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G. Romani

$\overset{1}{N}$-symmetric submanifolds

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$\overset{1}{N}$-Symmetric Submanifolds.

G. ROMANI (*)

SUNTO. - Nel presente lavoro si caratterizzano le sottovarietà di $\mathbf{R}^n$ che sono trasformate in sè dalla simmetria di $\mathbf{R}^n$ rispetto ad un qualunque loro primo spazio normale. Queste sottovarietà sono dette $\overset{1}{N}$-simmetriche e sono caratterizzate dall'avere la prima applicazione normale totalmente geodetica.

Introduction.

In [F] Ferus demonstrates that the submanifolds M of $\mathbf{R}^n$ having the second fundamental form, s_M, parallel, $\nabla s_M = 0$, are characterized in extrinsic terms as the submanifolds of $\mathbf{R}^n$ transformed locally in itself by the reflection of $\mathbf{R}^n$ with respect to any normal space of the submanifold; such submanifolds are called (locally) *symmetric submanifolds*.

In [R.V.] Ruh and Vilms show how the condition $\nabla s_M = 0$ is equivalent to that the, Gauss map, g_M, of the submanifold is totally geodesic: $\nabla (g_M)_* = 0$. One has, therefore, that

THEOREM. M is a (locally) symmetric submanifold of $\mathbf{R}^n$ if and only if $\nabla (g_M)_* = 0$.

Recently in [C.R.] there have been introduced for the sub-

(*) Indirizzo dell'A.: Dipartimento di Matematica « G. Castelnuovo », Piazzale A. Moro 5 - 00185 Roma (Italia).

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manifolds M of $\mathbb{R}^n$ some maps, $\overset{k}{\nu}_M$, that generalise the Gauss map in that, for $k = 0$, $\overset{\circ}{\nu}_M = g_M$.

In [C.R.] it is shown how $\nabla(\overset{k}{\nu}_M)_* = 0$ implies $\overset{k}{\nu}_M = \text{const}$ for $k > 1$ and how, therefore, the only significant conditions are the $\nabla(\overset{\circ}{\nu}_M)_* = \nabla(g_M)_* = 0$ already studied, and the $\nabla(\overset{1}{\nu}_M)_* = 0$.

It is precisely the condition $\nabla(\overset{1}{\nu}_M)_* = 0$ that will be dealt with here.

Called s_M the second fundamental form of the submanifold M of $\mathbb{R}^n$, the space generated by the vectors $s_M(X_p, X_p)$, $X_p \in T_p(M)$ is called *first normal space to M in p* and indicated with $\overset{1}{N}_p(M)$. If the submanifold M is *nicely curved* in a way that will later be explaned, the dimension on $\overset{1}{N}_p(M)$ does not depend from p and will be indicated with $\overset{1}{n}$.

The map $\overset{1}{\nu}_M$, that has been mentioned, is therefore defined as the map $\overset{1}{\nu}_M \colon M \to G(\overset{1}{n}, n - \overset{1}{n})$ that $p \in M$ associate $\overset{1}{\nu}_M(p) = \overset{1}{N}_p(M)$.

Therefore it will be demonstrated that

THEOREM. $\nabla(\overset{1}{\nu}_M)_* = 0$ if and only if for each $p \in M$ the reflection of $\mathbb{R}^n$ with respect to $\overset{1}{N}_p(M)$ transforms locally M in itself.

In analogy to the definition given by Ferus in [F] these other submanifolds which we have considered will be called *$\overset{1}{N}$-symmetric submanifolds.*

Still in [C.R.] sufficient and necessary conditions are given in order that one has $\nabla(\overset{\circ}{\nu}_M)_* (= \nabla(g_M)_* = \nabla s_M) = 0$, and $\nabla(\overset{1}{\nu}_M)_* = 0$. From such conditions one deduces at once that $\nabla(\overset{\circ}{\nu}_M)_* = 0 \Rightarrow \nabla(\overset{1}{\nu}_M)_* = 0$, and that, therefore, the (locally) symmetric submanifolds are a particular case of the $\overset{1}{N}$-symmetric submanifolds.

However, in [K.K] Kowalski and Külich present a notion of generalized k-symmetric submaniolfd that results in its turn to be a generalization of those $\overset{1}{N}$-symmetric submanifold: these last ones in fact appear as a particular case of generalized 2-symmetric submanifold, according to Kowalski and Külich [K.K].

I. Preliminaries.

Let M a m-dimensional submanifold of $\mathbb{R}^n$. Chosen a point p on M, the tangent space in p to M, $T_p(M)$, is also called the first osculator space to M in p and indicated with $\overset{1}{O}_p(M)$: $T_p(M) = \overset{1}{O}_p(M)$.

The second osculator space in p to M, $\overset{2}{O}_p(M)$, is defined as the subspace generated by $\overset{1}{X}_p$ and by $\overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{2}{X}$ when $\overset{1}{X}_p$varies in $T_p(M)$, and $\overset{2}{X}$ in $T(M)$ and where with $\overset{\mathbf{R}}{\nabla}$ is indicated the covariant derivative in $\mathbb{R}^n$.

In symbols:

$$\overset{2}{O}_p(M) = \{\overset{1}{X}_p, \overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{2}{X} : \overset{1}{X}_p \in T_p(M), \overset{2}{X} \in T(M)\} .$$

In general the k-th osculator space to M in p is defined putting:

$$\overset{k}{O}_p(M) = \{\overset{1}{X}_p, \overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{2}{X}, \overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{\mathbf{R}}{\nabla}_{\overset{2}{X}}\overset{3}{X}, \ldots, \overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{\mathbf{R}}{\nabla}_{\overset{2}{X}} \ldots \overset{\mathbf{R}}{\nabla}_{\overset{k-1}{X}}\overset{k}{X} : X_p \in T_p(M), \overset{2}{X}, \ldots, \overset{k}{X} \in T(M)\} . \tag{1}$$

If for each k the dimension of $\overset{k}{O}_p(M)$ does not depend on p it is said that the submanifold M is *nicely curved.*

Naturally $\overset{k}{O}_p(M) \subseteq \overset{k+1}{O}_p(M)$ and evidently if for a certain entire $l > 0$ $\overset{l}{O}_p(M) = \overset{l+1}{O}_p(M)$ then $\overset{l}{O}_p(M) = \overset{l'}{O}_p(M)$ for every $l' > l$. The orthogonal complement of $\overset{k}{O}_p(M)$ in $\overset{k+1}{O}_p(M)$ will be called k-th normal space to M in p and denoted with $\overset{k}{N}_p(M)$, in particular it will result

$$\overset{k+1}{O}_p(M) = \overset{k}{O}_p(M) \oplus \overset{k}{N}_p(M) \tag{2}$$

from which it follows at once that if M is nicely curved then also the dimensions of $\overset{k}{N}_p(M)$ is constant on M.

In the following we will place $\overset{0}{O}_p(M) = \{0\}$ and consequently

$\overset{0}{N}_p(M) = \overset{1}{O}_p(M) = T_p(M)$. From (2) it clearly follows

$$\overset{k+1}{O}_p(M) = \overset{0}{N}_p(M) \oplus \overset{1}{N}_p(M) \oplus \ldots \oplus \overset{k}{N}_p(M) . \tag{3}$$

For the notions stated up to here compare [Sp].

If we now suppose M nicely curved we can define for $k = 0, 1, \ldots, l-1$ the map $\overset{k}{\nu}_M\colon M \to G(\overset{k}{n}, n - \overset{k}{n})$, where with $\overset{k}{n}$ is indicated the constant dimension of $\overset{k}{N}_p(M)$, placing for $p \in M$,

$$\overset{k}{\nu}_M(p) = N_p(M) \qquad (\varepsilon\, G(n, n - \overset{k}{n})).$$

The differential of $\overset{k}{\nu}_M$ in the point p of M, $(\overset{k}{\nu}_M)_{*p}$, gives place to a homomorphism between $T_p(M)$ and $T_{\overset{1}{\nu}_M(p)}(G(\overset{k}{n}, n - \overset{k}{n}))$, but every tangent vector to the grassmannian, $G(n, n - \overset{k}{n})$ of the $\overset{k}{n}$-spaces of $\mathbb{R}^n$ in its point α, can be thought as a homomorphism between the point α considered as $\overset{k}{n}$-space and its orthogonal, $\alpha^\perp$ (cfr. [R.V.]).

From this it follows that $(\overset{k}{\nu}_M)_{*p}$ can be thought as a bilinear map between $T_p(M) \times \overset{1}{N}_p(M)$ and $\overset{1}{N}_p(M)^\perp$.

The covariant derivative of $(\overset{k}{\nu}_M)_*$ as bilinear map between $T_p(M) \times \times \overset{1}{N}_p(M)$ and $\overset{1}{N}_p(M)^\perp$ will be indicated with $\nabla(\overset{k}{\nu}_M)_*$. As already indicated in the introduction for every $k > 1$, $\nabla(\overset{k}{\nu}_M)_* = 0$ it implies $(\overset{k}{\nu}_M)_* = 0$ that is $\overset{k}{\nu}_M =$ constant (cfr. [C.R.]) for $k = 0$, $\overset{\circ}{\nu}_M = g_M$ (where g_M is the classical gauss map of M) and the condition $\nabla(g_M)_* (= \nabla(\overset{0}{\nu}_M)_*) = 0$ has been found equivalent to suppose M symmetric submanifold; for that reason we will limit ourselves to the study of the condition $\nabla(\overset{1}{\nu}_M)_* = 0$.

For that purpose, we observe that the first normal space, $\overset{1}{N}_p(M)$, in the point p to M coincides with the space generated by $s_M(X_p, X_p)$, where s_M is the second fundamental form of M, when X_p varies in $T_p(M)$ (cfr. [Sp]).

In the continuation the normal space to a submanifold M of $\mathbb{R}^n$ will be indicated with $N(M)$, furthermore given a subspace H of $\mathbb{R}^n$, with $P_H\colon \mathbb{R}^n \to H$ we will indicate the orthogonal projection of $\mathbb{R}^n$ on H; given a submanifold S of $\mathbb{R}^n$ with $\overset{S}{\nabla}$ we will indicate the connection on S induced by $\mathbb{R}^n$; given a vector sub-bundle $F \to S$ on S of the product bundle $S \times \mathbb{R}^n$, we will indicate with $\overset{F}{\nabla}$ the connection in F induced by $S \times \mathbb{R}^n$.

II. Let M a nicely curved submanifold of $\mathbb{R}^n$, of dimension m.

(1) Definition. M is a $\overset{1}{N}$-symmetric submanifold if for any $p \in M$,

the reflection of $\mathbb{R}^n$ with respect to $\overset{1}{N}_p$ transforms locally M in itself.
We prove the

(2) THEOREM. M is $\overset{1}{N}$-symmetric iff $\nabla(\overset{1}{\nu}_M)_* = 0$.
First we prove the implication

$$\nabla(\overset{1}{\nu}_M)_* = 0 \Rightarrow M \text{ is } \overset{1}{N}\text{-symmetric} \tag{3}$$

For each $p \in M$ we will define

$$(\overset{\perp}{NT})_p = (\overset{1}{N}_p(M) \oplus T_p(M))^\perp = \overset{1}{N}_p(M)^\perp \cap T_p(M)^\perp = \overset{1}{N}_p(M)^\perp \cap N_p(M) \tag{4}$$

and with $p + (\overset{\perp}{NT})_p$ the affine subspace of $\mathbb{R}^n$ through the point p parallel to $(\overset{\perp}{NT})_p$.

Let U an open set of $\mathbb{R}^n$ such that $U \cap \left(\bigcup_{p\in M} (p + (\overset{\perp}{NT})_p)\right)$ is a submanifold, $\tilde{M}$, or $\mathbb{R}^n$; for $p \in M \cap U (\subset \tilde{M})$ we will have

$$T_p(\tilde{M}) = T_p(M) \oplus (\overset{\perp}{NT})_p \tag{5}$$

and

$$N_p(\tilde{M}) = (T_p(\tilde{M}))^\perp = (T_p(M) \oplus (\overset{\perp}{NT})_p)^\perp = (T_p(M)^\perp \cap (\overset{\perp}{NT})_p^\perp = N_p(M) \cap \overset{1}{N}_p(M) = \overset{1}{N}_p(M) \tag{6}$$

and

$$T_p(\tilde{M}) = \overset{1}{N}_p(M)^\perp \tag{7}$$

moreover

(8) PROP. M is a totally geodesic submanifold of $\tilde{M}$

PROOF. If $X_p \in T_p(M)$, $Y \in T(M)$ then

$$\overset{\mathbf{R}}{\nabla}_{X_p} Y \in \overset{1}{O}_p(M) = T_p(M) \oplus \overset{1}{N}_p(M);$$

therefore, (4),

$$P_{(\overset{\perp}{NT})_p}\left(\overset{\mathbf{R}}{\nabla}_{X_p} Y\right) = P_{T_p(M)\oplus \overset{1}{N}_p(M)}\left(\overset{\mathbf{R}}{\nabla}_{X_p} Y\right) = 0;$$

it follows, (5),

$$\overset{\tilde{M}}{\nabla}_{X_p} Y = P_{T_p(\tilde{M}}\left(\overset{\mathbf{R}}{\nabla}_{X_p} Y\right) = P_{T_p(M)\oplus(\overset{1}{N}T)_p}\left(\nabla_{X_p} Y\right) = P_{T_p(M)}\left(\overset{\mathbf{R}}{\nabla}_{X_p} Y\right) = \overset{M}{\nabla}_{X_p} Y .$$

Consider now:

a) the gauss map of order zero $\overset{\circ}{\nu}_{\tilde{M}}\colon \tilde{M} \to G(n - \overset{1}{n}, \overset{1}{n})$ of $\tilde{M}$

$$\overset{\circ}{\nu}_{\tilde{M}}(q) = T_q(\tilde{M}) \qquad q \in \tilde{M}$$

b) the isometry

$$\mu\colon G(n - \overset{1}{n}, \overset{1}{n}) \to G(\overset{1}{n}, n - \overset{1}{n})$$

between the grasmannian $G(n - \overset{1}{n}, \overset{1}{n})$ of the $(n - \overset{1}{n})$-subspaces of $\mathbf{R}^n$ and the grasmannian $G(\overset{1}{n}, n - \overset{1}{n})$ of the $\overset{1}{n}$-subspaces of $\mathbf{R}^n$, that to a $(n - \overset{1}{n})$-subspace associates to it the orthogonal

c) the map

$$\overset{\circ}{\mu}_{\tilde{M}}\colon \tilde{M} \to G(\overset{1}{n}, n - \overset{1}{n})$$

defined by

$$\overset{\circ}{\mu}_{\tilde{M}} = \mu \circ \overset{\circ}{\nu}_{\tilde{M}}$$

therefore:

(9) Prop. For

$$p \in M \cap U(\subset \tilde{M}), \qquad X_p, Y_p \in T_p M(\subset T_p(\tilde{M})) , \qquad \xi_p \in \overset{1}{N}_p(M)(= N_p(\tilde{M}))$$

i) $\overset{\circ}{\mu}_{\tilde{M}}(p) = \overset{1}{\nu}_M(p)$

ii) $(\overset{\circ}{\mu}_{\tilde{M}})_*(X_p) = (\overset{1}{\nu}_M)_*(X_p)$

iii) $\nabla_{X_p}(\overset{\circ}{\mu}_{\tilde{M}})_*(Y_p, \xi_p) = \nabla_{X_p}(\overset{1}{\nu}_M)_*(Y_p, \xi_p)$

PROOF OF i).

$$\mathring{\mu}_{\tilde{M}}(p) = \mu(\mathring{\nu}_{\tilde{M}}(p)) \underset{a)}{=} \mu(T_p(\tilde{M})) \underset{b)}{=} T_p\overset{\perp}{(}\tilde{M}) \underset{(6)}{=} \overset{1}{N}_p(M) = \overset{1}{\nu}_M(p)$$

PROOF OF ii). It's a natural consequence of i).

PROOF OF iii).

$$\nabla_{X_p}(\mathring{\mu}_{\tilde{M}})_*(Y_p, \xi_p) =$$

$$= \overset{\overset{1}{N}\perp}{\nabla}_{X_p}[(\mathring{\mu}_{\tilde{M}})_*(Y, \xi)] - (\mathring{\mu}_{\tilde{M}})_*(\overset{\tilde{M}}{\nabla}_{X_p} Y, \xi_p) - (\mathring{\mu}_{\tilde{M}})_*(Y, \overset{\overset{1}{N}}{\nabla}_{X_p}\xi) \underset{\text{i}),(8)}{=}$$

$$= \overset{\overset{1}{N}\perp}{\nabla}_{X_p}((\overset{1}{\nu}_M)_*(Y, \xi)) - (\overset{1}{\nu}_M)_*(\overset{M}{\nabla}_{X_p} Y, \xi_p) - (\overset{1}{\nu}_M)_*(Y_p, \overset{\overset{1}{N}}{\nabla}_{X_p}\xi) =$$

$$= \nabla_{X_p}(\overset{1}{\nu}_M)_*(Y_p, \xi_p)$$

here with X, Y, ξ we indicate extensions of X_p, Y_p in $T(M)$ and of ξ_p in $\overset{1}{N}(M)$.

If we consider that μ is an isometry we will have

$$\text{(10)} \qquad \nabla(\mathring{\mu}_{\tilde{M}})_* \underset{b)}{=} \nabla(\mu_* \circ (\mathring{\nu}_{\tilde{M}})_*) = \mu_* \circ (\nabla(\mathring{\nu}_{\tilde{M}})_*)$$

therefore for the iii)

(11) PROP. For $X_p, Y_p \in T_p(M)$, $\xi_p \in \overset{1}{N}_p(M)$

$$\mu_*[(\nabla_{X_p}(\mathring{\nu}_{\tilde{M}})_*(Y_p)](\xi_p) = \nabla_{X_p}(\overset{1}{\nu}_M)_*(Y_p, \xi_p)\,.$$

Infact

$$\mu_*[(\nabla_{X_p}(\mathring{\nu}_{\tilde{M}})_*(Y_p)](\xi_p) \underset{(10)}{=} \nabla_{X_p}(\mathring{\mu}_{\tilde{M}})_*(Y_p, \xi_p) \underset{\text{iii})}{=} \nabla_{X_p}(\overset{1}{\nu}_M)(Y_p, \xi_p)\,.$$

In particular

(12) PROP. $\nabla(\overset{1}{\nu})_* = 0$ iff any $X_p, Y_p \in T_p(M) (\subset T_p(\tilde{M}))$

$$\nabla_{X_p}(\mathring{\nu}_{\tilde{M}})_*(Y_p) = 0$$

The (12) follows from (11) considering the fact that μ_* is an isomorfism.

We can now demonstrate that, indicated by $s_{\tilde{M}}$ the second fundamental form on $\tilde{M}$ and with $\overset{\tilde{N}}{\nabla} s_{\tilde{M}}$ its derivative considering it with values in the orthogonal, $N(\tilde{M})$, to $T(\tilde{M})$, one has

(13) Prop. If $\overset{1}{\nabla}\nu_* = 0$ then for any $p \in M$, $(\overset{\tilde{N}}{\nabla} s_{\tilde{M}})_p = 0$

(i.e. $\overset{\tilde{N}}{\nabla}_{\tilde{X}_p} s_{\tilde{M}}(\tilde{Y}_p, \tilde{Z}_p) = 0 \ \forall \tilde{X}_p, \tilde{Y}_p, \tilde{Z}_p \in T_p(\tilde{M})$)

Proof. First of all remember that $(\overset{\circ}{\nu}_{\tilde{M}})_* = s_{\tilde{M}}$ and that $\nabla(\overset{\circ}{\nu}_{\tilde{M}})_* = \overset{\tilde{N}}{\nabla} s_{\tilde{M}}$ [R.V.]. Then taken $X_p, Y_p \in T_p(M) (\subset T_p(\tilde{M}))$ and $\tilde{Z}_p \in T_p(\tilde{M}) \cdot (\supset T_p(M))$ one has

$$\text{j)} \ \overset{\tilde{N}}{\nabla}_{X_p} s_{\tilde{M}}(Y_p, \tilde{Z}_p) = \nabla_{X_p}(\overset{\circ}{\nu}_{\tilde{M}})_*(Y_p, \tilde{Z}_p) = \big(\nabla_{X_p}(\overset{\circ}{\nu}_{\tilde{M}})_*(Y_p)\big)(\tilde{Z}_p) \underset{(12)}{=} 0$$

and, for the symmetry of $s_{\tilde{M}}$

$$\text{jj)} \ \overset{\tilde{N}}{\nabla}_{X_p} s_{\tilde{M}}(\tilde{Z}_p, Y_p) = \overset{\tilde{N}}{\nabla}_{X_p} s_{\tilde{M}}(Y_p, \tilde{Z}_p) = 0$$

If now $\eta_p, \zeta_p \in (\overset{\perp}{NT})_p (\subset T_p(\tilde{M}))$ it results

$$\text{jjj)} \ (\overset{\tilde{N}}{\nabla}_{X_p} s_{\tilde{M}})(\eta_p, \xi_p) = 0$$

to prove jjj) let us begin by observing that $\overset{\tilde{N}}{\nabla}_{X_p} s_{\tilde{M}}(\eta_p, \xi_p)$ one calculates starting from two arbitrary vector fields, η, ζ tangent to $\tilde{M}$ and verifying the condition $\eta(p) = \eta_p$, $\xi(p) = \xi_p$. Moreover it will be sufficient to define η and ζ along any curve $\mathcal{C}$ for p having as tangent vector X_p, and in the points of $(\overset{\perp}{NT})_{p'}$ with $p' \in \mathcal{C}$.

We define η, ζ on $\mathcal{C}$ in the following manner: if $p' \in \mathcal{C}$, $\eta(p')$, $\xi(p')$, are the transported by parallelism of η_p and ζ_p in p' along $\mathcal{C}$ so that it results

$$\text{l)} \ \overset{\tilde{M}}{\nabla}_{X_p} \eta = \overset{\tilde{M}}{\nabla}_{X_p} \zeta = 0$$

$$\text{ll)} \ \eta(p'), \zeta(p') \in (\overset{\perp}{NT})_{p'} = \text{orthogonal in } T(\tilde{M}) \text{ of } T(M)$$

the second condition, ll), follows from the fact that $(\overset{\perp}{NT})_p$ is the orthogonal in $T(\tilde{M})$ of $T(M)$ and from the fact that M is totally geodesic in $\tilde{M}$; it assures that for every $p' \in \mathcal{C}$ and for every $q \in (\overset{\perp}{NT})_{p'}$ we can define $\eta(q) = \eta(p')$. $\zeta(q) = \zeta(p')$. So for each $p' \in \mathcal{C}$

$$\text{lll)} \ s_{\tilde{M}}(\eta(p'), \zeta(p')) = 0 \quad ,$$

We will therefore have

$$(\overset{\tilde N}{\nabla}_{X_p} s_{\tilde M})(\eta_p, \zeta_p) = \overset{\tilde N}{\nabla}_{X_p}(s_{\tilde M}(\eta, \zeta)) - s_{\tilde M}(\overset{\tilde M}{\nabla}_{X_p}\eta, \zeta_p) - s_{\tilde M}(\eta_p, \overset{\tilde M}{\nabla}_{X_p}\xi) = 0 .$$

They will be now $\tilde X_p, \tilde Y_p, \tilde Z_p \in T_p(\tilde M)$ and put $\tilde X_p = X_p + \eta_p$, $\tilde Y_p = Y_p + \theta_p$ $\tilde Z_p = \tilde Z_p + \zeta_p$ with $X_p, Y_p, Z_p \in T_p(M)$, $\eta_p, \theta_p, \zeta_p \in (\overset{\perp}{N}T)_p$: We will have

$$(\overset{\tilde N}{\nabla}_{\tilde X_p} s_{\tilde M})(\tilde Y_p, \tilde Z_p) = \overset{\tilde N}{\nabla}_{X_p} s_{\tilde M}(\tilde Y_p, \tilde Z_p) + \overset{\tilde N}{\nabla}_{X_p} s_{\tilde M}(\tilde Y_p, \tilde Z_p) \underset{\text{j),jj),jjj)}}{=} \overset{\tilde N}{\nabla}_{\eta_p} s_{\tilde M}(\tilde Y_p, \tilde Z_p)$$

but $\overset{\tilde N}{\nabla} s_{\tilde M}$ is symmetric with respect to its arguments, so

$$(\overset{\tilde N}{\nabla}_{\tilde X_p} s_{\tilde M})(\tilde Y_p, \tilde Z_p) = \overset{\tilde N}{\nabla}_{\tilde Y_p} s_{\tilde M}(\eta_p, \tilde Z_p) = \overset{\tilde N}{\nabla}_{Y_p} s_{\tilde M}(\eta_p, \tilde Z_p) + \overset{\tilde N}{\nabla}_{\theta_p} s_{\tilde M}(\eta_p, \tilde Z_p) \underset{\text{jj),jj}}{=}$$

$$= \overset{\tilde N}{\nabla}_{\theta_p} s_{\tilde M}(\eta_p, \tilde Z) = \overset{\tilde N}{\nabla}_{\tilde Z_p} s_{\tilde M}(\eta_p, \theta_p) \underset{\text{jjj)}}{=} \overset{\tilde N}{\nabla}_{\zeta_p} s_{\tilde M}(\eta_p, \theta_p) = 0$$

the last equality being the consequence of the fact that η_p and θ_p can be extended along ζ_p in parallel and constant vector fields.

From (13) it follows, in particular, that the second fundamental form of $\tilde M$, $s_{\tilde M}$, is parallel along each geodesic γ of M (and therefore of $\tilde M$ contained on M). From theorem 1 of Strübing [St] it follows, that all the curvatures of each geodesic γ of M are constant and that the vectors of Frenet of even order of γ are found in $N(\tilde M) = \overset{1}{N}(M)$, while the odd in $T(\tilde M)$. From the lemma 1 of Strübing it follows that the reflection of $\mathbb{R}^n$, with respect to $N(\tilde M) = \overset{1}{N}(M)$, changes each geodesic γ of M in itself for each $p \in M$ and therefore changes locally M in itself.

The part now demonstrated by the theorem has an immediate consequence:

(14) LEMMA. If $\nabla \overset{1}{\nu}_* = 0$ then M is locally symmetric.

PROOF. It is seen that if $\nabla \overset{1}{\nu}_* = 0$, M is locally $\overset{1}{N}$-symmetric; for each $p \in M$ there is, therefore, the reflection of $\mathbb{R}^n$ with respect to $\overset{1}{N}_p(M)$ that induces locally on M an involutive isometry that fixes a sole point of M, namely the point p. This is sufficient for the proof.

III. And now we can demonstrate

(1) $$M \text{ is } \overset{1}{N}\text{-symmetric} \Rightarrow \nabla \overset{1}{\nu}_* = 0 .$$

Let us begin by demonstrating the following proposition:

(2) PROP. If $\tau: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry that maps M in itself, then for each $k = 1, \dots, l$, $\tau(\overset{k}{O}_p(M)) = \overset{k}{O}_{\tau(p)}(M)$

PROOF. By definition

$$\overset{k}{O}_p(M) = \{\overset{1}{X}_p, \overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{2}{X}, \overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{\mathbf{R}}{\nabla}_{\overset{2}{X}} \dots \overset{\mathbf{R}}{\nabla}_{\overset{k-1}{X}}\overset{k}{X} : \overset{i}{X} \in T(M)\}$$

but being τ an isometry of $\mathbb{R}^n$ it results

$a)$ $\tau_*(\overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{2}{X}) = \overset{\mathbf{R}}{\nabla}_{\tau_*(\overset{1}{X}_p)}\tau_*(\overset{2}{X})$

$b)$ $\tau_*(\overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{\mathbf{R}}{\nabla}_{\overset{2}{X}}\overset{3}{X}) = \overset{\mathbf{R}}{\nabla}_{\tau_*(\overset{1}{X}_p)}\tau_*(\overset{\mathbf{R}}{\nabla}_{\overset{2}{X}}\overset{3}{X}) = \overset{\mathbf{R}}{\nabla}_{\tau_*(\overset{1}{X}_p)}\overset{\mathbf{R}}{\nabla}_{\tau_*(\overset{2}{X})}\tau_*(\overset{3}{X})$

$c)$ $\tau_*(\overset{\mathbf{R}}{\nabla}_{\overset{1}{X}_p}\overset{\mathbf{R}}{\nabla}_{\overset{2}{X}} \dots \overset{\mathbf{R}}{\nabla}_{\overset{k-1}{X}}\overset{k}{X}) = \overset{\mathbf{R}}{\nabla}_{\tau_*(\overset{1}{X}_p)}\overset{\mathbf{R}}{\nabla}_{\tau_*(\overset{2}{X})} \dots \overset{\mathbf{R}}{\nabla}_{\tau_*(\overset{k-1}{X})}\tau_*(\overset{k}{X})$

and as $\tau_*(\overset{i}{X}) \in T(M)$ because τ maps M in itself, the written equalities tell us that

$d)$ $\tau(\overset{k}{O}_p(M)) \subset \overset{k}{O}_{\tau(p)}(M)$

but τ is bijective, therefore conclusion.

From (2) follows

(3) PROP. If $\tau: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of M in itself, then for each $k = 0, 1, \dots, l-1$

$$\tau(\overset{k}{N}_p(M)) = \overset{k}{N}_{\tau(p)}(M) .$$

The (3) follows at once from (2) keeping in mind the fact that τ being an isometry preserves the angles and the fact that $\overset{k-1}{N}_p(M)$ is the orthogonal complement of $\overset{k-1}{O}_p(M)$ in $\overset{k}{O}_p(M)$.

Now, let us suppose that M is $\overset{1}{N}$-symmetric. That implies that for each $p \in M$ the reflection τ_p of $\mathbb{R}^n$ with respect to $\overset{k}{N}_p(M)$ maps locally M in itself. Because of (3) will map therefore in itself the above manifold $\tilde{M}$: infact, if $q \in \tilde{M}$, or $q \in M$ is then by hypothesis transformed in a point of $M \subset \tilde{M}$, or $q \in (\overset{\perp}{NT})_{p'}$ for a certain $p' \in M$. But

$$(\overset{\perp}{NT})_{p'} = \overset{2}{N}_{p'}(M) \oplus \ldots \oplus \overset{l-1}{N}_{p'}(M) \oplus [\overset{\circ}{N}_{p'}(M) \oplus \overset{1}{N}_{p'}(M) \oplus \ldots \oplus \overset{l-1}{N}_{p'}(M)]^{\perp}.$$

As for (3) $\overset{i}{N}_{p'}(M)$ $(i = 2, \ldots, l-1)$ is transformed in $\overset{i}{N}_{\tau_p(p')}(M) \subset \tilde{M}$ and obviously

$$\tau_p\big([\overset{\circ}{N}_{p'}(M) \oplus \ldots \oplus \overset{l-1}{N}_{p'}(M)]^{\perp}\big) = [\overset{\circ}{N}_{\tau_p(p')}(M) \oplus \ldots \oplus \overset{l-1}{N}_{\tau_p(p')}(M)]^{\perp}$$

since τ_p is an isometry, one has again the proof.

But we have already seen that $\overset{1}{N}_p(M) = N_p(\tilde{M}) = T_p(\tilde{M})^{\perp}$ for $p \in M$ therefore the fact that M is $\overset{1}{N}$-symmetric assures us, for a theorem of Ferus (Lemma 1 of [F]) that $(\overset{\tilde{N}}{\nabla} s_{\tilde{M}})_p = 0$ for $p \in M$.

That implies that for $X_p, Y_p \in T_p(M)$

$$a) \quad \overset{\tilde{N}}{\nabla}_{X_p} s_{\tilde{M}}(Y_p) = 0$$

but $s_{\tilde{M}} = (\overset{\circ}{\nu}_{\tilde{M}})_*$ for which

$$b) \quad \nabla_{X_p}(\overset{\circ}{\nu}_{\tilde{M}})_*(Y_p) = 0$$

and (12) paragraph II enables us to conclude with the thesis: $\nabla \overset{1}{\nu}_* = 0$.

If we now remember the condition in order that it results $\overset{N}{\nabla} s_M (= \nabla \overset{\circ}{\nu}_*) = 0$ and $\nabla \overset{1}{\nu}_* = 0$, [C.R.], we see at once that $\overset{N}{\nabla} s_M = 0 \Rightarrow$ $\Rightarrow \nabla \overset{1}{\nu}_* = 0$. It results therefore that the submanifolds $\overset{1}{N}$-symmetric constitute a generalization of the symmetric submanifolds. Moreover if $\overset{N}{\nabla} s_M = 0$ the osculating space $\overset{2}{O}_p(M) = T_p(M) \oplus \overset{1}{N}_p(M)$ is independent from $p \in M$ and M is all contained on it.

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