\dim V - \dim W = 1$ at most. This shows that there is at most one string, it contains all the solutions mentioned.

In the sequel we make use of the following well known fact: If G is a p-group of nilpotency class 2 and $G' = \langle a \rangle \times \langle b \rangle$ with $p^p = b^p = 1$, then the mapping

$$(x, y) \rightarrow [x, y] = a^{r(x,y)} b^{s(x,y)}$$

induces two alternating bilinear forms on $G/Z(G)$. This allows us to argue from vector spaces to groups and back. This argument can be found operating in Vishnevetskii [3], for instance.

PROPOSITION 3. If G is a finite group such that

$$C_p \times C_p = G' \subseteq Z(G)$$

and there is a group H such that $G = H/Z(H)$, then

$$p^2 < |G/Z(G)| < p^6 .$$

PROOF. The first part of the inequality $p^2 < G/Z(G)$ is obvious. For the other we begin with some preliminary statements. We assume that G is isomorphic to some quotient $H/Z(H)$ and deduce restrictions on G.

(a) If $G = UV$ and $[U, V] = 1$, then $U' \cap V' = 1$.

If, on the contrary, $G = UV$ with $[U, V] = 1$ and $U' \cap V' \neq 1$ and $G = H/Z(H)$, we choose a basis $u_1, \dots, u_r, v_1, \dots, v_s$ of G such that the elements u_i belong to U and the v_j's to V. Since $U \cap V \subseteq Z(G)$ we have $\langle u_1, \dots, u_r \rangle Z(G) = UZ(G)$ and $\langle v_1, \dots, v_s \rangle Z(G) = VZ(G)$.

The pre-image of the element x of G with respect to the mapping of H onto $G = H/Z(H)$ shall be denoted by $\tilde{x}$.

Now

$$[[v_i, v_j], u_k] = [[u_k, v_i], v_j]^{-1}[[v_j, u_k], v_i]^{-1} = 1 \qquad \text{for all } i, j, k .$$

The same holds if the roles of U and V are interchanged. Take

$$\prod [u_i, u_j]^{n_{ij}} = \prod [v_i, v_j]^{m_{ij}} = c \neq 1 .$$

Then $\tilde{c} \notin Z(H)$ and $[\tilde{c}, \tilde{v}_k] = [\tilde{c}, \tilde{u}_k] = 1$ for all k, a contradiction. So (a) is true.

(b) If $G = UV$ with $[U, V] = 1$ and $UZ(G) \neq Z(G) \neq VZ(G)$, then $G/G' = p^4$.

By (a) we have $U' \cap V' = 1$, and the hypothesis yields $U' \neq 1 \neq V'$. So both commutator subgroups U', V' have order p and $UZ(G)/Z(G)$ and $VZ(G)/Z(G)$ are elementary abelian of even rank. Using (a) again we see that both these quotient groups must be of order p^2 and (b) follows.

From now on we consider $G/Z(G)$ as a F_p-vector space V with two alternating forms, as outlined just before this Proposition. By (b) we have

(c) If $\dim V > 4$, there is no proper decomposition of V into an orthogonal sum.

We assume $\dim V = m > 5$. If there is a linear combination $rf_1 + sf_2$ $((r, s) \neq (0, 0))$ which is degenerate on V, then V is a string with respect to $rf_1 + sf_2 = g$ and one of f_1, f_2, say f. So there are generators $x_1Z(G), \dots, x_mZ(G)$ of $G/Z(G)$ and a, b of G' such

$$[x_{2t-1}, x_{2t}] = a , \quad [x_{2t}, x_{2t+1}] = b , \quad [x_h, x_k] = 1 \quad \text{for } |h - k| > 1 .$$

Using pre-images as before we have

$$[[\tilde{x}_1, \tilde{x}_2], \tilde{x}_k] = [[\tilde{x}_2, \tilde{x}_k], \tilde{x}_1]^{-1}[[\tilde{x}_k, \tilde{x}_1], \tilde{x}_2]^{-1} = 1 \quad \text{for } k > 3 ,$$

and, using the same argument,

$$[[\tilde{x}_1, \tilde{x}_2], \tilde{x}_k] = [[\tilde{x}_5, \tilde{x}_6], \tilde{x}_k] = 1 \quad \text{for } k \leqslant 3 ,$$

$$[[\tilde{x}_2, \tilde{x}_3], \tilde{x}_k] = 1 \quad \text{for } k > 4 ,$$

$$[[\tilde{x}_2, \tilde{x}_3], \tilde{x}_k] = [[\tilde{x}_4, \tilde{x}_5], \tilde{x}_k] = 1 \quad \text{for } k \leqslant 2 ,$$

$$[[\tilde{x}_2, \tilde{x}_3], \tilde{x}_3] = [[\tilde{x}_4, \tilde{x}_5], \tilde{x}_3] = [[\tilde{x}_5, \tilde{x}_3], \tilde{x}_4]^{-1}[[\tilde{x}_3, \tilde{x}_4], \tilde{x}_5]^{-1} = 1 ,$$

$$[[\tilde{x}_2, \tilde{x}_3], \tilde{x}_4] = [[\tilde{x}_3, \tilde{x}_4], \tilde{x}_2]^{-1}[[\tilde{x}_4, \tilde{x}_2], \tilde{x}_3]^{-1} = 1 ,$$

and neither $\tilde{a}$ nor $\tilde{b}$ are outside $Z(H)$, a contradiction. This shows

(d) If V is a string, $\dim V \leqslant 5$.

Assume now that V is not a string but all bilinear forms are nondegenerate on V and $\dim V \geqslant 6$. Consider a subspace W of codimension 1 of V; by Proposition 2 we know that W is the orthogonal sum of a completely nondegenerate part and a string. If $\dim V \geqslant 8$ either the orthogonal sum is nontrivial and $\tilde{a}$, $\tilde{b}$ commute with all elements of the pre-image of W, or W is a string of dimension 7 at least, with the same consequence. Since this holds for all W, this also holds for V, a contradiction. We have found

(e) $\dim V \geqslant 7$.

If $\dim V = 6$, each W must be a string by Proposition 2 (i). We choose a basis $x_1 Z(G), \ldots, x_6 Z(G)$ of $G/Z(G)$ and determine the maximal subgroups U_i of G such that $[U_i, x_i] = \langle b \rangle$.

We have corresponding subspaces W_i of codimension 1 of V. These subspaces are strings and allow a basis as a string such that x_i appears as the first basis element y_1. Now

$$[[\tilde{y}_1, \tilde{y}_2], \tilde{y}_1] = [[\tilde{y}_3, \tilde{y}_4], \tilde{y}_1] = 1 ,$$

$$[[\tilde{y}_2, \tilde{y}_3], \tilde{y}_1] = [[\tilde{y}_4, \tilde{y}_5], \tilde{y}_1] = 1 ,$$

and $\tilde{a}$, $\tilde{b}$ commute with every of the $\tilde{x}_i$, the final contradiction

(*f*) $\dim V < 6$,

and this proves the Proposition.

2. Construction of some groups H.

To show that Proposition 3 is in a sense bestpossible we construct groups H for the case

$$G' = Z(G) = C_p \times C_p\,, \qquad G^p = 1\,.$$

This excludes $p = 2$, where more scrutinous observations are necessary. In each case a basis of $H_3 \cap Z(H)$ will be given such that the order of this characteristic subgroup is maximal. It is not too difficult to determine all $T \subset H_3 \cap Z(H)$ such that $(H/T)/Z(H/T)$ is still isomorphic to G; for brevity we do not concern ourselves with this task.

Case A: $|G/G'| = p^3$.
Here we have

$$G = \left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} [x_1, x_2] = a,\ [x_2, x_3] = b \\ x_i^p = [[x_i, x_j], x_k] = [x_1, x_3] = 1 \end{array} \right\rangle .$$

In the notation as before we find

$$\begin{aligned} H_3 \cap Z(H) = \langle [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_1], [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_3], [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_2], \\ [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_2[, [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_3] = [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_1]^{-1}\rangle\,. \end{aligned}$$

Case B: $|G/G'| = p^4$, the string case.
Then

$$G = \left\langle x_1, x_2, x_3, x_4 \;\middle|\; \begin{array}{l} [x_1, x_2] = [x_3, x_4] = a \\ [x_2, x_3] = b \\ x_i^p = [[x_i, x_j], x_k] = 1 \\ [x_i, x_j] = 1 \text{ for } |i-j| > 1 \end{array} \right\rangle .$$

and

$$H_3 \cap Z(H) = \left\langle \begin{matrix} [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_3] = [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_1]^{-1} \\ [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_4] = [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_2]^{-1} \\ [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_2], \ [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_3] \end{matrix} \right\rangle .$$

Case C: $|G/G'| = p^4$ and G is a direct product $\langle x_1, x_2 \rangle \times \langle x_3, x_4 \rangle$.
Then

$$H_3 \cap Z(H) = \langle [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_1], [[\tilde{x}_1, \tilde{x}_2], \tilde{x}_2], [[\tilde{x}_3, \tilde{x}_4], \tilde{x}_3], [[\tilde{x}_3, \tilde{x}_4], \tilde{x}_4] \rangle .$$

Case D: $|G/G'| = p^4$, completely nondegenerate case.
Here G can be described as a group with the galois field of order p^2 as operator domain, and $H_3 \cap Z(H) \leqslant p^4$.
The actual description would depend on the prime p.

Case E: $G/G' = p^5$.
Here

$$G = \left\langle x_1, x_2, x_3, x_4, x_5 \;\middle|\; \begin{matrix} [x_1, x_2] = [x_3, x_4] = a \\ [x_2, x_3] = [x_4, x_5] = b \\ x_i^p = 1 = [[x_i, x_j], x_k] \\ [x_i, x_j] = 1 \text{ for } |i-j| > 1 \end{matrix} \right\rangle .$$

and

$$H_3 \cap Z(H) = \langle [[\tilde{x}_2, \tilde{x}_3], \tilde{x}_4] = [[\tilde{x}_3, \tilde{x}_4], \tilde{x}_2]^{-1} \rangle .$$

(In particular H does not exist if $x_2^p = [x_3, x_4]$.)

REMARKS. (1) The quotient groups $H_2 \cap Z(H)/H_3 \cap Z(H)$ have orders bounded by p in Case A, p^4 in Cases B, C, D and p^8 in Case E.

(2) If $G = H/Z(H)$ and G is a p-group, if further $|\langle x \rangle| = p$, then also $G \times \langle x \rangle$ is a central quotient: Choose a maximal subgroup M of G and an element y such that $G = \langle M, y \rangle$, and form the extension K of $\langle x, z \rangle$ by H such that $z^p = 1 = [x, z]$, $[x, \tilde{y}] = z$, $[x, \tilde{t}] = [z, \tilde{t}] = 1$ for all $\tilde{t}$ in the pre-image $\tilde{M}$ of M in H. Now $K/Z(K)$ is isomorphic to $G \times \langle x \rangle$.

This shows that groups H do exist as constructed in this section as long as $G^p = 1$, even if $G' \not\subseteq Z(G)$.

3. An example for higher rank.

Consider

$$G = \left\langle \begin{array}{l|l} s_i, t_i, \ i \leqslant n & s_i^p = t_i^p = m_{ij}^p = 1 \\ m_{ij} = m_{ji}, \ i \neq j, \ i, j \leqslant n & [s_i, t_i] = [s_j, s_j, m_{ij}] = c_i \\ & [s_i, t_j] = [s_i, s_j] = [t_i, t_j] = 1 \text{ for } i \neq j \\ & [m_{ij}, t_k] = 1 \text{ for all } i, j, k \\ & [m_{ij}, s_k] = 1 \text{ for } i \neq k \neq j \\ & [m_{jk}, m_{55}] = 1 \text{ for all } i, j, u, v \\ & [[g_1, g_2], g_3] = 1 \text{ for all } g_1 \text{ in } G \end{array} \right\rangle$$

This group is isomorphic to a central quotient $H/Z(H)$ where

$$H_3 \cap Z(H) = \langle [[\tilde{s}_i, \tilde{t}_i], \tilde{s}_i] = [[\tilde{s}_j, \tilde{m}_{ij}], \tilde{s}_i] = \\ = [[\tilde{s}_i, \tilde{m}_{ij}], \tilde{s}_j] = [[\tilde{s}_j, \tilde{t}_j], \tilde{s}_j] \quad \text{for all } i, j \rangle .$$

This follows from the fact that the vector space corresponding to the subgroup $\langle t_i, s_i, m_{ij}, s_j t_j, Z(G) \rangle$ is a string.

REFERENCES

[1] F. R. Beyl - J. Tappe, *Extensions, representations, and the Schur multiplicator*, Lecture Notes in Mathematics, **958**, Berlin, Heidelberg, New York, 1989.

[2] R. Scharlau, *Paare alternierender Formen*, Math. Z., **147** (1976), pp. 13-19.

[3] A. L. Wishnevetskii, *Groups of class 2 and exponent p with commutator subgroups of order p^2*, Doklady Akad. Nauk Ukrain. S.S.R. Ser. A, **103**, 9 (1980), pp. 9-11 (Russian).

Manoscritto pervenuto in redazione il 15 settembre 1989 e in forma revisionata l'8 novembre 1989.