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Stationary Spatially Periodic Compressible Flows at High Mach Number.

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ABSTRACT - We prove the existence of stationary spatially periodic solutions to the Navier-Stokes equations for compressible flows even for the case of high Mach number, under the assumption that the external force field is small in a suitable sense. The proof of this result is based on a existence result for a convenient linearized problem, followed by a fixed point argument.

1. Introduction.

A computational technique frequently used to obtain stationary flow is based on perturbation schemes; that is, we consider the flow to be computed as a perturbations of a known « mean flow », and, then, we use a successive approximation method to compute it. The theoretical counterpart of this technique is to use a fixed point argument for a suitable operator obtained by rewriting the equations for the flow in a convenient way. The task, therefore, is to obtain certain a priori estimates that are used to guarantee that the associated operator has the required properties for a fixed point theorem be applied.

Usually, the difficulties that appear during the process of the derivations of the a priori estimates are such that the hypothesis of the smallness of the speed of the mean flow is required, ruling out

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in this way the possibility of studying of solutions corresponding to high Mach numbers.

The aim of this work is to show the existence of stationary flows in the whole space of a viscous compressible fluid in the case of spatially periodic perturbations of a constant mean flow (see (1.2)), even in the case of high Mach numbers.

We recall that the equations for the motion in R^n, $n = 2$ or 3, of a viscous compressible barotropic fluid in the stationary case can be written as:

$$
(1.1)\quad \begin{cases} \varrho[(v\cdot\nabla)v - f] - \mu\Delta v - (\zeta + \mu/3)\nabla \operatorname{div} v + \nabla[p(\varrho)] = 0 \quad \text{in } R^n, \\ \operatorname{div}(\varrho v) = 0 \quad \text{in } R^n. \end{cases}
$$

The first equation corresponds to the conservation of momentum, the second to the conservation of mass. In these equations v and ϱ are the velocity and the density of the fluid, respectively; p is the pressure, which is assumed to be a known increasing function of ϱ; f is the assigned external force field; the constants $0 < \mu$ and $0 \leqslant \zeta$ are the viscosity coefficients (see Serrin [5] for details); ∇, Δ, div are, respectively, the gradient, Laplacian and divergence operators. In this paper the external force field is assumed to be periodic in space and small in a norm to be defined later on.

We look for a spatially periodic solution of (1.1) by using a variant of the method introduced by Valli in the paper [7], for the case of bounded domain and small Mach number. We prove first a existence theorem for a suitable linearization of (1.1) around the given mean flow, followed by a fixed point argument using Schauder fixed point theorem. In order to cope with the high speed of the mean flow, we have to retain in the linearized operator certain terms that did not show up in Valli's work. We observe that the linearized problem is solved via the continuation method and that we need a priori estimates in Sobolev spaces of sufficiently high order to be able to handle the nonlinear terms.

Let us now describe the main result. We will be working in the important case of perturbation of a uniform stream; that is, we will be searching for a nontrivial solution in the neighborhood of:

$$
(1.2)\quad \begin{cases} \varrho_0 \text{ a positive constant}; \\ v_0 = (\omega, 0) \text{ if } n = 2 \quad \text{or} \quad v_0 = (\omega, 0, 0) \text{ if } n = 3, \\ \qquad\qquad\qquad\qquad \text{where } \omega \text{ is a positive constant}, \end{cases}
$$

which is a solution of (1.1) when $f = 0$. We observe that since we do not impose restrictions on the magnitudes of ϱ_0 and ω, this mean flow can have high Mach number. The case of other mean flows is still under investigation.

Now we introduce the variables:

$$w = v - v_0\,, \qquad \eta = \varrho - \varrho_0\,, \tag{1.3}$$

and the equations of motion become:

$$\left\{\begin{aligned} &(\varrho_0 + \eta)\{[(v_0 + w)\cdot\nabla](v_0 + w) - f\} - \mu\Delta(v_0 + w) + \\ &\qquad\qquad + (\zeta + \mu/3)\nabla \operatorname{div}(v_0 + w) + \nabla[p(\varrho_0 + \eta)] = 0\,,\\ &\operatorname{div}[(\varrho_0 + \eta)(v_0 + w)] = 0\,. \end{aligned}\right. \tag{1.4}$$

Denoting $H_p^{m,0}(Q)$ the Sobolev space, with norm $\|\cdot\|_m$, of the Q-periodic functions with mean value zero (see the section (2.1) for the precise definitions), we can state the following theorem:

THEOREM 1.1. Consider ϱ_0 and v_0 as in (1.3), and suppose that $p(\cdot)$ is a C^2-function such that $p'(\varrho_0) > 0$, and $f \in (H_p^{1,0}(Q))^n$ with $\|f\|_1$ sufficiently small. Then there exists a solution (w, η) $(H_p^{3,0}(Q))^n \times \times H_p^{2,0}(Q)$ to (1.4) such that $\|w\|_3 + \|\eta\|_2 \leqslant C\|f\|_1$, where C is independent of f.

Finally, we should mention that the method of the proof of the theorem sugests that, for purposes of numerical computation using perturbation schemes of flows at high Mach numbers, it is important to keep certain terms in the linearized equations to be used. Some of these terms come from the equation of conservation of mass; they are the term $\operatorname{div}(v\eta)$ (actually $(v\cdot\nabla)\eta$) to ensure that there is no loss of regularity at each iteration of the schem, and the term $(v_0\cdot\nabla)\eta$ to guarantee high Mach numbers.

2. The linearized problem.

2.1. *The functional setting of the equations.*

We consider the same functional setting as Temam does in [6]; briefly, given a positive constant L, we take the n-cube $Q = [0, L]^n$

and denote by $H_p^m(Q)$, $m \in N$, the space of functions which are in $H_{\mathrm{loc}}^m(R^n)$ (i.e., the Lebesgue measurable functions defined in R^n such that their restrictions to any bounded open set Ω in R^n belong to the usual Sobolev space $H^m(\Omega)$), and which are periodic with period Q, that is, functions $u(\cdot)$ such that $u(x + Le_i) = u(x)$ for all $x \in R^n$, $i = 1, \ldots, n$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of R^n.

For an arbitrary $m \in N$, $H_p^m(Q)$ is a Hilbert space for the scalar product:

$$(u, v)_m = \sum_{[\alpha] \leqslant m} \int_Q D^\alpha u(x) D^\alpha v(x)\, dx .$$

The functions in $H_p^m(Q)$ are caracterized by their Fourier series expansions:

$$H_p^m(Q) = \left\{u,\ u = \sum c_k \exp(2ik\cdot x/L),\ c_k = c_{-k},\ \sum |k|^{2m} |c_k|^2 < \infty,\ (k \in Z^n)\right\}$$

where $|\cdot|$ denotes the euclidean norm in R^n. The norm $\|u\|_m$ induced by the scalar product is equivalent to the norm given by

$$\left\{\sum_k (1 + |k|^{2m})\, |c_k|^2\right\}^{\frac{1}{2}} .$$

We remark that to easy the notation we will denote by $\| \ \|_m$ also the norm in the space $(H_p^m(Q))^n$ (cartesian product of n copies of $H_p^m(Q)$). We also set:

$$H_p^{m,0}(Q) = \left\{u \in H_p^m(Q) \text{ such that in its Fourier expansion } c_0 = 0\right\},$$

that is, we are considering functions with mean value zero.

Now, given $F \in H_p^{1,0}(Q))^n$, $G \in H_p^{2,0}(Q)$, v_0 as before, $p_1 > 0$ and $v \in (H_p^{3,0}(Q))^n$ such that $\|v\|_3$ is sufficiently small (this will be precised later on), we want to find a solution $(w, \eta) \in (H_p^{3,0}(Q))^n \times H_p^{2,0}(Q)$ to the system:

$$\text{(2.1)} \quad \left\{\begin{array}{l} -\mu\, \Delta w - (\zeta + \mu/3) \nabla \operatorname{div} w + (\varrho_0 v_0 \cdot \nabla) w + p_1 \nabla \eta = F\ , \\ \varrho_0 \operatorname{div} w + (v_0 \cdot \nabla)\eta + \operatorname{div}(\eta v) = G\ . \end{array}\right.$$

Concerning this problem, we have the following result:

THEOREM 2.1. If $v \in (H_p^{3,0}(Q))^n$ is such that $\|v\|_3$ is sufficiently small (see (2.7)) then there is a unique solution $(w, \eta) \in (H_p^{3,0}(Q))^n \times \times H_p^{2,0}(Q)$ to the system (2.1). Moreover, this solution satisfies the estimate

$$\|w\|_3^2 + \|\eta\|_2^2 \leqslant C[\|F\|_1^2 + \|G\|_2^2], \tag{2.2}$$

where C does not depend on F, G.

REMARK 2.1. Suppose for a moment that the above line theorem is true. Let us take $p_1 = p'(\varrho_0)$ in (2.1) and, for given $(v, \sigma) \in \in (H_p^{3,0}(Q))^n \times H_p^{2,0}(Q)$, call

$$\Phi(v, \sigma) = (w, \eta)$$

the solution of (2.1) which corresponds to F and G given by

$$F = F(v, \sigma) = -\sigma(v_0 \cdot \nabla)v - \sigma(v \cdot \nabla)v - \varrho_0(v \cdot \nabla)v + \\ + \sigma f + \varrho_0 f - [p_1 - p'(\varrho_0 + \sigma)]\nabla\sigma,$$

$$G = 0.$$

These functions have mean value zero if the mean value of f is zero, and $F \in (H_p^{3,0}(Q))^n$, $G \in H_p^{2,0}(Q)$ because of the usual Sobolev imbeddings theorems (it is to handle terms like $\sigma(v \cdot \nabla)v$ that we require Sobolev spaces of relative high order).

Then, a fixed point of the map

$$\Phi\colon (v, \sigma) \to (w, \eta)$$

is a solution of the problem (1.4) (it is used the fact that (ϱ_0, v_0) is a solution to (1.1) with $f = 0$).

In what follows we will prove that (2.1) has a solution satisfying (2.2) by a continuation argument, and secondly that Φ has a fixed point via the Schauder fixed point theorem (we need (2.2) to control the nonlinear terms in the above F).

2.2. *A priori estimates for the linearized problem.*

The necessary a priori estimates will be obtained in the sequence of lemmas that follow. We begin by observing that the following periodic Stokes Problem can be treated exactly as in Temam [6], and as we did in Section 2.2, by using Fourier series. We conclude that the unique solution $(u, q) \in (H_p^{m+2,0}(Q))^n \times H_p^{m+1,0}(Q)$ to

$$-\mu \Delta u + \nabla q = h_1 , \qquad \operatorname{div} u = h_2 ,$$

for $h_1 \in (H_p^{m,0}(Q))^n$, $h_2 \in H_p^{m+1,0}(Q)$, satisfies the estimate

$$\|u\|_{m+2} + \|q\|_{m+1} \leqslant C[\|h_1\|_m + \|h_2\|_{m+1}] ,$$

for any $m \in N$, with C independent of h_1, h_2.

Using this and the form of the equations in (2.1), one quickly obtains the

LEMMA 2.1. A solution $(w, \eta) \in (H_p^{3,0}(Q))^n \times H_p^{2,0}(Q)$ of (2.1) satisfies

$$(2.3) \qquad \begin{cases} \text{(i)} & \|w\|_2^2 + \|\eta\|_1^2 \leqslant C[\|F\|_0^2 + \|\operatorname{div} w\|_1^2] , \\ \text{(ii)} & \|w\|_3^2 + \|\eta\|_2^2 \leqslant C[\|F\|_1^2 + \|\operatorname{div} w\|_2^2] . \end{cases}$$

Thus, from (2.3) (ii) it is enough to estimate $\|\operatorname{div} w\|_2$ to prove (2.2). For this, we enunciate

LEMMA 2.2. For any $0 < \varepsilon_1$, there is a positive constant C, independent of ε_1, such that any solution of $(w, \eta) \in (H_p^{3,0}(Q))^n \times H_p^{2,0}(Q)$ of (2.1) satisfies:

$$(2.4) \qquad \|w\|_1^2 + \|\operatorname{div} w\|_0^2 \leqslant C\left[\|F\|_{-1} + \frac{1}{\varepsilon_1}\|G\|_0^2 + \|v\|_3\|\eta\|_0^2 + \varepsilon_1\|\eta\|_0^2\right] .$$

PROOF. We multiply (2.1) (i) scalarly by w, (2.1) (ii) by $p_1\eta/\varrho_0$, integrate in Q, and add the two resulting equations. After some integrations by parts, using of the periodic boundary conditions, the fact that w has mean value zero and also

$$\int_Q p_1 \nabla\eta \cdot w = -\int_Q p_1 \eta \operatorname{div} w ,$$

one is left with:

$$\mu\|\nabla w\|_0^2 + (\zeta + \mu/3)\|\operatorname{div} w\|_0^2 \leqslant C\left[\|F\|_{-1}^2 + \frac{1}{\varepsilon_1}\|G\|_0^2 + \right.$$
$$\left. + (\mu/2)\|\nabla w\|_0^2 + \varepsilon_1\|\eta\|_0^2\right] - \int_Q (\varrho_0 v_0\cdot\nabla)w\cdot w -$$
$$- \int_Q v_0\cdot\nabla\eta(p_1/\varrho_0)\eta - \int_Q \operatorname{div}(\eta v)(p_1/\varrho_0)\eta\,.$$

But, we have

$$\int_Q (v_0\cdot\nabla)w\cdot w = \int_Q v_0\cdot\nabla(|v|^2/2) = -\int_Q \operatorname{div} v_0(|w|^2/2) = 0\,,$$
$$\int_Q (v_0\cdot\nabla)\eta\cdot\eta = \int_Q v_0\cdot\nabla(\eta^2/2) = -\int_Q \operatorname{div} v_0(\eta^2/2) = 0\,,$$
$$\int_Q \operatorname{div}(\eta v)\eta = -\int_Q \eta v\cdot\nabla\eta = -\int_Q v\cdot\nabla(\eta^2/2) = \int_Q \operatorname{div} v(\eta^2/2)\,,$$

and since $\|\operatorname{div} v\|_\infty \leqslant C\|v\|_3$, v and η have mean value zero, the stated result follows.

To obtain higher order estimates, we apply the operator D_i (which denotes $\partial/\partial x^i$), $i = 1, \dots, n$, to (2.1); then we multiply the first of the resulting equations by $D_i w$, the seconj by $(p_1/\varrho_0)D_i\eta$, and integrate on Q. Proceeding exactly as before, we obtain that there is a positive constant C such that for any $0 < \varepsilon_2$

$$\|w\|_2^2 \leqslant C\left[\|F\|_0^2 + \frac{1}{\varepsilon_2}\|G\|_1^2 + \|v\|_3\|\eta\|_1^2 + \left(1 + \frac{1}{\varepsilon_2}\right)\|w\|_1^2 + \varepsilon_2\|\eta\|_1^2\right].$$

By using (2.4), we obtain

LEMMA 2.3. *Any solution* $(w, \eta) \in (H_p^{3,0}(Q))^n \times H_p^{2,0}(Q)$ *satisfies*

$$\|w\|_2^2 \leqslant C\left[\left(1 + \frac{1}{\varepsilon_2}\right)\|F\|_0^2 + \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_1\varepsilon_2}\right)\|G\|_1^2 + \right.$$
$$\left. + \left(1 + \frac{1}{\varepsilon_2}\right)\|v\|_3\|\eta\|_1^2 + \left(\varepsilon_1 + \varepsilon_2 + \frac{\varepsilon_1}{\varepsilon_2}\right)\|\eta\|_1^2\right], \tag{2.5}$$

for any positive ε_1, ε_2, *where* C *is independent of* ε_1, ε_2.

Analogously, applying $D_{\hat{i}}D_{\hat{j}}$, $i, j = 1, \ldots, n$, to (2.1), and proceeding as before, one obtains that for any $0 < \varepsilon_3$ it holds

$$\|w\|_3^2 \leqslant C\left[\|F\|_1^2 + \frac{1}{\varepsilon_3}\|G\|_2^2 + \|v\|_3\|\eta\|_2^2 + \left(1 + \frac{1}{\varepsilon_3}\right)\|w\|_2^2 + \varepsilon_3\|\eta\|_2^2\right]$$

with C independent of ε_3. By using (2.5), one obtains

LEMMA 2.4. Any solution $(w, \eta) \in (H_p^{3,0}(Q))^n \times H_p^{2,0}(Q)$ to (2.1) satisfies

$$\begin{aligned}(2.6)\qquad \|w\|_3^2 \leqslant C\Bigg[&\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2\varepsilon_3}\right)\|F\|_2^2 + \\ &+ \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_1\varepsilon_2} + \frac{1}{\varepsilon_2\varepsilon_3} + \frac{1}{\varepsilon_2\varepsilon_3} + \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3}\right)\|G\|_3^2 + \\ &+ \left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2\varepsilon_3}\right)\|v\|_3\|\eta\|_2^2 + \\ &+ \left(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_3} + \frac{\varepsilon_2}{\varepsilon_3} + \frac{\varepsilon_1}{\varepsilon_2\varepsilon_3}\right)\|\eta\|_2^2\Bigg],\end{aligned}$$

for any positive ε_1, ε_2 and ε_3, with C independent of ε_1, ε_2, ε_3.

By taking $\varepsilon_3 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$, $\varepsilon_1 = \varepsilon^4$, $\varepsilon \in (0, 1]$, in (2.6), and using the result in (2.3) (ii), one obtains (where we can assume $C \geqslant \frac{1}{2}$)

$$\|w\|_3^2 + \|\eta\|_2^2 \leqslant C\left[\frac{1}{\varepsilon^3}\|F\|_1^2 + \frac{1}{\varepsilon^7}\|G\|_2^2 + \frac{1}{\varepsilon^3}\|v\|_3\|\eta\|_2^2 + \varepsilon\|\eta\|_2^2\right].$$

By taking $\varepsilon = 1/(2C)$ in this last inequality, it results

$$\|w\|_3^2 + \|\eta\|_2^2 \leqslant C_1[\|F\|_1^2 + \|G\|_1^2 + \|v\|_3\|\eta\|_1^2].$$

Finally, by taking

$$(2.7)\qquad \|v\|_3 \leqslant 1/(2C_1),$$

we obtain the estimate (2.2).

If we had taken $\varepsilon_2 = \varepsilon$, $\varepsilon_1 = \varepsilon^2$, $\varepsilon \in (0, 1]$, in (2.5), proceeding as before under a condition like (2.7), and finally using (2.3) (i), we could obtain

$$\|w\|_2^2 + \|\eta\|_1^2 \leq C[\|F\|_0^2 + \|G\|_1^2] . \tag{2.8}$$

This last estimate will be explicitly used later on. From now on, we assume that condition (2.7) is such that both estimates (2.2) and (2.8) are valid.

2.3. *Existence of solutions to the linear problem.*

We follow Padula [4] and Valli [7], and introduce the « modified pressure »

$$\pi = (p_1/\varrho_0)\eta - (\zeta/\mu + \tfrac{1}{3})\operatorname{div} w .$$

System (2.1) is the transformed into

$$\begin{cases} -\Delta w + (\mu^{-1}\varrho_0 v_0\cdot\nabla)w + \nabla\pi = F/\mu , \\ \operatorname{div} w = (\zeta/\mu + \tfrac{1}{3})^{-1}[(p_1/\mu)\eta - \pi] , \\ [\varrho_0(\zeta/\mu + \tfrac{1}{3})^{-1}(p_1/\mu)]\eta + \operatorname{div}[(v_0 + v)\eta] = \varrho_0(\zeta/\mu + \tfrac{1}{3})^{-1}\pi + G . \end{cases} \tag{2.9}$$

Obviously, for a given $F \in (H_p^{3,0}(Q))^n$, $G \in H_p^{2,0}(Q)$, to each solutions $(w, \eta, \pi) \in (H_p^{3,0}(Q))^n \times H_p^{2,0}(Q) \times H_p^{2,0}(Q)$ to (2.9) corresponds to a solution (w, η) to (2.1).

To solve (2.9), we first make the following observations:

(i) The first two equations in (2.9) are related to the modified periodic Stokes problem

$$-\Delta w + (\mu^{-1}\varrho_0 v_0\cdot\nabla)w + \nabla\pi = h_1 , \qquad \operatorname{div} w = h_2 , \tag{2.10}$$

For $h_1 \in (H_p^{3,0}(Q))^n$, $h_2 \in H_p^{2,0}(Q)$, by the use of Fourier series exactly as in Temam [6], it is easyly shown that the coefficients

$$\{w^k, k \in Z^n - \{0\}\} \qquad \text{and} \qquad \{\pi^k, k \in Z^n - \{0\}\}$$

of the Fourier expansions of w and π, respectively, satisfy

$$|w^k| = O(|h_1^k|/|k|^2) + O(|h_2^k|/|k|) , \qquad |\pi^k| = O(|h_1^k|/|k|) + O(|h_2^k|) ,$$

where $\{h_1^k, k \in Z^n - \{0\}\}$, $\{h_2^k, k \in Z^n - \{0\}\}$ are the Fourier coefficients of h_1 and h_2, respectively.

Thus, we conclude that (2.10) has a unique solution satisfying the estimate

$$(2.11) \qquad \|w\|_3^2 + \|\pi\|_2^2 \leqslant C_2[\|h_1\|_1^2 + \|h_2\|_2^2] , \qquad C_2 = C_2(Q, \mu) .$$

(ii) The third equation in (2.9) is related to the stationary transport equation

$$(2.12) \qquad \lambda\eta + \operatorname{div}(V\eta) = h .$$

For $0 < \lambda$, $V \in \left(H_p^{3,0}(Q)\right)^n$, $h \in H_p^{2,0}(Q)$, it can be shown that, under the condition

$$(2.13) \qquad \|\nabla V\|_2 \leqslant C_3\lambda , \qquad C_3 = C_3(Q) ,$$

the (2.12) has a unique solution $\eta \in H_p^{2,0}(Q)$ satisfying the estimate

$$(2.14) \qquad \|\eta\|_2^2 \leqslant C_4\|h\|_2^2\lambda^{-1} , \qquad C_4 = C_4(Q) .$$

In fact, these results can be proved exactly as was done in the paper of Beirão da Veiga [1], [2], [3]. There, these results were proved for a bounded domain with a certain boundary condition by the use of elliptic regularization; that is, one approaches the solution of (2.12) by the solutions of the equation

$$-\varepsilon\Delta\eta_\varepsilon + \lambda\eta_\varepsilon + \operatorname{div}(V\eta_\varepsilon) = h$$

as $0 < \varepsilon$ approaches zero. In our case, exactly the same procedure applies: we just coppy his estimates by taking in consideration that our problem is actually simpler since all boundary terms in the derivation of the estimates automatically desappear due to the periodicity condition.

With these observations, the existence of solutions to (2.9) is proved by arguments similar to the ones of Valli [7]. As a first step one considers (2.9) in a special case of $\mu = \mu_0$, $\zeta = \zeta_0$ to be described below. In this case a solution of (2.9) will be solved by a fixed point argument as follows: given $(\eta_1, \pi_1) \in H_p^{2,0}(Q) \times H_p^{2,0}(Q)$, we solve the periodic Stokes problem (2.10) with $h_1 = F/\mu$ and $h_2 = (\zeta_0/\mu_0 + \frac{1}{3})^{-1} \cdot [(p_1/\mu_0)\eta_1 - \pi_1]$. In this way we obtain a solution (w, π_2). Then, we solve (2.12) with

$$\lambda = \varrho_0(\zeta_0/\mu_0 + \tfrac{1}{3})^{-1}(p_1/\mu_0) , \; V = v_0 + v , \; h = \varrho_0(\zeta_0/\mu_0 + \tfrac{1}{3})^{-1}\pi_2 + G ,$$

and we obtain a solution η_2 (the condition (2.15) below will be used here to guarantee the solvability of (2.12)).

This defines a map

$$\Gamma\colon H_p^{2,0}(Q)\times H_p^{2,0}(Q)\to H_p^{2,0}(Q)\times H_p^{2,0}(Q)\,,$$

given by $\Gamma(\eta_1, \pi_1) = (\eta_2, \pi_2)$. Obviously, a fixed point of the map Γ, together with the corresponding w that comes from solving the periodic Stokes problem, constitutes a solution to (2.9). Now, we consider the compact convex set of $H_p^{1,0}(Q)\times H_p^{1,0}(Q)$:

$$K = \{(\eta, \pi)\in H_p^{2,0}(Q)\times H_p^{2,0}(Q),\ \|\eta\|_2^2\leqslant R_1,\ \|\pi\|_2^2\leqslant R_1\}\,.$$

Here, $0 < R_1$ will be chosen suitably. It is shown easyly that Γ is continuous in the topology of $H_p^{1,0}(Q)$; by using estimate (2.11), the above described Stokes problem furnishes

$$\begin{aligned}\|w\|_3^2 + \|\pi_2\|_2^2&\leqslant\\ &\leqslant C_2(Q, \mu_0)[\mu_0^{-2}\|F\|_1^2 + (\zeta_0/\mu_0 + \tfrac{1}{3})^{-2}(p_1^2\mu_0^{-2}\|\eta_1\|_2^2 + \|\pi_1\|_2^2)]\leqslant\\ &\leqslant C_5(Q, \mu_0)\|F\|_1^2 + (\zeta_0 + \mu_0/3)^{-2}[C_6(Q, \mu_0)\|\eta_1\|_2^2 + C_7(Q, \mu_0)\|\pi_1\|_2^2]\leqslant\\ &\qquad\leqslant C_5(Q, \mu_0)\|F\|_1^2 + (\zeta_0 + \mu_0/3)^{-2}C_8(Q, \mu_0)R_1\,.\end{aligned}$$

If we take v such that

$$\|v\|_3\leqslant\min\{1/(2C_1),\ C_3(Q)\varrho_0(\zeta_0/\mu_0 + \tfrac{1}{3})^{-1}(p_1/\mu_0)\} = D\,, \tag{2.15}$$

then we have

$$\|\nabla(v_0 + v)\|_2\leqslant\|\nabla v\|_2\leqslant\|v\|_3\leqslant C_3(Q)\varrho_0(\zeta_0/\mu_0 + \tfrac{1}{3})^{-1}(p_1/\mu_0)\,,$$

and we can apply estimate (2.14) to the above transport equation,

$$\begin{aligned}\|\eta_2\|_2^2&\leqslant C_4(Q)[(\zeta_0 + \mu_0/3)^2\varrho_0^{-2}p_1^{-2}\|G\|_2^2 + \mu_0^2p_1^{-2}\|\pi_2\|_2^2]\leqslant\\ &\leqslant C_4(Q)\{(\zeta_0 + \mu_0/3)^2\varrho_0^{-2}p_1^{-2}\|G\|_2^2 +\\ &+ \mu_0^2p_1^2[C_5(Q)\|F\|_1^2 + (\zeta_0 + \mu_0/3)^{-2}C_8(Q, \mu_0)R_1]\}\leqslant\\ &\leqslant C_9(Q, \mu_0, \zeta_0)\|G\|_2^2 + C_{10}(Q, \mu_0)\|F\|_1^2 + (\zeta_0 + \mu_0/3)^{-2}C_{11}(Q, \mu_0)R_1\,.\end{aligned}$$

Now, if we take ζ_0 large enough such that

$$(2.16) \qquad (\zeta_0 + \mu_0/3)^{-2} C_8(Q, \mu_0) \leqslant \tfrac{1}{2} \text{ and } (\zeta_0 + \mu_0/3)^{-2} C_{11}(Q, \mu_0) \leqslant \tfrac{1}{2}\,,$$

and take

$$R_1 \geqslant \max\left\{2C_5(Q, \mu_0)\|F\|_1^2,\ 2C_9(Q, \mu_0, \zeta_0)\|G\|_2^2 + C_{10}(Q, \mu_0)\|F\|_1^2\right\},$$

then we obtain $\|\pi_2\|_2^2 \leqslant R_1$ and $\|\eta_2\|_2^2 \leqslant R_1$. Thus, $K \supset \Gamma(K)$, and we can apply Schauder Fixed Point Theorem to conclude that Γ has a fixed point. Hence, (2.9) has a unique solution due to estimate (2.2).

The proof of existence of solutions to (2.1) for other μ and ζ is done by a continuation argument. For this, we take fixed μ_0 and ζ_0 satisfying (2.16); assume v satisfying (2.15), and introduce the parametrization

$$\mu_t = (1-t)\mu_0 + t\mu\,; \qquad \zeta_t = (1-t)\zeta_0 + t\zeta\,; \qquad t \in [0, 1]\,,$$

as well as the corresponding unbounded operators

$$T(t)\colon \left(H_p^{3,0}(Q)\right)^n \times H_p^{2,0}(Q) \to \left(H_p^{1,0}(Q)\right)^n \times H_p^{2,0}(Q)\,,$$

defined by

$$\begin{aligned} T(t)(w, \eta) = \big(&-\mu_t \Delta w - (\zeta_t + \mu_t/3)\nabla \operatorname{div} w + \\ &+ (\varrho_0 v_0 \cdot \nabla) w + p_1 \nabla \eta,\ \varrho_0 \operatorname{div} w + (v_0 \cdot \nabla)\eta + \operatorname{div}(\eta v)\big)\,. \end{aligned}$$

It is enough to prove that 1 belongs to the set

$$\begin{aligned} \Lambda = \big\{t \in [0, 1],\ &\text{for any } (F, G) \in \left(H_p^{1,0}(Q)\right)^n \times H_p^{2,0}(Q), \\ &\text{there is a unique solution} \\ &(w, \eta) \in \left(H_p^{3,0}(Q)\right)^n \times H_p^{2,0}(Q) \text{ of } T(t)(w, \eta) = (F, G)\big\}\,. \end{aligned}$$

Actually, we prove that $\Lambda = [0, 1]$ by showing that Λ is open, closed and non void. By the previous argument, $0 \in \Lambda$; now, consider $\tau \in \Lambda$; there exists $T(\tau)^{-1}$. From its derivation, it is seen that estimate (2.2) can be taken with the constant C independent of $t \in [0, 1]$, and we conclude that $\|T(\tau)^{-1}\|$ is bounded independent of

$\tau \in \Lambda$. Moreover, the equation $T(t)(w, \eta) = (F, G)$ can be rewritten as

$$\{I - (t - \tau)T(\tau)^{-1}[T(0) - T(1)]\}(w, \eta) = T(\tau)^{-1}(F, G) .$$

Now, observing that although $T(t)$ is not bounded because of the term $v \cdot \nabla\eta$ in $\operatorname{div}(\eta v)$, the difference $T(0) - T(1)$ is a bounded operator; therefore, the last equation has a unique solution if $|t - \tau|$ is sufficiently small, and Λ is open. The proof that Λ is closed is easyly done if one observes that, from its derivation, estimate (2.8) holds independent of $t \in [0, 1]$; thus, $\|T(t)^{-1}\| \leqslant C$, with C independent of $t \in \Lambda$. Therefore, if $t_n \to t$, $t_n \in \Lambda$, and $T(t_n)(w, \eta_n) = (F, G)$, we conclude that the sequence $\{(w_n, \eta_n)\}$ is bounded in $(H_p^{3,0}(Q))^n \times H_p^{2,0}(Q)$, and, consequently, convergent in $(H_p^{2,0}(Q))^n \times H_p^{1,0}(Q)$ to (w, η). Now, it is easy to show that $T(t)(w, \eta) = (F, G)$; hence, $t \in \Lambda$, and we conclude that Λ is closed.

Thus, (2.1) has a unique solution satisfying (2.2).

3. The non linear problem.

Now, we can proof Theorem 1.1. We use again Schauder fixed point theorem for the map Φ defined in the Remark 2.1 just after the statement of the Theorem 2.1. By this last theorem, the map Φ is well defined in the set

$$M = \{(v, \sigma) \in (H_p^{3,0}(Q))^n \times H_p^{2,0}(Q),\ \|v\|_3 + \|\sigma\|_2 \leqslant R_2\}$$

if $0 < R_2 < \min\{D, 1\}$, where D is the constant that appears in (2.15). Also, for $(v, \sigma) \in M$ if we call $(w, \eta) = \Phi(v, \sigma)$, by using (2.2) we have

$$\begin{aligned}
(\|w\|_3 + \|\eta\|_2)^2 &\leqslant 2(\|w\|_3^2 + \|\eta\|_2^2) \leqslant \\
&\leqslant C[\| - \sigma(v_0 \cdot \nabla)v - \sigma(v \cdot \nabla)v - \varrho_0(v \cdot \nabla)v + \\
&+ \sigma f + \varrho_0 f - [p_1 - p'(\varrho_0 + \sigma)]\nabla\sigma\|_1^2 + \|0\|_2^2] \leqslant \\
&\leqslant C_{12}[\|\sigma\|_2^2\,\|v\|_3^2 + \|\sigma\|_2^2\,\|v\|_3^4 + \|v\|_3^4 + (\|\sigma\|_2 + 1)^2\,\|f\|_1^2] \leqslant \\
&\qquad \leqslant C_{12}[2R_2^4 + R_2^6 + (1 + R_2)^2\,\|f\|_1^2] \leqslant C_{13}[R_2^4 + \|f\|_1^2] ,
\end{aligned}$$

where C_{12} and C_{13} indicate constants depending on the constants appearing in the Sobolev imbeddings of $H_p^{3,0}(Q)$ and $H_p^{2,0}(Q)$ in $L^\infty(Q)$, $n=2$ or 3.

Thus, if we take $R_2 = \min\{D, (2C_{13})^{-\frac{1}{2}}, 1\}$ and $f \in (H_p^{1,0}(Q))^n$ such that $\|f\|_1 \leqslant (2C_{13})^{-\frac{1}{2}}$ we obtain $\|w\|_3 + \|\eta\|_2 \leqslant R_2$, and therefore $M \supseteq \Phi(M)$.

Moreover, M is a convex compact set of $W = (H_p^{2,0}(Q))^n \times H_p^{1,0}(Q)$, and $\Phi\colon W \to W$ is continuous in the topology of W. In fact, let $\{(v_n, \sigma_n)\}$ be a sequence in M convergent to (v, σ) in the topology of W. By the definition of M, we can extract a subsequence that converges weakly in $(H_p^{3,0}(Q))^n \times H_p^{2,0}(Q)$ to (v, σ), and since M is closed and convex in this space, we conclude that $(v, \sigma) \in M$. Now, let $(w_n, \eta_n) = \Phi(v_n, \sigma_n)$ and $(w, \eta) = \Phi(v, \sigma)$ that also belong to M. Now, the difference $(w, \eta) - (w_n, \eta_n)$ is a solution of the linear system (2.1) for right-hand sides $F_n = F(v, \sigma) - F(v_n, \sigma_n)$, $G_n = 0$ (the expression for $F(v, \sigma)$ is given in Remark 2.1). Using the fact that (v_n, σ_n), (w_n, η_n), (v, σ), (w, η) all belong to M, we obtain that F_n converges to zero in $L^2(Q)$. Since by (2.8) $\|w - w_n\|_2^2 + \|\eta - \eta_n\|_1^2 \leqslant C\|F_n\|_0^2$, we obtain that (w_n, η_n) converges to (w, η) in W. Thus, we can use Schauder fixed point theorem, and Theorem 1.1 is proved.

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