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Well-posed minimum problems for preorders

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Well-Posed Minimum Problems for Preorders.

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SUMMARY - It is introduced Tikhonov well-posedness for minimum problems for preorders. In case of total preorders represented by real valued functions, it is investigated the relationship between well-posedness for the preorder and for the function. A characterization, resembling that one given by Furi and Vignoli, is given for well-posed preorders.

1. Introduction.

Given a topological space X and $f: X \to \mathbb{R}$, the minimum problem associated to f is said to be well-posed in Tikhonov sense or, briefly, f is said to be well-posed (wp) if f has a unique point of minimum to which every minimizing sequence converges ([Ti], [DoZ]). We notice that f induces on X a total preorder by defining $x \leqslant y$ iff $f(x) \leqslant f(y)$. Obviously, another $g: X \to \mathbb{R}$ may identify the same preorder on X. Note that in this case the mimimum points of f and g (and of $\leqslant$) are the same. The question that can be raised is the following: does wp of f guarantee wp of g? Put in other words: is wp in some sense a property intrinsic to the preorder $\leqslant$, or it does depend on its different representatives? This is of some interest, since in (neoclassical) economics what is considered usually to be given is the « preference ysstem » of the agent, and not some specific « utility function » that does represent it. Partial results on this topic are in [Pa] (where the emphasis is on saddle points and Nash equilibria, instead of minima)

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and in [Lu3]. What is shown here is that wp is essentially intrinsic to the preorder, in the following sense: if f is wp, any other function representing the same preorder as f is wp too (provided that some non-degeneracy conditions are respected).

The above mentioned result provides some hope that it is possible to define wp for a given preorder $\preccurlyeq$, without any function as an intermediate. This is actually the main topic of this paper: it is proposed a definition of wp for a preorder (not necessarily total), implicitly suggested in [Lu3], and some properties are investigated. In particular, it is shown that in this context we must distinguish between wp through minimizing sequences or nets (contrary to the case of real-valued functions: see [Pa]). Furthermore, it is proved a characterization of sequential wp for preorders resembling that one given in [FV]. We take also the occasion for some general considerations about Tikhonov wp (not only for minimum problems, but also for variational inequalities, saddle points, Stackelberg equilibria, etc.).

2. Notations and hypotheses.

Let X be a non empty set.

A relation $\preccurlyeq$ on X which is reflexive and transitive is said to be a preorder on X. We use the standard abbreviation « $x \prec y$ » to mean « $x \preccurlyeq y$ and not $y \preccurlyeq x$ ». If $\preccurlyeq$ is also total (i.e.: $x \preccurlyeq y$ or $y \preccurlyeq x$ is true for every $x, y \in X$), we say that the preorder is total. As usual, $x_0 \in X$ is said to be a minimum for $\preccurlyeq$ on X if $x_0 \preccurlyeq x$ for every $x \in X$.

Let $\preccurlyeq$ be a total preorder on X. $f\colon X \to \mathbb{R}$ is said to represent $\preccurlyeq$ if

$$(2.1) \qquad x \preccurlyeq y \text{ if and only if } f(x) \leq f(y)$$

or, equivalently:

$$(2.2) \qquad x \prec y \text{ if and only if } f(x) < f(y) .$$

Note that if a relation can be represented by some f accordingly to (2.1), then it must be a total preorder. Obviously, given $f\colon X \to \mathbb{R}$, we may define a total preorder through (2.1): in this case we say that f induces the preorder on X.

Given $f\colon X \to \mathbf{R}$ and $Y \subseteq X$, we use the following notation:

$$\inf(f;\, Y) = \inf\{f(x)\colon x \in Y\} \in [-\infty, +\infty]\,,$$

$$\min(f;\, Y) = \min\{f(x)\colon x \in Y\} \in \mathbf{R}\,,$$

$$\operatorname{argmin}(f;\, Y) = \{x \in Y\colon f(x) = \min(f;\, Y)\}\,.$$

A sequence $x_n \in X$ is said to be minimizing for f if $f(x_n) \to \inf(f;\, X)$ If $f(x_n) = \min(f;\, X)$ for all large n, then the minimizing sequence will be said to be trivial.

When X is a topological space, we say that f is Tikhonov well-posed (briefly: « wp ») if:

1) there is a unique minimum point $x_0 \in X$,

2) every minimizing sequence converges to x_0.

3. Ordinal character of well-posedness.

Definition 3.1. Let X be a non empty set, and $\preccurlyeq$ be a preorder on X. A net $x_\nu \in X$ is said to be a « minimizing net for $\preccurlyeq$ » if:

(3.1) for every $y \in X$ which is not a minimum on X for $\preccurlyeq$, eventually $x_\nu \prec y$. //

Remark 3.2. For terminology on nets, we follow [Kel].

It is easy to see that any subnet of a minimizing net is minimizing, and that a net s.t. eventually x_ν is a minimum for $\preccurlyeq$ is a minimizing net (and will be called trivial minimizing net). //

Definition 3.3. Let X be a topological space and $\preccurlyeq$ be a preorder on X. The minimum problem for $\preccurlyeq$ is said to be wp (respectively: seq-wp) if:

1) there is a unique minimum x_0 for $\preccurlyeq$.

2) every minimizing net (respectively: sequence) converges to x_0. //

Remark 3.4. Of course, a minimizing sequence is also a minimizing net, so if $\preccurlyeq$ is wp then it is also seq-wp. Note also that wp (seq-wp) w.r.t. some topology implies wp (seq-wp) w.r.t. any coarser topology. //

THEOREM 3.5. Let $f\colon X \to \mathbb{R}$ be non constant and let $\preccurlyeq$ be the total preorder induced by f on X.

If a sequence x_n is minimizing for f, then x_n is minimizing for $\preccurlyeq$.

Conversely, we have:

1) if f has minimum on $X \setminus \operatorname{argmin}(f; X)$, then f has minimum and the only minimizing sequences are the trivial ones; moreover, minimizing sequences for f and $\preccurlyeq$ are the same.

2) if f has not minimum on $X \setminus \operatorname{argmin}(f; X)$, then $\tilde{f}$, defined as:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus \operatorname{argmin}(f; X)\,, \\ \inf(f; X \setminus \operatorname{argmin}(f; X)) & \text{if } x \in \operatorname{argmin}(f; X)\,, \end{cases}$$

represents $\preccurlyeq$ and minimizing sequences for $\tilde{f}$ and for $\preccurlyeq$ are the same. //

REMARK 3.6. The hypothesis f non constant implies that

$$X \setminus \operatorname{argmin}(f; X) \neq \emptyset\,.$$

Of course, the case in which f is constant is trivial. //

PROOF (of Theorem 3.5). For the first assertion, take x_n minimizing for f and let y be not a minimum for $\preccurlyeq$. Since $\preccurlyeq$ is a total preorder, there is $x \in X$ s.t. $x \prec y$. Since f represents $\preccurlyeq$, we have that $f(x) < f(y)$, so $\inf(f; X) < f(y)$: hence, $f(x_n) < f(y)$ for all large n, i.e. $x_n \prec y$ for all large n. So, x_n is minimizing for $\preccurlyeq$.

For the converse, let us see case 1) first. That f has minimum on X is trivial by contradiction and the verification of all of the other assertions is straightforward.

Let's consider case 2). To prove that $\tilde{f}$ represents $\preccurlyeq$ it is only needed to show that $\tilde{f}(x) < \tilde{f}(y)$ for $x \in \operatorname{argmin}(f; X)$ and $y \in X \setminus \operatorname{argmin}(f; X)$. If we assume the contrary, this means that there exists $\bar{y} \in X \setminus \operatorname{argmin}(f; X)$ s.t. $f(\bar{y}) = \tilde{f}(x) = \inf(f; X \setminus \operatorname{argmin}(f; X))$. But this amounts to say that f has minimum on $X \setminus \operatorname{argmin}(f; X)$, contrary to the hypothesis.

To prove the last assertion, we confine us to show that if we take a minimizing sequence x_n for $\preccurlyeq$, s.t. x_n is not a minimum for f, then x_n is minimizing for f (from this, the general result follows immediately). Now, such x_n is minimizing for $f|_{X\setminus\operatorname{argmin}(f;X)}$. Namely, take $y \in X\setminus\operatorname{argmin}(f;X)$: there is $x \in X$ s.t. $x \prec y$, so eventually $x_n \prec y$. This means that for every $y \in X\setminus\operatorname{argmin}(f;X)$ we have eventually $f(x_n) < f(y)$. Since for every $\varepsilon > 0$ there is $y \in X\setminus\operatorname{argmin}(f;X)$ s.t.

$$\inf(f; X\setminus\operatorname{argmin}(f;X)) \leqslant f(y) < \inf(f; X\setminus\operatorname{argmin}(f;X)) + \varepsilon\,,$$

we have proved that $f(x_n) \to \inf(f; X\setminus\operatorname{argmin}(f;X))$, i.e. x_n is minimizing for $f|_{X\setminus\operatorname{argmin}(f;X)}$: From this it is patent that such x_n is minimizing for $\tilde{f}$. //

COROLLARY 3.7. Let X be a topological space, $f\colon X \to \mathbb{R}$ and $\preccurlyeq$ be the preorder induced by f on X.

If $\preccurlyeq$ is seq-wp, then f is wp.

Conversely, assuming f is not constant, we have:

1) if f has minimum on $X\setminus\operatorname{argmin}(f;X)$, then f wp implies $\preccurlyeq$ wp.

2) if f has not minimum on $X\setminus\operatorname{argmin}(f;X)$, and $\inf(f;X) = \inf(f; X\setminus\operatorname{argmin}(f;X))$, then f wp implies $\preccurlyeq$ wp. //

PROOF. It is obvious that existence and uniqueness of a minimum for $\preccurlyeq$ is equivalent to existence and uniqueness of a minimum for f (and they are the same). So, the result follows from the fact that under the hypotheses above f and have the same minimizing sequences (of course, the hypotheses in case 2) guarantee that $f = \tilde{f}$ in the notation of Theorem 3.5). //

REMARK 3.8. Proposition 3.1 of [Pa] is a corollary of this result, since its hypotheses guarantee that $f(X)$ is an interval in $\mathbb{R}$, and this implies $f = \tilde{f}$ (and that f has not minimum on $X\setminus\operatorname{argmin}(f;X)$). Also Theorem 2.13 of [Lu3] follows: its assumption that f has non trivial minimizing sequences clearly implies that f has not minimum on $X\setminus\operatorname{argmin}(f;X)$ and that $f = \tilde{f}$. //

REMARK 3.9. Note that if f is lower semicontinuous, the same is true for $\tilde{f}$: this may be considered as a nice feature, having in mind the role of lower semicontinuity in the context of minimum problems. //

4. Well-posedness: sequences and nets.

In the previous section we have investigated the relationship between minimizing sequences for preorders and functions representing them. In section 3, however, we introduced minimizing nets for preorders: it has already been remarked, however, that for functions there is no need to introduce nets as long as wp is concerned [Pa] (the situation is different, of course, when generalized wp is considered: see [ČKenR]). We find convenient to restate the result of [Pa] in a more abstract setting, that will prove to be useful later.

We assume the following hypotheses:

$$(\bigstar)\quad \begin{cases} E \text{ and } X \text{ are topological spaces; } \varphi\colon E \to 2^X_\blacksquare = \mathcal{P}(X)\setminus\{\emptyset\}. \\ \text{For some } e_0 \in E\,, \quad \varphi(e_0) = \{x_0\}\,, \quad x_0 \in X\,. \end{cases}$$

We topologize $2^X_\blacksquare$ with the upper Vietoris topology [KlTh]. Of course, continuity of φ in e_0 is equivalent to:

$$\text{for every net } e_\alpha \to e_0\,,\ \varphi(e_\alpha) \to \{x_0\}$$

(and in case E is 1st countable, continuity can be characterized using sequences only).

It is easy to see that, given a net Z_α in $2^X_\blacksquare$, we have $Z_\alpha \to \{x_0\}$ in the upper Vietoris topology if and only if:

$$(4.1)\qquad \text{for every net } x_\alpha \text{ s.t. } x_\alpha \in Z \text{ for every } \alpha\,, \quad x_\alpha \to x_0\,.$$

Note, moreover, that the equivalence above remains true if we replace on both sides nets by sequences.

Combining all of the preceding remarks, we get that, when E is 1st countable, the following are equivalent:

$$(4.2_n)\qquad \begin{cases} \text{for every sequence } x_n\,, \\ (x_n \in \varphi(e_n) \text{ and } e_n \to e_0) \text{ implies } x_n \to x_0\,, \end{cases}$$

$$(4.2)\qquad \begin{cases} \text{for every net } x_\alpha\,, \\ (x_\alpha \in \varphi(e_\alpha) \text{ and } e_\alpha \to e_0) \text{ implies } x_\alpha \to x_0\,. \end{cases}$$

This implies that for minimum problems seq-wp and wp are equivalent (simply take $E = [0, +\infty[$, $e_0 = 0$ and $\varphi(e) = \{x \in X: f(x) \leqslant \leqslant \min(f; X) + e\}$, or $E = f(X)$, $e_0 = \min(f; X)$ and $\varphi(e) = \{x \in X: f(x) = e\}$ as in [Pa]).

We want to stress the fact that the same remark (for an appropriate but straightforward choice of E and e_0) applies to: wp for variational inequalities (see [LuPa1] and [LuPa3]; see also [R] for an alternative definition); wp saddle point problems (see [CM1], where wp is reduced to wp for an associated minimum problem, or see [PaTo] and [Pa] where a more « direct » approach is used); Stackelberg problems (see [M], where wp for Stackelberg problems is defined in a way such that it does not reduce to wp for the minimization of the functional of the leader, incorporated with the best reply of the follower); wp for constrained minimum problems (in the sense of [LePo], or considering the so called strong wp, as suggested in [BLu]); wp Nash equilibria (as introduced by [CM2]; see [Pa]).

Nothing of the above applies, however, to wp for preorders, as can be seen easily from examples (we refer to [Kel], in general for terminology, and in particular for problem 2.*B* on p. 76).

EXAMPLE 4.1. Let Ω be the first uncountable ordinal: consider $[0, \Omega[$ with the usual ordering on it. Standard considerations about cardinality show that we cannot have maximizing sequences: if x_n is a maximizing sequence, then we should have

$$\bigcup_{n \in \mathbf{N}} \{x \in [0, \Omega[: x \leqslant x_n\} = [0, \Omega[$$

which is impossible since x_n has only countably many predecessors, while $[0, \Omega[$ is uncountable. //

EXAMPLE 4.2. Take $[0, \Omega]$ with the order topology: since there are no nontrivial maximizing sequences, while on the contrary the identity map on $[0, \Omega]$ provides an example of a maximizing net, we may see that w.r.t. the discrete topology $[0, \Omega]$ with the usual ordering is seq-wp but not wp. //

For what concerns wp, we have the following:

PROPOSITION 4.3. Let X be totally ordered by $\leqslant$, and assume that there exists a (necessarily unique) minimum x_0 on X. Then, the minimum problem for $\leqslant$ on X is wp w.r.t. the order topology on X. //

PROOF. Take a minimizing net x_α. Given a neighborhood U of x_0, there exists $y \in X$ with $x_0 \prec y$ and s.t. the interval $[x_0, y[$ is contained in U. But this means that eventually we have $x_\alpha \in [x_0, y[$ (because x_α is a minimizing net). hence x_α is eventually in U. So, the net x_α converges to x_0. //

EXAMPLE 4.4. Consider $X = \{(x, y) \in \mathbb{R}^2 \colon x, y \geqslant 0\}$, with the lexicographic ordering $\leqslant_L$ on it. Then $(X, \leqslant_L)$ is wp for the order topology, hence seq-wp (and so wp and seq-wp w.r.t. the usual topology on X, since order topology induced by $\leqslant_L$ is finer than euclidean topology). The interesting feature of this example is that there is no $f \colon X \to \mathbb{R}$ representing $\leqslant_L$ (see e.g. [De]): so, we can see that wp for preorders is a nontrivial extension of wp for functions. //

5. A characterization of well-posedness for preorders.

Despite of (or due to) its semplicity, a characterization of wp for functions that has proved to be useful is due to [FV]: for lower semicontinuous and lower bounded functions defined on a complete metric space wp is equivalent to

$$\inf_{\varepsilon > 0} \operatorname{diam} \{x \colon f(x) \leqslant \inf(f; X) + \varepsilon\} = 0 .$$

We shall prove something similar to this in the context of seq-wp for preorders. The main handicap from which we suffer is that we do not have level sets indexed by some $\varepsilon \in \mathbb{R}$, as has been already noticed in the previous section. We shall try to provide some surrogate for them, but before passing to the main point we would like to provide an « abstract » version of the Furi-Vignoli result, which applies to the case in which we have some « level sets » that can be parametrized in a reasonable way.

THEOREM 5.1. Let $n \in \mathbb{N}$, $\varepsilon_i \in [0, +\infty]$ for $i \in \{1, \dots, n\}$, $E = \prod_{i=1}^{n} [0, \varepsilon_i]$, X be a complete metric space. Let $\varphi \colon E \to 2^X_\blacksquare$ be closed valued, s.t. $0 \leqslant \varepsilon^1 \leqslant \varepsilon^2$ implies $\varphi(\varepsilon^1) \subseteq \varphi(\varepsilon^2)$ and assume that $\varphi(0) = \bigcap_{\substack{\varepsilon \gg 0 \\ \varepsilon \in E}} \varphi(\varepsilon)$. Then, $\inf_{\varepsilon \gg 0} \operatorname{diam} \varphi(\varepsilon) = 0$ if and only if

(5.1) there is $x_0 \in X$ s.t. $\varphi(0) = \{x_0\}$ and φ is continuous in 0 w.r.t. upper Vietoris topology on $2^X_\blacksquare$. //

PROOF. For the « if » part, assume that the infimum is positive. Taking $\varepsilon^k = (1/k, \ldots, 1/k)$, we can find $x_k', x_k'' \in \varphi(\varepsilon^k)$ and $\lambda > 0$ such that $d(x_k', x_k'') \geqslant \lambda$. This is not possible since the hypotheses guarantee that $x_0 \in \varphi(\varepsilon^k)$ and that $\varphi(\varepsilon^k)$ is contained in a ball of center x_0 and radius $\lambda/3$, for large k.

To prove the « only if », first of all we prove that there is $x_0 \in X$ s.t. $\bigcap_{\substack{\varepsilon \gg 0 \\ \varepsilon \in E}} \varphi(\varepsilon) = \{x_0\}$. Take $\varepsilon^k = (1/k, \ldots, 1/k)$, and $x_k \in \varphi(\varepsilon^k)$: x_k is a Cauchy sequence, so x_k converges to some x_0 due to completeness of X. Because $\varphi(\varepsilon^k)$ is closed, $x_0 \in \varphi(\varepsilon^k)$ for every $k \in \mathbf{N}$, so

$$x_0 \in \bigcap_{k \in \mathbf{N}} \varphi(\varepsilon^k) = \bigcap_{\varepsilon \gg 0} \varphi(\varepsilon) .$$

Since $\inf_{\varepsilon \gg 0} \operatorname{diam} \varphi(\varepsilon) = 0$, $\bigcap_{\varepsilon \gg 0} \varphi(\varepsilon)$ cannot contain more than one point. So, $\varphi(0) = \{x_0\}$. For what concerns continuity of φ in 0, take any open set G containing x_0: because $\inf_{\varepsilon \gg 0} \operatorname{diam} \varphi(\varepsilon) = 0$, $\varphi(\varepsilon)$ is contained in G for ε sufficiently close to zero. //

Since (5.1) is equivalent to wp (as it has been noticed in sect. 4), the theorem above provides a generalization of the result of Furi and Vignoli, and may be applied to all of the cases considered in the previous paragraph, under appropriate conditions.

We confine ourselves to the case of strong wp [BLu]. Take $f: X \to \mathbf{R}$, and consider the problem of minimizing f on $K \subseteq X$. A sequence x_n is a «generalized minimizing sequence» ([BLu]) if $\limsup_{n \to \infty} f(x_n) \leqslant \inf(f; K)$ and $d(x_n, K) \to 0$. Asking for the convergence of these « generalized minimizing sequences » to the unique minimizer of f on K, we have the idea of f being strongly wp (relatively to K: [BLu]). Let us assume X completely metrizable, f lower bounded and lower semicontinuous, K closed and non empty. If we define $\varphi: [0, +\infty] \times [0, +\infty] \to 2^X_{\blacksquare}$ as

$$\varphi(\varepsilon^1, \varepsilon^2) = \{x \in X: d(x, K) \leqslant \varepsilon^1, f(x) \leqslant \inf(f; K) + \varepsilon^2\} ,$$

from Theorem 5.1 we get that f is strongly wp if and only if

$$\inf_{\varepsilon \gg 0} \operatorname{diam} \varphi(\varepsilon) = 0 ,$$

which is the result proved in [BLu].

What is behind this kind of reasoning is that in many cases a minimizing sequence can be described as an $x_n \in \varphi(\varepsilon_n)$ with $\varepsilon_n \to 0$: in other words, we have naturally at our disposal some φ to which we can apply Theorem 5.1. However, in the case of preorders we saw that this is not possible (since otherwise seq-wp and wp would coincide), if we wish that φ is defined on a space as « good » as 1st countable. In any case, it is possible to conceive for preorders some mathematical object which can be considered as intermediate between minimizing sequence and the « level sets » $\varphi(\varepsilon)$. We shall deal with a preorder $\leqslant$ on X, in the sequel.

DEFINITION 5.2. A set-valued minimizing sequence is a sequence $\Omega_n \subseteq X$ s.t. there is a minimizing sequence x_n, with $x_{n+1} \leqslant x_n$ and $\Omega_n = \{x\colon x \leqslant x_n\}$ for every n. We say that Ω_n is « determined » by the minimizing sequence x_n. //

REMARK 5.3. For a set-valued minimizing sequence we have:

1) $x_n \in \Omega_n$ for every n, hence $\Omega_n \neq \emptyset$,
2) $\Omega_n \supseteq \Omega_{n+1}$ (for the transitivity of $\leqslant$),
3) $\bigcap_{n\in\mathbb{N}} \Omega_n = \{x\colon x \leqslant x_n \text{ for every } n\} = \{\text{minima for } \leqslant \text{ on } X\}$,
4) if $x'_n \in \Omega_n$ for every n, then x'_n is a minimizing sequence. //

THEOREM 5.4. Let X be a complete metric space, $\leqslant$ a preorder on X s.t. $\{x \in X.\ x \leqslant \bar{x}\}$ is closed for every $\bar{x} \in X$. Assume moreover that there exist minimizing sequences for $\leqslant$. Then, $\leqslant$ is seq-wp if and only if $\operatorname{diam} \Omega_n \to 0$ for every set valued minimizing sequence. //

PROOF. « Only if »: by contradiction, assume there is Ω_n s.t. $\operatorname{diam} \Omega_n \not\to 0$. So we have $x'_n, x''_n \in \Omega_n$ s.t. $d(x'_n, x''_n) \geqslant \lambda > 0$ for every n, But $d(x'_n, x''_n) \leqslant d(x'_n, x_0) + d(x''_n, x_0) \to 0$ since x'_n, x''_n are minimizing, for Remark 5.3 (x_0 is the unique minimum for $\leqslant$).

« If ». Let x_n be minimizing. Consider the subsequence ζ_n of x_n s.t. $\zeta_{n+1} \leqslant \zeta_n$ (see Lemma 5.6 below) and the Ω_n associated to it. From Cantor's theorem, we have that $\bigcap_{n\in\mathbb{N}} \Omega_n = \{x_0\}$, i.e. is a singleton. For Remark 5.3, x_0 is (the unique) minimum. So, thanks to Lemma 5.5 below, we can restrict ourselves to decreasing sequences to test seq-wp.

Given a minimizing sequence y_n s.t. $y_{n+1} \preccurlyeq y_n$, consider Ω_n associated to it: we have $d(y_n, x_0) \leqslant \operatorname{diam} \Omega_n \to 0$. Hence, $\preccurlyeq$ is seq-wp. //

LEMMA 5.5. Let X be a topological space and $\preccurlyeq$ a preorder on X. If there is a unique minimum point x_0 and every minimizing sequence x_n s.t. $x_{n+1} \preccurlyeq x_n$ converges to x_0, then $\preccurlyeq$ is seq-wp. //

PROOF. Take x_n minimizing. Assume that it does not converge to x_0. So, there is a neighborhood V of x_0 s.t. $x_n \notin V$ for infinite indexes. Let n_1 be one of these indexes and define $\xi_1 = x_{n_1}$. Since x_{n_1} cannot be a minimum ($x_0 \in V$, $x_{n_1} \notin V$ and the minimum is unique) and since x_n is minimizing, there is n_2 s.t. $x_{n_2} \prec x_{n_1}$, and we may take $n_2 > n_1$. Let $\xi_2 = x_{n_2}$. And so on .... We get a decreasing sequence ξ_n that is minimizing since it is a subsequence of x_n. Of course ξ_n cannot converge to x_0. So, we have a contradiction. //

LEMMA 5.6. If x_n is a minimizing sequence for a preorder $\preccurlyeq$, there is a subsequence ζ_n s.t. $\zeta_{n+1} \preccurlyeq \zeta_n$. //

PROOF. If x_n is a minimum for infinite indexes, obviously we have a subsequence whose values are minimum points, so we are done. If x_n is not a minimum for all $n \geqslant k_1$, define $\zeta_1 = x_{k_1}$, and repeat the construction in the proof of Lemma 5.5 to get the result. //

6. Final remarks.

As seen in the introduction, the motivation to introduce « Tikhonov wp » for preorders has its roots in the fact that in some instances what is given is a preorder and not the function representing it.

So, after having seen that wp for a function is essentially a property of the preorder represented by it, it has been a natural step to propose what could be considered wp for a preorder and study its main features. Of course, many other investigations could be done about wp for preorders, but in the author's opinion it could be much more interesting to see whether this notion does have some real interest at least in the field (neoclassical economics) from which it has been indirectly generated. A typical test for the interest of wp for preorders could be to see whether wp has some useful consequence for what concerns stability w.r.t. data perturbations (that sometimes has been termed Hadamard wp). Let's mention that this fact is known for wp of functions (see [LuPa2] and [Lu2]); on the other hand, stability w.r.t. data perturbation has been studied for a long time e.g. in equilibrium theory, so that basic books like [De] and [H] deal extensively with that.

Let's conclude saying that at least two other problems could be interesting to study from the point of view of wp for preorders: the search for saddle points (or Nash equilibria) and that of minimal points (i.e.: vector or Pareto optimization). For these classes of problems very little is known (see [Pa] for saddle points and Nash equilibria), even from the point of view of functions and not of preorders (see references in [Pa] for saddle points and Nash equilibria, and [Lu3] for vector optimization).

REFERENCES

[BLu] G. Beer - L. Lucchetti, *Continuity results for the epi-distance topology with applications to constrained convex optimization problems*, preprint, 1989.

[CM1] E. Cavazzuti - J. Morgan, *Well-posed saddle point problems*, Lect. Notes in Pure and Applied Mathematics, Vol. 86, Dekker, New York, 1983.

[CM2] E. Cavazzuti - J. Morgan, personal communication.

[ČKenR] M. M. Čoban - P. S. Kenderov - J. Revalski, *Generic well-posedness of optimization problems in topological spaces*, C. R. Acad. Bulg. Sci., **42**, 1 (1989), pp. 11-14.

[De] G. Debreu, *Theory of Value*, Wiley, New York, 1959.

[DoZ] A. Dontchev - T. Zolezzi, book in preparation.

[FV] M. Furi - A. Vignoli, *About well-posed optimization problems for functionals in metric spaces*, J. Optimiz. Theory Appl., **5** (1970), pp. 225-229.

[H] W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton Univ. Press, Princeton, 1974.

[Kel] J. L. Kelley, *General Topology*, Van Nostrand, Princeton, 1955.

[KlTh] E. Klein - A. C. Thompson, *Theory of Correspondences*, Wiley, New York, 1984.

[LePo] E. S. Levitin - B. T. Polijak, *Constrained minimization methods*, USSR Comput. Math. and Math. Phys., **6** (1968), pp. 1-50.

[Lu1] R. Lucchetti, *Some aspects of the connection between Hadamard and Tyhonov well-posedness of convex programs*, Boll. Un. Mat. Ital. Anal. Funz. Appl., **1** (1982), pp. 337-344.

[Lu2] R. LUCCHETTI, *Hadamard and Tyhonov well-posedness in optimal control*, Methods Oper. Res., **45** (1983).

[Lu3] R. LUCCHETTI, *Well-posedness, towards vector optimization*, Lecture Notes in Econ. and Math. Systems, Vol. 294, Springer, Berlin, 1983.

[LuPa1] R. LUCCHETTI - F. PATRONE, *A characterization of Tyhonov well-posedness for minimum problems, with applications to variational inequalities*, Numer. Funct. Analysis Optimiz., **3** (1981), pp. 461-476.

[LuPa2] R. LUCCHETTI - F. PATRONE, *Hadamard and Tyhonov well-posedness of a certain class of convex functions*, J. Math. Anal. Appl., **88** (1982), pp. 204-215.

[LuPa3] R. LUCCHETTI - F. PATRONE, *Some properties of « well-posed » variational inequalities governed by linear operators*, Numer. Funct. Anal. Iptimiz., **5** (1982-83), pp. 349-361.

[M] J. MORGAN, *Constrained well-posed two-level optimization problems*, preprint, Universitè de Montréal, Centre de Recherches Mathématiques, 1988.

[Pa] F. PATRONE, *Well-posedness as an ordinal property*, Riv. Mat. Pura Appl., **1** (1987), pp. 95-104.

[PaTo] F. PATRONE - A. TORRE, *Characterization of existence and uniqueness for saddle point problems and related topics*, Boll. Un. Mat. Ital. Anal. Funz. Appl., **5**-C (1986), pp. 175-184.

[R] J. REVALSKI, *Variational inequalities with unique solution*, in: *Proc. 14th Spring Conference of the Union of Bulgarian Mathematicians*, Bulg. Acad. of Sciences, Sofia (1985), pp. 534-541.

[Ti] A. N. TIKHONOV, *On the stability of the functional optimization problem*, USSR Comput. Math. and Math. Phys., **6** (1966), pp. 28-33.

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