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Classification of Projective Surfaces with Small Sectional Genus: char $p \geqslant 0$.

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0. Introduction.

The purpose of this note is to give a biregular classification of smooth connected projective surfaces, S, such that the genus of a smooth hyperplane section, C, is less or equal to five.

Over the field of the complex numbers this is a very classical problem posed and studied first by Enriques and Castelnuovo, later by Del Pezzo, Scorza and Roth. As they pointed out this classification is strongly related to the study of the adjoint system, $K_S + [C]$; in recent time this system and the map associated to it was well studied, in char $= 0$, mostly by A. Sommese and A. Van de Ven ([So1], [VdV]). Using their results a good classification was given by L. Livorni [Li] and P. Ionescu [Io].

Over an algebraically closed field F of arbitrary characteristic, $p = \text{char}\,(G) \geqslant 0$, in a previous paper [A-B], we studied and obtained nice results for the system $2(K_S + [C])$.

In the present paper using these results we obtain a complete classification in char. $p \geqslant 0$ (apart some instances in the char. 2 case) which is described in detail in the section 1.8; we do not claim the existence of all the pairs mentioned.

Finally we point out that in the complex case results on surfaces with sectional genus six and seven are known (due mainly to L. Livorni

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and M. Biancofiore): in the positive characteristic case we do not have a comparable classification and that is due to the lack of complete results about $K_S + [C]$ in this case.

1. Notations and preliminaries.

All through the paper S will be a non singular projective surface over an algebraically closed field, F, of characteristic $p = \mathrm{char}\,(F) \geqslant 0$ and L will be a very ample line bundle on S.

Let K_S be the canonical bundle of S, we do not distinguish nationally between a locally free coherent analytic sheaf and its associated holomorphic vector bundle. Moreover we frequently shift from the multiplicative to the additive notation.

We use the following notations:

$k(S) =: \mathrm{tr}\deg_F \bigoplus_{n=0}^{\infty} H^0(S, \mathcal{O}_S(nK_S)) - 1$,	the « kodaira dimension » of S,
$q = : \dim \mathrm{Pic}\ S = \dim \mathrm{Alb}\ S$,	the « irregularity » of S,
$p_g(S) =: \dim_F H^0(S, \mathcal{O}_S(K_S))$,	the « geometric genus » of S,
$g(D) =: 1 + \frac{1}{2}(D\cdot D + K_S\cdot D)$,	for D a curve or a line bundle on S, the « (arithmetic) genus » of D.
$\chi(L) =: \Sigma(-1)^i \dim_F H^i(S, L)$	for L a line bundle on S, the « euler characteristic of L ».

If L is a line bundle and $C \in |L|$ is a smooth element then, of course, $g(L) = \dim H^1(C, \mathcal{O}_C)$.

We will call the pair (S, L) a *quadric*, $(Q, \mathcal{O}_Q(1))$, if S is a quadric in $\boldsymbol{P}_{F^3}$ and L is the restriction of the hyperplane bundle, $\mathcal{O}_{P^3}(1)$, to S.

If $\pi\colon S \to C$ is a ruled surface, we write $S = P(\xi)$, where ξ is a rank two vector bundle over C satisfying the condition 2.8, pag. 372, in [Ha], i.e. ξ is normalized. We call ζ_E the tautological line bundle over S, $e = -\zeta_E\cdot\zeta_E = -\deg\xi$, f a general fibre of π, and E the section corresponding to ζ_E.

We denote by F_r the geometrically ruled surface $\boldsymbol{P}(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(r))$, and by E and f respectively the section satisfying $E\cdot E = -r$ and the general fibre.

We will call the pair (S, L) a *scroll*, respectively a *conic bundle*, if S is a ruled surface over a smooth curve $R, p\colon S \to R$, and $K_S \otimes L^{\otimes 2} \cong p^*\mathcal{M}$, respectively $K_S \otimes L \cong p^*\mathcal{M}$, for an ample line bundle $\mathcal{M}$ on R. Finally a pair (S, L) such that $K_S^{-1} \cong L$ is called a *Del Pezzo surface* (see [De] for a classification of these surfaces).

The following theorems were proved in our previous paper [A-B] and, as in the char. 0 case, they are the principal tools in working out our classifications.

Theorem A. *If (S, L) is neither $(\boldsymbol{P}^2, \mathcal{O}(e))$, $e = 1, 2$, nor a* quadric, *nor a* scroll *then $(K_S \otimes L)^{\otimes 2}$ is spanned at each point by global sections (i.e. $(K_S \otimes L)^{\otimes 2}$ is semiample and therefore nef).*

Theorem B. *Suppose $(K_S \otimes L)^{\otimes 2}$ is semiample and suppose also (S, L) is neither a* Del Pezzo *surface nor a* conic bundle.

Then the map given by the sections $\Gamma(K_S \otimes L)^{\otimes 2}$, $\phi\colon S \to \boldsymbol{P}^{N_F}$, has a two dimensional image (equivalently $(K_S \otimes L)$ is big).

Moreover if $x, y \in S$ are two points not on a line $P \subset S$ such that $P \cdot P = K_S \cdot P = -L \cdot P = -1$, then ϕ separates x and y, i.e. $\phi(x) \neq \phi(y)$.

If $\phi = s \circ r$ is the Stein factorization of ϕ, then theorem B say that, except for the obvious cases, $r\colon S \to \hat{S} = r(S)$ is the contraction of a finite number of lines, E_i for $i = 1, \ldots, r$, such that $E_i \cdot E_i = K_S \cdot E_i = -L \cdot E_i = -1$. Call $\hat{L} = r(L)$. In particular $K_{\hat{S}} \otimes \hat{L}$ is ample.

Theorem C. *Suppose to be in the hypothesis of the theorem B, i.e. (S, L) is not one of the following pairs*: $(\boldsymbol{P}^2, \mathcal{O}(e))$, $e = 1, 2$, a quadric, a scroll, a Del Pezzo surface, a conic bundle.

If $h^0(L) \geqslant 6$ and $L \cdot L \geqslant 9$, then s (which is the map given by the sections $\Gamma(K_{\hat{S}} \otimes \hat{L})^{\otimes 2}$) is an embedding.

The map given by the sections $\Gamma(K_{\hat{S}} \otimes \hat{L})^{\otimes 3}$) is an embedding.

1.1. Remark. The proof of our theorem C actually gives, without the hypothesis $L^2 \geqslant 9$ and $h^0(L) \geqslant 6$, that s is generically an embedding.

We will use also the following vanishing theorems (see [Bo], p. 71 and [Ek], theorem 1.6).

1.2. Theorem. *a)* [Bo] *Let L be a nef and big line bundle on a surface S and suppose that the Frobenius cohomology operation is injective on $H^1(S, \mathcal{O}_S)$ (that is the case if for instance $H^1(S, \mathcal{O}_S) = 0$);*

b) [Ek] *Let L be an ample line bundle on a surface S and suppose* $k(S) < 1$.

If we are in the hypothesis a) or b), then $H^1(S, L^{-1}) = 0$.

1.3. REMARK. If $h^1(S, \mathcal{O}_S) = 0$ then the Van de Ven's results [VdV] are true in each characteristic since his proofs are based on the Ramanujam vanishing theorem which helds in this case, i.e. the theorem 1.2.*a*) (use also our proposition 1, [A-B]).

Therefore for $h^1(S, \mathcal{O}_S) = 0$ the theorems A, B, C are true also for the sheaf $K_S \otimes L$, i.e. the adjunction sheaf, with the only stronger hypothesis in the theorem C that $L^2 \geqslant 10$ and $h^0(L) \geqslant 7$.

1.4. CASTELNUOVO'S LEMMA (see [Ra]). *If C is an irreducible curve embedded in $\boldsymbol{P}^{l-1}$ and C belongs to no linear hyperplane $\boldsymbol{P}^{l-2}$, then, with d the degree of C and g the genys of the desingularization of C*:

$$g \leqslant [(d-2)/(l-2)] \cdot (d-l+1-[d-l/l-2] \cdot (l-2/2))$$

where [ ] *is the least integer function.*

We now state the case $g(L) = 0$, which is very well known, in a slightly weaker form.

1.5. THEOREM. *If L is an* ample *line bundle on S and $g(L) = 0$, then L is very ample and (S, L) is one of the following pairs*: $(\boldsymbol{P}^2, \mathcal{O}(1))$, $(Q, \mathcal{O}_Q(1))$, $(F_r, E + kf)$ *with* $k \geqslant r + 1$.

1.6. If S is a smooth surface embedded in $\boldsymbol{P}^4$ by a very ample line bundle L of degree $d = L \cdot L$, then we have the following equality

$$d^2 - 10d + 12\chi(\mathcal{O}_S) = 2K_S \cdot K_S + 5K_S \cdot L .$$

1.7. Suppose L is a very ample line bundle on a ruled surface S, $S \neq \boldsymbol{P}^2$, such that $K_S \otimes L$ is semiample (i.e. suppose (S, L) is not one of the pair in the theorem A).

Then $(K_S \otimes L)^2 \geqslant 0$ and, since $K_{S^2} \leqslant 8 - 8q$, we have

$$L^2/8 + q \leqslant (g+1)/2 . \tag{1.7.1}$$

1.8. We finally state the classification proved in the paper.

First for *every g we have the scrolls over a curve of genus g; and for every genus $g \geqslant 2$ we have the conic bundles over $\boldsymbol{P}^1$, (S, L), such that S is obtained by F_r blowing up at most seven point for $g = 2$ and of $3 \cdot g$ points for $g \geqslant 5 > 2$, and, if $r': S \to F_r$ is this map,*

$$r'_* L = 2E + (g + a + 1)f\,, \qquad a = 0, \ldots, g\,.$$

Besides those, using the previous notation, we have the following table

$g(L)$	(S, L)
0	See the theorem 1.5.
1	Del Pezzo surface (see [De] for a classification of these surfaces).
2	$\varnothing$
3	i) degree 4 hypersurface in $\boldsymbol{P}^3$ ii) $(\hat{S}, \hat{L}) = (\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(4))$, $r: S \to \hat{S}$ is the blowing up of at most 10 points iii) $(\hat{S}, \hat{L})$ is a two sheeted branched cover of $\boldsymbol{P}^2$ with (for $p \neq 2$) a smooth quartic curve as branch locus, $K_{\hat{S}}^{-2} = \hat{L}$; r is the identity or one blow up. iv) a conic bundle over an elliptic curve, more precisely $S = P(\xi)$, $e = -1$ and $L = \zeta_E^2 \otimes [f]$,
4	i) S is embedded in $\boldsymbol{P}^4$ by $\lvert L\rvert$ as the intersection of a quadric and a cubic hypersuface. ii) $\hat{S}$ is a quadric in $\boldsymbol{P}^3$, $\hat{L} = \mathcal{O}_{\boldsymbol{P}^3}(3)$ and $r: S \to \hat{S}$ is the blowing up of at most 10 points. iii) $\hat{S}$ is a cubic in $\boldsymbol{P}^3$, $\hat{L} = \mathcal{O}_{\boldsymbol{P}^3}(2)$ and $r: S \to \hat{S}$ is the blowing up of at most 5 points (more precisely if we blown up 5 points we obtain S, a rational surface of degree 7 represented on $\boldsymbol{P}^2$ by a linear system, $\lvert C_6 - 2_{p_1} \ldots - 2_{p_6} - q_1 \ldots q_5\rvert$, of sextic with six double base point and five single base points. iv) (S, L) is a two sheeted branched cover over a quadric Q in $\boldsymbol{P}^3$ and $K_S^{-3} = L$; if $p \neq 2$ the quadric has an isolated singularity e and it meets transversally a cubic surface in a curve B such that (B, e) is the branch locus $r: S \to \hat{S}$ is the identity or one blowing up.

v) A conic bundle over an elliptic curve, more precisely $\hat{S} = \boldsymbol{P}(\xi)$, $e = 0$ or -1, $\hat{L} = \zeta_E^2 \otimes b[f]$ with $b = 3$ or 2, $S = \hat{S}$ or the blowing up of at most 4 points.

vi) S is geometrically ruled over an elliptic curve, $S = \boldsymbol{P}(\xi)$, $e = -1$ and $L = 3\zeta_E$.

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5 i) S is the complete intersection of three quadrics in $\boldsymbol{P}^5$ ($\deg S = 8$) or S is the $K-3$ surface obtained from projecting this complete intersection from a proper point in $\boldsymbol{P}^4$ ($\deg S = 7$). S is a smooth divisor $\in |-K_V|$ where V is a smooth three dimension scroll of degree 3 contained in $\boldsymbol{P}^5$ (i.e. $V = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(1))$)

ii) $(\hat{S}, \hat{L}) = (F_1, E^3 + f^5)$ and (S, L) is obtained by $(\hat{S}, \hat{L})$ blowing up at most 12 points.

iii) $(\hat{S}, \hat{L})$ is the blowing up at seven points of $(F_r, 2E + (r+3)f)$ with $r = 0, 1, 2$ and (S, L) is obtained by $(\hat{S}, \hat{L})$ blowing up at most 3 points.

iv) $(\hat{S}, \hat{L})$ is the blowing up of $\boldsymbol{P}^2$ in 5 points, $\hat{L} = \pi^*\mathcal{O}_{\boldsymbol{P}^2}(3) \otimes K_{\hat{S}}^{-1}$ and (S, L) is obtained by $(\hat{S}, \hat{L})$ blowing up at most seven points.

v) $\hat{S}$ is geometrically ruled over an elliptic curve $\hat{S} = \boldsymbol{P}(\xi)$, $e = -1, 0, 1$ and $\hat{L} = 2\zeta_E + bf$ with $b = 4 + e = 5, 4, 3$.

vi) S is a rational surface of degree 8 or 9 in $\boldsymbol{P}^4$ or in $\boldsymbol{P}^5$.

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2. Some special lower degree surfaces.

We denote from now on by $d =: \deg L = L \cdot L$, the degree of L, and by $g =: g(L)$, the sectional genus.

From the lemma 1.4 we have that

2.1. $$d \geqslant g + 3 \quad \text{for } g \leqslant 5\,,$$

with the following exceptions.

2.2.a $g = 3$ and S is a surface of degree 4 in $\boldsymbol{P}^3$.

2.2.b $g = 4$, $h^0(L) = 5$, $d = 6$ and $K_{S|C} \cong \mathcal{O}_C$ for a generic $C \in |L|$.

Therefore either S is ruled or $K_S \sim \mathcal{O}_S$ (see for instance the appendix of Mumford, p. 120 [Za]). In the last case, by the theorem 1.2.b), we have that $h^i(L^t) = 0$ for $i > 0$ and $t > 0$. Moreover in this case we have also the inequality

$$h^0(K_S) + h^0(L) - 1 \leqslant h^0(K_S \otimes L) = g + h^0(K_S) - h^1(\mathcal{O}_S) = 0 .$$

A standard argument (see for instance the proof of the theorem 3.4 of [So2], or p. 342 [Ha]) gives that S is embedded in $\boldsymbol{P}^4$ by $|L|$ as the intersection of a quadric and a cubic hypersurfaces.

Suppose S is ruled, then $\chi = 1 - q$ and $K_S^2 = 8(1 - q) - s$, where s is the number of blowing up from a geometrically ruled model to S. Together with 1.6 these give $s + 2q - 14 = 0$. By 1.7.1 we have $q \leqslant 1$. If $q = 0$, then $K_S^2 = -6$ and $(K_S + L)^2 = 0$, which implies, by the theorem B, that (S, L) is a conic bundle. If $q = 1$ then $K_S^2 = -12$ and therefore $(K_S + L)^2 = K_S^2 + L^2 < 0$, which is absurd.

2.2.c $g = 5$ and $d = 7$ in this case S is a $K\cdot 3$ surface which is the projection in $\boldsymbol{P}^4$ of the complete intersection of three quadrics of $\boldsymbol{P}^5$ from a proper point. The proof of this is exactly the same of the one in char. 0 given in the section 5 of [La] (We use the remark 1.3 giving the spannedness of $K_S \otimes L$ if $g > q = 0$, and the vanishing theorem 1.1.b) since $k(S) = 0$).

2.3. In the case $g = 4$ we can moreover suppose $d \geqslant 8$. In fact using the section 4 of [La] we can prove in the same way that the only surface of sectional genus 4 and degree 7 is a rational surface represented on $\boldsymbol{P}^2$ by a linear system, $|C_6 - 2p_1 - \ldots - 2p_6 - q_1 \ldots - q_5|$ of sextics with six double base point and five simple base point (For the existence of such a surface, at least in char $= 0$, we can see the paper [Ok]).

2.4. We consider now the case $g = 5$, $d = 8$ and therefore $h^0(L) = 5$ or $h^1(L) = 6$.

If $h^0(L) = 5$ then, by 1.6, we have

$$(*) \qquad -16 + 12\chi = 2K^2 .$$

Since $K_S \cdot L = 0$, we have that, if $k(S) = 0$ then S is minimal, therefore $K_S^2 = 0$ and $(*)$ gives an absurd. If instead $k(S) = -\infty$ then, as in the case 2.2.b, we have by 1.7.1 $q \leqslant 2$. If $q = 1$ then (S, L) is a conic bundle, while if $q = 2$ then $(K_S + L)^2 < 0$ which is absurd. The case $q = 0$ gives no contradiction and will be considered later (in this case $s = 10$ $K^2 = -2$, $(K_S + L)^2 = 6$).

If $h^0(L) = 6$, then using again the result of Weil (p. 120 [Za]) we have that either $K_S \sim \mathcal{O}_S$ or $k(S) = -\infty$.

In both cases, since the general section $C \in |L|$ is canonical, either it is the complete intersection of three quadrics, and the same follows for S, either it is trigonal and the intersection of the three quadrics containing S is a scroll (possibly singular) of dimension 3 and minimal degree 3 (therefore a scroll over $\boldsymbol{P}^1$). Suppose first that this scroll in smooth; using the notation $\boldsymbol{P}(e_1, \ldots, e_n) =: \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(e_1) \oplus \ldots \oplus \mathcal{O}_{\boldsymbol{P}^1}(e_n))$ we have $V = \boldsymbol{P}(e_1, e_2, e_3) \subset \boldsymbol{P}^5$. Since V is smooth, $0 < e_1 \leqslant e_2 \leqslant e_3$, and since degree $V = 3$, $e_1 + e_2 + e_3 = 3$: therefore $V = \boldsymbol{P}(1, 1, 1)$.

Let H be a generic smooth hyperplane section of V, that is a smooth surface of degree 3 in $\boldsymbol{P}^4$, $H = \boldsymbol{P}(1, 2)$, and F a generic fibre of V. Since our surface S is a divisor of $VS \approx aH + bF$; moreover a hyperplane section of S has to be trigonal and therefore $S \approx \approx 3H + bF$. Finally we require $\deg S = 8$, that is $8 = (3H + bF) \cdot \cdot H \cdot H = 3H^3 + bF \cdot H \cdot H = 9 + b$. So if S exists, $S \in |3H - F|$. We observe that $3H - F = 2H + (H - F)$ is very ample and therefore such a smooth irreducible S exists; moreover it is easy to show that $K_V = -3H + F$ and therefore $K_S \sim \mathcal{O}_S$.

Suppose now that this scroll is singular and first that the vertex has o-dimension. Let H a generic hyperplane through the vertex, therefore $H \cap V$ is again a cone and its desingularization is the rational surface F_3. Consider now $S \cap H$; it is irreducible (use our proposition 1 [A-B]) and its strict transform in F_3 is in $|3E + bf|$. $\mathrm{Deg}\,(D \cap H) = 8 = (3E + bf) \cdot (E + 3F)$ implies $b = 8$ and we have the absurd since $|3E + 8f|$ has no irreducible curve in F_3.

If the vertex has 1-dimension the proof is analogous, i.e. take a generic hyperplane section H and consider $H \cap V$.

2.5 If we exclude the surfaces in this section we can therefore suppose $d \geqslant g + 3$ for $g \leqslant 3$, $d \geqslant 8$ for $g = 4$ and, for $g = 5$, $d \geqslant 9$ or $d = 8$ and $q = 0$, $k(S) = -\infty$.

3. Rational surfaces.

3.0. We consider now S a rational surface, therefore $k(S) = -\infty$ and $q = h^1(O_S) = 0$, embedded by a very ample line bundle L with $g(L) \leqslant 5$. Let $C \in |L|$ a general section. Suppose also (S, L) is not one of the special pairs in § 2. For $g(L) = 0$ we have the theorem 1.5.

By the remark 1.3 if $g > q = 0$, $(K_S + L)$ is spanned by global sections. Moreover $h^0(K_S + L) = g$. (All this follows immediately in this case from the exact sequence $0 \to K_S \to K_S + L \to K_C \to 0$, Catanese's theorem D [Ca] and the theorem 1.2).

Suppose also that (S, L) is not a Del Pezzo surface $(g(L) = 1)$.

3.1. We first consider the case $(K_S + L)^2 = 0$. In this case by the theorem B (S, L) is a conic bundle over $\boldsymbol{P}^1$ and $g \geqslant 2$. A routine computation gives more precisely that $(\hat{S}, \hat{L}) = (F_r, 2E + (g + a + 1)f)$ $a = 0, \dots, g$ and the map $r\colon S \to \hat{S}$ in the blowing up of seven points for $g = 2$ and of $3g$ points for $g > 2$.

3.2. We now suppose $(K_S + L) = d' > 0$. By the Hodge Index theorem and 2.5 we have that:

a) the case $g = 1, 2$ is impossible

b) $g = 3$ implies $d' = 1$ or 2

c) $g = 4$ implies $d' = 1, 2, 3$ or 4

d) $g = 5$ implies $0 < d' \leqslant 7$ (or $d = 8$).

Let $D \in |K_S + L|$ be a generic element. Since $(K_S + L)$ is spanned and $d' > 0$, by the theorem 6.3.4 of [Ja], D is geometrically irreducible.

3.3. If it is not reduced then $D = qD'$ with $q = p^e$ a power of the characteristic. If $h^1(D') = 0$ then we have the following equalities:

$$g = h^0(K_S + L) = h^0(\mathcal{O}(D') = \chi(\mathcal{O}(D')) = 1 + \frac{D'^2 - D'K}{2};$$

multiplying by $2q$ by both parts we have

$$2q^2(g - 1) = (K + L)^2 - q(K + L)\cdot K = (K + L)^2(1 - q) + q(2g - 2)$$

that is $(K + L)^2(1 - q) = 2(q^2 - q)(g - 1)$. If $q \neq 1$ then $(K + L)^2 = - 2q(g - 1) < 0$ that is absurd.

To show that $h^1(D') = 0$ we use the following exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D') \to \mathcal{O}_{D'}(D') \to 0$$

and we prove that $h^1 =: h^1(\mathcal{O}_{D'}(D')) = 0$.

Since $2g-2=(K_S+L)\cdot L=qD'\cdot L$, if $g=3$ or 4 then $D'\cdot L=2$ or 3 and $g(D')\leqslant 1$: this implies, since $D'^2>0$, $h^1=0$. If $g=5$ then $qD'\cdot L=8$ implies $q=2, 4$ or 8. If $q=4$ or 8 then $g(D')=0$ and we are done as before, therefore we can suppose $q=2=p$ and $g(D')=2$ or 3.

If $D'^2\geqslant 5$ we have immediately that $h^1=0$, then consider the cases $D'^2\leqslant 4$. Moreover we have that $h^1\leqslant 1$, or $h^1=2$ and $g(D')=3$, $D'^2=1$. Using the Riemann Roch theorem we can show that $h^1=0$.

Therefore the generic section $D\in|K_S+L|$ is reduced and irreducible and the map given by the section $\Gamma(K_S+L)$, « ϕ » is unseparable. This implies also that $g'=:g(D)\geqslant 0$.

3.4. Since $2g'-2=(K_S+L)(2K_S+L)=2(K_S^2-L^2)+6(g-1)$ and

$(*)$ $K_S^2-L^2=-4(g-1)+d'(=K_{\hat{S}}^2-\hat{L}^2)$, we have the equality

$(**)$ $g'=-g+2+d'$

3.5. If $g=3$ then we have the two cases

i) $g'=0$, $d'=1$

ii) $g'=1$, $d'=2$

i) In this case the map ϕ is a birational map into $\boldsymbol{P}^2$. By $(*)$ we have $9\geqslant K_S^2=L^2-7\geqslant -1$ and therefore that (S,L) is obtain by $(\boldsymbol{P}^2, \mathcal{O}(4))$ through $\leqslant 10$ blowing ups given by ϕ.

ii) If $\phi=s\circ r$ is the Stein factorization of ϕ then s is a 2 to 1 map, $s\colon \hat{S}\to\boldsymbol{P}^2$. By the Riemann-Hurwitz formula, for $p\neq 2$, it is easy to prove that s ramified along a smooth quartic and $\hat{L}=-2K_{\hat{S}}$, $\hat{L}^2=8$. Moreover the map s is the blowing up of $\hat{S}$ in one point or the identity (use again $(*)$)

3.6. $g=4$; then $g'=-2+d'$ and we leave the following possibilities

i) $g'=0$ $d'=2$

ii) $g'=1$ $d'=3$

iii) $g'=2$ $d'=4$.

Since $h^0(K_S+L)=4$, the first two imply that $\hat{S}$ is a quadric in $\boldsymbol{P}^3$ (respectively a cubic) and $\hat{L}=\mathcal{O}_{\boldsymbol{P}^3}(3)$ (respectively $\hat{L}=\mathcal{O}_{\boldsymbol{P}^3}(2)$). In the first case $\hat{L}^2=8$ and since $\hat{L}^2\geqslant 8$ we have that r is the identity.

In the second case $\hat{L}^2 = 12$ and therefore is the blowing up of at most 5 points (in the case of exactly five points we obtain the surface in 2.3.).

In the case iii) either $\hat{S}$ is a singular quartic in $\boldsymbol{P}^3$ or $K_{\hat{S}} + \hat{L}$ gives a 2 to 1 map from S' to a quadric in $\boldsymbol{P}^3$, $s\colon \hat{S} \to \boldsymbol{P}^3$.

Since if $\hat{L}^2 \geqslant 10$ (by Riemann Roch $h^0(L) \geqslant 8$) the Remark 1.2 implies that $K_{\hat{S}} + \hat{L}$ gives an embedding, the first possibility can happen only for $d = 8$ or 9.

If $d = 8$, then $K_{\hat{S}}^2 = 0$ and, by the Riemann-Roch theorem, $h^0(-K_S) > 0$. Moreover, $-L\cdot K_S = 2$ and $p_a(-K_S) = 1$, and since L is very ample this gives the absurd.

If $d = 9$, then $K_{\hat{S}}^2 = 1$ and, by the Riemann-Roch theorem, $h^0(L + 3K_S) > 0$. Since $L\cdot(L + 3K_S) = 0$, it follows that $L = -3K_S$. We have also $h^0(-K_S) \geqslant 2$ and if $A \in h^0(K_S)$ then, since $-K_S\cdot A = 1$, A is reduced and irreducible. Finally, since

$$h_0(K_S + L) = 4,, \qquad (K_S + L)\cdot A = 2$$

and $p_a(A) = 1$, $K_S + L$ is 2 to 1 on A, and therefore on S.

By the Riemann-Hurwitz theorem we have that (for $p \neq 2$) the image of $\hat{S}$ by s is a quadric in $\boldsymbol{P}^3$ with an isolated singularity e and such that it meets transversally a cubic surface C in a curve B. Such that B and e are the branch locus of s. A routine computation using (*) shows that S is obtained by $\hat{S}$ in this last case blowing up at most one point.

3.7. $g = 5$. Since we have $\chi(O_S) = 1$, by the Riemann-Roch theorem $\chi(L) = L^2 - 3$.

Let $L^2 \geqslant 10$, then $h^1(L) = 0$ and therefore $h^0(L) \geqslant 7$. By the [VdV]'s theorem 2, $K_{\hat{S}} + \hat{L}$ is very ample; by the Hodge Index theorem $d' \leqslant 6$.

Using the formula 1.6 (for the very ample line bundle $K_{\hat{S}} + \hat{L}$, $h^0(K_{\hat{S}} + \hat{L}) = 5$) and the equalities

$$g' = d' - 3\,, \qquad \hat{d} + d' = \hat{c}_1^2 + 16 \qquad (\hat{c}_1^2 = K_{\hat{S}}^2)$$

we have the following three possibilities

$$\begin{array}{rllll} \text{i)} & g' = 0 & d' = 3 & \hat{c}_1^2 = 8 & \hat{d} = 21 \\ \text{ii)} & = 1 & = 4 & \phantom{\hat{c}_1^2} = 4 & \phantom{\hat{d}} = 16 \\ \text{iii)} & = 2 & = 5 & \phantom{\hat{c}_1^2} = 1 & \phantom{\hat{d}} = 12 \\ \text{iv)} & = 3 & = 6 & \phantom{\hat{c}_1^2} = -1 & \phantom{\hat{d}} = 9\,. \end{array}$$

The last is excluded since $\hat{d} \geqslant d \geqslant 10$ in our hypothesis. For the other three cases we can apply our previous results to $(\hat{S}, K_S + \hat{L})$ and with some easy computation we have respectively that $(\hat{S}, \hat{L})$ are as in ii), iii), iv) of the table in section 1.8.1 (see the pages 108-109 of Li1] for a detailed proof).

We have to consider two remaining cases, i.e. $g = 5$, $d' > 0$ and $L^2 = 8$ or 9. Since for char$(F) = p \neq 0$ we do not know the very ampleness of $(K_S + \hat{L})$ in this case, we cannot go much further. We just point out that in the char $=0$ case there exist at most a pair (S, L) for $L^2 = 9$ and its blowing up for $L^2 = 8$ (see p. 108-109 of [Li1]).

4. The non rational surfaces.

We now consider projective smooth surfaces which are not rational, embedded by a very ample line bundle L with $g(L) \leqslant 5$. The case $g(L) = 0$ implies S is rational and therefore we suppose $g(L) > 0$.

For each $0 < g(L) \leqslant 5$ we have the scroll over a curve of genus g. From now on, by our theorem A, we can therefore suppose that $2K_S + 2L$ is semiample.

4.1. $g(L) = 1$. This implies $(K_S + L)\cdot L = 0$ and since $2K_S + 2L$ is semiample $2K_S + 2L = O_S$. Using the Riemann Roch theorem we have actually that $K_S + L = \mathcal{O}_S$ i.e. $K_S \approx L^{-1}$ and therefore we are in the rational case.

4.2. $g(L) = 2$. By the Hodge Index theorem $(K_S + L)^2 L^2 = d'\, d \leqslant (2g-2)^2 = 4$. Since $L^2 \geqslant 5$, (sect. 2), we have that $d' = 0$ and by the theorem B this implies that (S, L) is a conic bundle. By the Hurwitz lemma $q(S) = 0$ or 1. If $q(S) = 1$ then $K_S^2 \leqslant 0$ and we have the absurdity

$$0 = (K_S + L)^2 = K_S^2 + 2(K_S + L)\cdot L - L\cdot L \leqslant 4 - L\cdot L < 0 .$$

If $q(S) = 0$, then we are in the rational case.

4.3. $g(L) = 3$. Suppose first $(K_S + L)^2 = 0$, that is (S, L) is a conic bundle. By Hurwitz formula $q(S) = 1$ or 2. If $q(S) = 2$ then $-2 \leqslant L\cdot L - 8 = K_S\cdot K_S \leqslant 8\,(1 - q(S)) < 0$, which is absurd. If $q(S) = 1$ then (S, L) is a conic bundle over an elliptic curve. Let

$\sigma\colon S \to S'$ be the minimal model and let $L' = \sigma^*(L) \equiv 2E + bf$ for some b. Since L' is smooth we get by adjunction $b = e + 2$ (E is a section with minimal $E^2 = -e$). If $\tilde{E}$ is the strict transform of E by σ we have $3 \leqslant \tilde{E}\cdot L \leqslant -2e + b = -e + 2$ (since L is very ample and E is an elliptic curve) since $e \geqslant -1$ we have $e = -1$ $b = 1$ and σ is an isomorphism (otherwise there would be at least two values of e).

Now suppose $(K_S + L)^2 > 0$. Gy the Hodge Index theorem we have $(K_S + L)^2 = 1$ or 2. Since $L^2 \geqslant 6$, (sect. 2), $k(S) = -\infty$, and by Hurwitz formula $q(S) = (0$ or$)\ 1$. Note also $h^0(K_S + L) = g - q \geqslant 2$ and that

$$g(K_S + L) = (K_S + L)^2 - 1\,.$$

Suppose $(K_S + L)^2 = 1$ and take $D \in |K_S + L|$. Using the exact sequence

$$0 \to -K_S - L \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$

and the Kodaira vanishing theorem we have $h^1(\mathcal{O}_S) = q = 0$. If instead $(K_S + L)^2 = 2$ then we have

$$8(1 - q(S)) \geqslant K_S\cdot K_S = \\ = (K_S + L)^2 - 2(K_S + L)\cdot L + L\cdot L = -6 + L\cdot L \geqslant 0\,.$$

Therefore $q(S) = 1$, $K_S^2 = 0$, $L^2 = 6$, S is minimal. It is easy to prove that $L = 2E + f$, $E^2 = 0$ and therefore that $L\cdot E = 1$, i.e. we have the absurd that L is not very ample.

4.4. $g(L) = 4$. We can suppose $L^2 = d \geqslant 8$, therefore that $k(S) = -\infty$.

For $g = 4$, 1.7 implies $q = 1$. If $d' = 0$ then (S, L) is a conic bundle over an elliptic curve, and as in 4.3. we have $\hat{L} = 2E + bf$ with $-E^2 = e = 0$ or -1 and $b = e + 3 = 3$ or 2.

Suppose now $d' > 0$. Call $g'' = g(2K_S + 2L)$. By the theorem of [Ja] the generic section of $2K_S + 2L$ is geometrically irreducible and by the remark 1.1. it is reduced, therefore

4.4.2. $g'' \geqslant 0$.

By the adjunction formula we have

4.4.3. $$2g'' - 2 = (2K_S + 2L)(3K_S + 2L) = \\ = 6K_S^2 - 6L^2 + 20(g - 1) = 6(K_S^2 - L^2) + 60\,.$$

Since $K_S^2 \leqslant 0$ and $K_S^2 = 0$ if and only if S is a minimal model we have that $L^2 \geqslant 11$ is impossible.

If $L^2 = 10$ (by Riemann-Roch $h^0(L) \geqslant 6$) then $S = \hat{S}$ and $g'' = 1$. By our previous classification, since $2K_S + 2L = \mathfrak{L}$ is very ample by theorem C, $(S, 2K_S + 2L)$ is either a scroll or a Del Pezzo surface. They are both impossible since the first will imply $2(K_S + 2L)\cdot F = 1$ and the second $3K_S \sim -2L$ and $q = 0$.

If $L^2 = 9$ then either $g'' = 1$ and S is the blown up of $\hat{S}$ in one point or $g'' = 4$ and $S = \hat{S}$. The first case is ruled out as before (working in $\hat{S}$). In the second case we see that if $\mathfrak{L} = K_S + 2L$, then $\mathfrak{L}^2 = 12$ and $(K_S + \mathfrak{L})^2 = 0$.

As we have seen in the previous part we have that $E^2 = -e = 0$ or 1 and $\mathfrak{L} = 2E + bf$, $b = e + 3$. If $e = 0$ then $2K_S + 2L = 2E + 3f$, and since $K_S = -2E$ this implies the absurdity $2L = 6E + 3f$. Therefore $e = -1$, $2K_S + 2L = 2E + 2f$ and $K_S = -2E + f$. We have finally that S is a geometrically ruled surface over an elliptic curve and $L = 3E$.

If $L^2 = 8$, this implies $k(S) = -\infty$ and $K\cdot L = -2$. By $(*)$ we have $2g'' - 2 = 6K_S^2 + 12$, that implies $K_S^2 = -2, -1$ or 0.

If $K_S^2 = 0$ then S is minimal; let E be the section with minimal autointersection and let $L = aE + bf$. Since (S, L) is neither a scroll neither a conic bundle $a \geqslant 3$. $L\cdot E \geqslant 3$ implies $b - ea \geqslant 3$. If $e \geqslant 0$ then $b \geqslant 0$ and these imply the absurd

$$8 = L^2 = a(-ea + 2b) \geqslant 9 .$$

If $e = -1$ then, since $K_S = -2E - ef = -2E + f$, the following equalities:

$$-2 = K\cdot L = -a - 2b , \qquad 8 = L^2 = -a(ae - 2b) = a(a + 2b)$$

imply $a = 4$, $b = -1$.

Since by Riemann-Roch $h^0(L) - h^1(L) = 5$ and by 1.6. $h^0(L) > 5$ we have that $h^1(L) > 0$. Moreover, since $L\cdot(2E - f) = -L\cdot K_S = 2$, $h^0(2E - f) = 0$ otherwise they are rational sections. This, together with $h^2(2E - f) = h^2(-K_S) = h^0(2K_S) = 0$, implies by Riemann Roch that $h^1(2E - f) = 0$. Now use the exact sequences.

$$0 \to 2E - f \to 3E - f \to (3E - f)|_E \to 0$$

and

$$0 \to 3E - f \to 4E - f \to (4E - f)|_E \to 0$$

to have the contradiction $h^2(4E - f) = h^1(L) = 0$.

If $K_S^2 = -1$, then S is obtained by the minimal model blowing up one point. If $P \cup Q$ is the exceptional fibre ($P^2 = Q^2 = -1$) we can suppose $P \cdot E = 0$, $Q \cdot E = 1$ and $L = (aE + bf) + cP$ with $c < 0$ therefore we have again $L \cdot E = b - ea \geqslant 3$ and:

$$8 = L^2 = -a(ae - 2b) - c^2\,, \qquad -2 = KL = (ae - 2b) - c\,.$$

The last two implies

$$8 = -a(c-2)c^2 = 2a - c(a + c)\,.$$

Since $(a + c) > 0$ $\big((a + c) = L \cdot (f - P) > 0\big)$ and $a \geqslant 3$, $c < 0$ we have the only possible solution $a = 3$, $c = -1$. $L \cdot E = b - ca = b - 3e \geqslant 3$ implies $b \geqslant 0$ ($e \geqslant -1$) and the above equalities give $-2 = (3e - 2b) + 1$, i.e. $2b - 3e - 3 = 0$. But $0 = 2b - 3e - 3 = b + b - 3e - 3 \geqslant b + 3 - 3 = b$, therefore $b = 0$, $e = -1$. Therefore (S, L) is the blow up in one point of the pair in the case $L^2 = 9$.

But then, $h^0(L) = h^0(L) - 1 = 5$, and we have a contradiction using the Riemann-Roch formula for a surface in $\boldsymbol{P}^4$.

Suppose fiinally $K_S^2 = -2$. Since $(2K_S + 2L)^2 = 8$ and

$$h^0(2K_S + 2L) = \chi(2K_S + 2L) = \tfrac{1}{2}(2K + 2L)(K + 2L) = 8\,,$$

we have that map φ given by the sections of $2K_S + 2L$ is birational. Call $S' = \varphi(S)$. Let F the generic fibre of the map $S \to \mathrm{Alb}\,(S)$; $(2K_S + 2L) \cdot F \geqslant 2$ since (S, L) is not a conic bundle and to reach on absurd we will prove that S' contains ∞ many lines which are necessarly image of curves of $|F|$. Suppose first S' is smooth; then $H \cong \mathcal{O}_{S'}(1)$ has genus $\leqslant 2$ (a degree 8 curve spanning $\boldsymbol{P}^6$ has genus $\leqslant 2$). Therefore $(K_{S'} + H')^2 = K_{S'}^2 + 2K_{S'} \cdot H + H^2 \leqslant -4$, that implies that $(S', K_{S'} + H')$ is a scroll; but $F \cdot (2K_S + 2L) = 1$ is absurd. If S' is singular and does not contain ∞ many lines then, projecting from $x \in \mathrm{Sing}\,(S')$ we find a new surface S'' which spans $\boldsymbol{P}^6$ and with $\deg S'' \leqslant 6$. If S'' is singular and not containing ∞ many lines, we could project from $x' \in \mathrm{Sing}\,(S'')$ obtaining a surface $S''' \subset \boldsymbol{P}^5$ and with

$\deg S'' \leq 4$. S'' would be a minimal degree surface, and S'' and S would be rational which is absurd. Therefore S'' is smooth; let $H = \mathcal{O}_{S''}(1)$. We have that $g(H) \leq 1$ and that $(K_{S''} + H)^2 < 0$, which imply that (S'', H) is a scroll and there exist ∞ many lines.

4.5. $g(L) = 5$. We can suppose $d \geq 9$ and therefore that $k(S) = -\infty$.

Using the inequality 1.7. we have $q \leq (24 - d)/8 \leq 15/8$, i.e. $q = 1$. Again if $d' = 0$ then (S, L) is conic bundle over an elliptic curve and $\hat{L} = 2E + bf$ with $-E^2 = e = 1, 0$ or -1, $b = 4 + e = 5, 4$ or 3.

Suppose therefore $d' > 0$. We have, as in 4.4.2, that $g'' \geq 0$ and, as in 4.4.3, that

4.5.1. $2g'' - 2 = 6(K_S^2 - L^2) + 80$.

Since $K_S^2 \leq 0$, $L^2 \geq 14$ is absurd.

If $L^2 = 13$ then $g'' = 2$ and $K_S^2 = 0$ that implies $S = \hat{S}$. Since by theorem C, $\mathfrak{L} = 2(K_S + L)$ is very ample we can use our previous results and have that $(S, \mathfrak{L})$ is either a scroll or a conic bundle over $\boldsymbol{P}^1$. They are both absurd since the first implies $2(K_S + L)\cdot F = 1$ and the second implies $q = 0$.

If $L^2 = 12$ then either

i) $g'' = 5$, $K_S^2 = 0$ and $S = \hat{S}$

or

ii) $g'' = 4$, $K_S^2 = -1$ and $S = \hat{S}$ or the blown up of $\hat{S}$ in one point.

In the first cases, since $d'' = (2K_S + 2L)^2 = 16$, using our previous, results, $(S, \mathfrak{L})$ has to be a conic bundle and therefore $(2K_S + 2L)\cdot F = 2$ for a generic fibre. This implies $(K_S + L)\cdot F = 1$, that is $K_S \cdot F = 0$. Since $g(F) = 0$ and $F \cdot F = 0$ this is absurd. In the second case $(K_{\hat{S}} + 2\hat{L}) = 12$ and $(\hat{S}, \hat{\mathfrak{L}} = 2K_{\hat{S}} + 2\hat{L})$ has to be a conic bundle giving again an absurd.

If $L^2 = 11$ then either

i) $g'' = 8$ $\quad K_S^2 = 0$ $\quad S = \hat{S}$ $\quad$ or

ii) $g'' = 5$ $\quad K_S^2 = -1$ $\quad$ or

iii) $g'' = 2$ $\quad K_S^2 = -2$

The second case is impossible since $(\hat{S}, \hat{\mathfrak{L}} = 2K_{\hat{S}} + 2\hat{L})$ is such that $\hat{\mathfrak{L}}$ is very ample, $g(\hat{\mathfrak{L}}) = g'' = 5$ and $\hat{\mathfrak{L}}^2 = d'' = 12$, and we have just seen this cannot happen.

The third case also does not happen since $g(\hat{\mathfrak{C}}) = g'' = 2$ implies that either $(\hat{S}, \hat{\mathfrak{C}})$ is a scroll or S is rational (see 4.2).

To rule out the case i), since $S = \hat{S}$ and $L = aE + bf$ with $a \geqslant 3$, observe that $3 \leqslant L \cdot E = -ea + b$ and $-3 = K_S \cdot L = ea - 2b$. If $e \geqslant 0$, then $b \geqslant 3 + ea > 0$ and we have the absurd $-3 = ea - 2b = ea - b - b \leqslant -3 - b < -3$. If $e = -1$ then $-3 = K \cdot L = -a = 2b$, i.e. $3 = a + 2b$ and $11 = L^2 = a^2 + 2ab = a(a + 2b) = a \cdot 3$ which is absurd.

Let now $L^2 = d = 10$: as before, by 4.5.1, we have the following cases

K_S^2	0	-1	-2	-3
g''	11	8	5	2

The two last cases are ruled out as in the previous paragraphs. If $K_S^2 = 0$ $S = \hat{S}$ we have again an absurd if $e \geqslant 0$, and if $e = -1$ we have: $-2 = K \cdot L = -a - 2b$, $10 = L^2 = a(a + 2b)$ which give $a = 5$, $2b = -3$ which is absurd.

If $K_S^2 = -1$ then, as in the genus 4 case, $K_S = -2E - ef + P$ and $L = aE + bf + \alpha P$ with $\alpha < 0$.

As usual $L \cdot E = -ea + b \geqslant 3$ and we have the equalities: $10 = L^2 = -a(ae - 2b) - \alpha^2$, $-2 = K_S \cdot L = (ae - 2b) - \alpha$. These give $10 = -a(-2 + \alpha) - \alpha^2 = 2a - \alpha(\alpha + a)$. Since $a \geqslant 3$, $\alpha < 0$, $a + \alpha = L \cdot (f - P) > 0$, there are no solutions.

We finally discuss the case $L^2 = 9$; by 4.5.1 we have the following possibilities:

K_S^2	0	-1	-2	-3	-4
g''	14	11	8	5	2

The last case, as already observed, implies $q = 0$. $g'' = 5$ and $K_S^2 = -3$ implies $\hat{\mathfrak{C}}^2 = (2K_{\hat{S}} + 2\hat{L})^2 = 4(K_S + L)^2 = 16$ which is absurd (see 4.5.1).

If $K_S^2 = 0$ then $S = \hat{S}$ and $L = aE + bf$, $K_S = -2E - ef$. If $e \geqslant 0$ then, since $L \cdot E = -ea + b \geqslant 3$, we have the absurd $-1 = K \cdot L = ea - b - b \leqslant -3 - b < -3$. If $e = -1$ we have:

$$1 = -K \cdot L = a + 2b\,, \qquad 9 = L^2 = a(a + 2b)\,,$$

which imply $a = 9$ $b = -4$, i.e. $L = 9E - 4f$. Therefore $S = \boldsymbol{P}(\zeta)$ and $L = 9E - 4f$ with $\zeta =$ degree 1 rank 2 bundle over an elliptic curve.

But then, $h^0(L) = \chi(L) = 5$, and we have a contradiction using 1.6.

(We thank Ionescu for the last observation given us by private communication).

If $K_S^2 = -1$ then $L = (aE + bf) + cP$ with $c < 0$ and: $9 = L^2 = -a(ae - 2b) - c^2$, $-1 = (ae - 2b) - c$. These imply

$$9 = -a(c-1) - c^2 = a - c(a + c).$$

Since $a \geqslant 3$, $c < 0$ and $(a + c) = L(f - P) > 0$, the only possible solution is $a = 5$ $c = -1$ and the above inequalities give $5e - 2b + 2 = 0$.

$$0 = 2b - 5e - 2 = b + (b - 5e) - 2 \geqslant b + 3 - 2 = b + 1 \Rightarrow -1 \geqslant b\,.$$

But also $b - 5e \geqslant 3 \Rightarrow b \geqslant 3 + 5e$. The only possibility is $e = -1$ $b = -1$ or -2, which do not satisfy $5e - 2b + 2 = 0$ above.

The case $K_S^2 = -2$ also does not occur as can be showed by a straightforward computation similar to the previous but blowing up two points from the minimal model in the (three) possible ways: we omit it.

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