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Some commutativity criteria

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Some Commutativity Criteria.

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1. Introduction.

In [5] B. H. Neumann proved the following elegant result as an application of Ramsey's Theorem [6]:

Theorem A. All sets of mutually non-commuting elements of a group G are finite if and only if G is centre-by-finite.

See [1], [3] and [4] for some variations on this theme. We present here some further results of the same general type, the difference being that the outcome of imposing this or that condition turns out to be commutativity.

To state our first theorem, we define an n-set in a group G to be any subset of cardinality exactly n: G is said to be a P_n-group if $XY = YX$ for all n-sets X and Y in G.

Theorem B. For every positive integer n, infinite P_n-groups are abelian.

Following the spirit of [4], we consider the class P_n^* of groups G such that every infinite set of n-sets in G contains a pair X, Y of different members such that $XY = YX$. The content of Theorem A is that the P_1^*-groups are precisely the centre-by-finite groups.

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We have been able to show that infinite P_2^*-groups and infinite P_3^*-groups are commutative, and believe that our methods will show that infinite P_n^*-groups are abelian when n is not too large. Indeed, non-abelian infinite P_n^*-groups must be something like Tarski groups, as Theorems C and D below show.

THEOREM C. For every $n \geqslant 2$, P_n^*-groups possessing infinite abelian subgroups are abelian. In particular, non-periodic and infinite locally finite P_n^*-groups are abelian for $n \geqslant 2$.

COROLLARY 1. Every infinite P_2^*-group is abelian.

Theorem C tells us that to prove that all infinite P_n^*-groups are abelian, we need do it only for finitely generated periodic groups. In this context, we have:

THEOREM D. For every $n \geqslant 2$, every finitely generated P_n^*-group with a proper infinite subgroup is abelian.

COROLLARY 2. Every infinite P_3^*-group is abelian.

2. Proofs.

To prove Theorem B, assume for a contradiction that G is an infinite P_n-group with elements x and y such that $xy \neq yx$. Choose elements $a_1, a_2, \ldots, a_{n-1}$ which are different from each other and from x and y. Then

$$\{a_1, a_2, \ldots, a_{n-1}, x\}\{a_1, a_2, \ldots, a_{n-1}, y\} = \\ = \{a_1, a_2, \ldots, a_{n-1}, y\}\{a_1, a_2, \ldots, a_{n-1}, x\},$$

whence we get:

$$xy \text{ is one of } a_i a_j,\ y a_k,\ a_m x$$

for suitable i, j, k, m. Thus $a_i a_j = xy$, or $a_k = x^y$ or $a_m = y^{x^{-1}}$ for some i, j, k, m. Now put $X = G \setminus \{x, y, x^y, y^{x^{-1}}\}$, and define a graph Γ on the vertex set X by declaring that there is an edge joining u to v if and only if $uv \neq xy$ and $vu \neq xy$.

By Ramsey's Theorem [6], Γ contains an infinite complete subgraph or an infinite totally disconnected subgraph. If the latter holds, we have an infinite sequence $u_1, \ldots, u_n, \ldots$ of vertices with no edges between them. Without loss of generality, we many assume that $u_1u_2 = xy$. Then $u_1u_3 \neq xy$, since $u_2 \neq u_3$. Thus $u_3u_1 = xy$. Similarly $u_1u_4 \neq xy$, so that $u_4u_1 = xy$, contradicting the fact that $u_3 \neq u_4$. Thus we conclude that there is an infinite sequence $t_1, t_2, \ldots$ of elements of X such that $t_it_j \neq xy$ for all i and j. All we need is the first $n-1$ of these to falsify the equation

$$\{t_1, t_2, \ldots, t_{n-1}, x\}\{t_1, t_2, \ldots, t_{n-1}, y\} = \{t_1, t_2, \ldots, t_{n-1}, y\}\{t_1, t_2, \ldots, t_{n-1}, x\},$$

so that G is not a P_n-group after all.

The reader will observe that Theorem B holds for quasigroups as well as for groups. Probably it fails for semigroups.

Our proof of Theorem C rests on the following technical lemma, whose proof we omit.

LEMMA 1. Let T be an infinite subset of an abelian group. Then there exist two infinite sequences $a_1, a_2, \ldots, a_n, \ldots$ and $a'_1, a'_2, \ldots, a'_n, \ldots$ of elements of T such that $a_ia'_j \neq a_ja'_i$ whenever $i \neq j$.

To prove Theorem C, we first prove it in the case where G has infinite centre. Let n be an integer greater than 1 and G an infinite P^*_n group with elements x, y such that $xy \neq yx$. Let $t_1, t_2, \ldots, t_{n-2}$ be fixed distinct elements of the centre Z of G; then, for any $z, w \in Z \setminus \{t_1, t_2, \ldots, t_{n-2}\}$, $X_{z,w} := \{t_1, t_2, \ldots, t_{n-2}, zx, wy\}$ is an n-set. Further, $X_{z,w} = X_{z',w'}$ if and only if $z = z'$ and $w = w'$, since $xy \neq yx$.

Since we are assuming Z to be infinite, we can use the Lemma to construct infinite sequences $z_1, z_2, \ldots$ and $w_1, w_2, \ldots$ of elements of Z different from $t_1, t_2, \ldots, t_{n-2}$ and such that $z_iw_j \neq w_iz_j$ whenever $i \neq j$. Now consider the n sets X_{z_i,w_i}. There are infinitely many of them, so that

$$X_{z_i,w_i} \cdot X_{z_j,w_j} = X_{z_j,w_j} \cdot X_{z_i,w_i}$$

for some different i and j. Writing out this equality yields, after a small calculation, the contradiction that $z_iw_j = w_iz_j$. This completes the proof of our special case of Theorem C.

To complete the proof in general, we shall show that the existence of an infinite abelian subgroup implies that the centre is infinite. First, another technical result.

LEMMA 2. Let G be a group, T an infinite subset of G and g an element of G not commuting with any element of T. Then T has an infinite subset S such that $S \cap S^g = \emptyset$.

PROOF. We construct the elements of S by induction. Choose any element s_1 of T, when $\{s_1\} \cap \{s_1\}^g = \emptyset$. Assume that we have found different elements $s_1, s_2, \ldots, s_n$ of T such that $\{s_1, s_2, \ldots, s_n\} \cap \cap \{s_1, s_2, \ldots, s_n\}^g = \emptyset$. For s_{n+1} we choose any element in

$$T \setminus \{s_1^g, s_2^g, \ldots, s_n^g, s_1^{g^{-1}}, s_2^{g^{-1}}, \ldots, s_n^{g^{-1}}\};$$

then $\{s_1, s_2, \ldots, s_{n+1}\}$ clearly fails to intersect $\{s_1, s_2, \ldots, s_{n+1}\}^g$, and the induction is complete.

The situation is now that G is a P_n^* group, $n \geqslant 2$, having an infinite abelian subgroup A. To show that G is abelian, it is enough to show that $\langle A, g \rangle$ has infinite centre for every g in A, and apply the special case done above to establish that A centralizes g, so that A is central in G and again the special case applies. For this it is enough to show that every element g of G commutes with some element of every infinite subset of A; for we can express A as a disjoint union of infinitely many infinite subsets, thus producing infinitely many elements of A commuting with g. This will show that $\langle A, g \rangle$ has infinite centre, since A is abelian.

Thus, for a contradiction, let T be an infinite subset of A having no element commuting with g. Choose an infinite subset S of T according to Lemma 2; by the P_n^*-property, there exist different elements $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}$ of S such that

$$(*) \qquad \{a_1, a_2, \ldots, a_{n-1}, g\}\{b_1, b_2, \ldots, b_{n-1}, g\} = \\ = \{b_1, b_2, \ldots, b_{n-1}, g\}\{a_1, a_2, \ldots, a_{n-1}, g\}.$$

The element $a_1 g$ must appear on the right-hand side of this equation. It must be of the form $b_i a_j$, g^2, $b_r g$ or $g a_s$. The first three cases are impossible since g would be in A and thus commute with everything in A; and it is not $g a_s$ since $S \cap S^g = \emptyset$. Thus T must have a non-trivial elemenet commuting with g after all, and this completes the proof of Theorem C.

To prove Corollary 1, let G be an infinite P_2^*-group. It will be enough if we show that G is centre-by-finite, for then G has infinite centre and Theorem C applies. This is where Theorem A comes in: we prove that G is a P_1^*-group. To this end, let X be an infinite subset of G. Then there are infinitely many 2-sets $\{1, x\}$ with $x \in X \setminus \{1\}$, so that

$$\{1, x\}\{1, y\} = \{1, y\}\{1, x\}$$

for some different x and y in G, that is,

$$\{1, x, y, xy\} = \{1, y, x, yx\} .$$

But then xy is one of $1, x, y$ or yx and in all cases $xy = yx$. Thus G is a P_1^*-group, as required.

For Theorem D, let G be a finitely generated P_n^*-group with a proper infinite subgroup H, and let g be any element outside H. It will be enough for us to show that $C_H(g)$ is infinite. This is because G is generated by finitely many elements $g_1, g_2, \ldots, g_n$ outside H, and we have $C_H(g_1) := H_1$ is infinite so that by the same argument $C_{H_1}(g_2) := H_2$ is infinite, and so on. This procedure leads to an infinite subgroup centralizing $g_1, g_2, \ldots, g_n$, that is, to an infinite central subgroup. Theorem C now applies.

To show that $C_H(g)$ is infinite when $g \notin H$, we prove that every infinite subset T of H contains an element centralizing g, and reason as before. If T contains no element centralizing g, choose an infinite subset S of T with $S \cap S^g = \emptyset$; since G is in P_n^*, there are different elements $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}$ satisfying equation $(*)$. Now g is outside H, whereas the a_i and b_j are inside H; since $S \cap S^g = \emptyset$, equation $(*)$ forces us to deduce that $a_1 g = g a_1$, just as in the proof of Theorem C; and the proof of Theorem D is complete.

Finally, we establish Corollary 2 by proving that every infinite periodic P_3^*-group contains a non-central element with infinite centralizer. Theorem D then applies to show that every finitely generated infinite periodic P_3^*-group is abelian, which is enough for our purposes.

Thus let g be an element of an infinite periodic P_3^*-group G. Firstly, we shown hat g cannot have cofinite conjugacy class. In fact, more is true:

Lemma 3. Let G be a periodic group with a cofinite conjugacy class of elements. Then G is finite.

PROOF. Let g be an element with cofinite conjugacy class. By Ditsman's Lemma, the subgroup N generated by the elements not conjugate to g is finite, and G/N is a periodic group in which the non-trivial elements are mutually conjugate. But then, G/N is of prime exponent, p say, and there exist elements x, y of G/N with $x^y = x^{-1}$. Since $x^{y^2} = x$, this means that $p = 2$, G/N is elementary abelian and of order at most 2.

Returning now to the proof of Corollary 2, let G be an infinite periodic P_3^*-group and g an element outside the centre of G. Lemma 3 tells us that g does not have cofinite conjugacy class, so that infinitely many elements of G fail to be conjugate to g. Our aim of proving $C_G(g)$ infinite will be realized by showing that every infinite set T of elements not conjugate to g contains an element centralizing g. Any T without such centralizing elements has an infinite subset S with $S \cap S^g = \emptyset$, and, since G is in P_3^*, there are different elements a, b of S such that

$$\{1, a, g\}\{1, b, g\} = \{1, b, g\}\{1, a, g\} .$$

What are the possibilities on the right hand side for the element ag of the left hand side? Not 1, a, g, g^2 or ga, since g does not centralize a. Not ba, since g is not conjugate to b. Thus ag must be b. The very same reasoning gives that the element ga of the right-hand side must also be b; and g does commute with a after all. This completes the proof of Corollary 2.

Finally, we shall deal with finite P_n-groups in a later paper. It turns our that almost all finite P_n-groups for given n are commutative, though not all: every group of order t is in P_n whenever $n > t/2$.

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