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An Application of Fractional Calculus.

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SUMMARY - The object of this note is to show how fractional calculus approach can be employed to derive a formula expressing a generalized hypergeometric function in terms of a finite sum of lower order functions. Another representation of this formula is also given.

1. Introduction.

In his paper Karlsson [2] gave a formula for the generalized hypergeometric function ${}_pF_q$ with positive integral differences between certain numerator and denominator parameters in terms of a finite sum of lower order functions. This result [2, p. 270, eqn. (1)] is given by

$$(1.1)\qquad {}_pF_q\left[\begin{matrix} b_1+m_1, \ldots, b_n+m_n, a_{n+1}, \ldots, a_p; \\ b_1, \ldots, b_n, b_{n+1}, \ldots, b_q \quad ; \end{matrix}\; z\right] =$$

$$= \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} C(j_1, \ldots, j_n)\, z^{J_n}\, {}_{p-n}F_{q-n}\left[\begin{matrix} a_{n+1}+J_n, \ldots, a_p+J_n; \\ b_{n+1}+J_n, \ldots, b_q+J_n; \end{matrix}\; z\right],$$

where

$$(1.2)\qquad C(j_1, \ldots, j_n) = \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \cdot \frac{(b_2+m_2)_{J_1} \cdots (b_n+m_n)_{J_{n-1}} (a_{n+1})_{J_n} \cdots (a_p)_{J_n}}{(b_1)_{J_1} \cdots (b_n)_{J_n} (b_{n+1})_{J_n} \cdots (b_q)_{J_n}},$$

and

$$(1.3)\qquad J_n = j_1 + \ldots + j_n ,$$

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m_i $(i = 1, \ldots, n)$ are positive integers, and $n \leqslant \min(p, q)$, provided that $p \leqslant q$ $(|z| < 1$, if $p = q + 1)$, and no denominator parameter is a negative integer or zero.

A simpler proof of (1.1) was given by Srivastava [7]. Our purpose in this paper is to extend (1.1) to the case when the differences m_i $(i = 1, \ldots, n)$ between the numerator and denominator parameters are arbitrary. This formula is obtainable in a straightforward manner by applying the results well known in the theory of fractional calculus. A new representation of the result is also considered.

2. Extension of (1.1).

For any bounded sequence of real or complex numbers $\{A_n\}$, define a function $f(z)$ by the power series

$$f(z) = \sum_{n=0}^{\infty} A_n z^n , \qquad |z| < R . \tag{2.1}$$

A widely used basic formula in the fractional calculus is the formula

$$D_z^{\beta}\{z^{\alpha-1}\} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} z^{\alpha-\beta-1} , \qquad \operatorname{Re}(\alpha) > 0 , \tag{2.2}$$

valid for all values of β.

Also, for arbitrary β, the Leibniz rule is given by ([3, p. 263])

$$D_z^{\beta}\{u(z)v(z)\} = \sum_{n=0}^{\infty} \binom{\beta}{n} D_z^{\beta-n}\{u(z)\} D_z^{n}\{v(z)\} , \tag{2.3}$$

where $u(z)$ and $v(z)$ are functions of the form $z^{a-1}h(z)$ and $z^{b-1}k(z)$, respectively, where $h(z)$ and $k(z)$ are analytic in the disc $|z| < \varrho$, provided that $\operatorname{Re}(a) > 0$ and $\operatorname{Re}(a + b) > 0$.

Let us write

$$z^{\alpha+\beta-1} f(z) = \sum_{n=0}^{\infty} A_n z^{\alpha+\beta+n-1} , \tag{2.4}$$

and operate D_z^{β} on both the sides, then (2.4) in conjuction with (2.2) and (2.3) gives

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n}{(\alpha)_n} A_n z^n = \sum_{n=0}^{\infty} \binom{\beta}{n} \frac{z^n}{(\alpha)_n} D_z^n\{f(z)\} , \tag{2.5}$$

provided that $\operatorname{Re}(\alpha+\beta)>0$, the arbitrary sequence of real or complex numbers is bounded, and $|z|<R$.

The fractional derivative operator D_z^β can be applied term-wise to the right side of the series of (2.4) in view of the result mentioned in a recent monograph of Srivastava and Monacha [9, p. 289, Theorem 2], under, of course, the hypotheses surrounding the above equation (2.5).

As a consequence of the relation (2.5), we now derive a formula expressing a generalized hypergeometric function in terms of a finite sum of lower order. To this end, if we spacialize the arbitrary sequence $\{A_n\}$ by setting

$$A_n=\frac{\prod_{i=2}^{p}(a_i)_n}{n!\prod_{i=2}^{q}(b_i)_n},\qquad n\geqslant 0, \tag{2.6}$$

then (2.5) is seen (on slight adjustment of parameters) to yield the result

$$\begin{aligned}{}_pF_q\left[\begin{matrix}b_1+\beta_1,a_2,\dots,a_p;\\ b_1,\dots,b_q\quad;\end{matrix}\; z\right]&=\sum_{j_1=0}^{\infty}\binom{\beta_1}{j_1}\frac{(a_2)_{j_1}\dots(a_p)_{j_1}}{(b_2)_{j_1}\dots(b_q)_{j_1}}z^{j_1}\cdot\\ &\cdot{}_{p-1}F_{q-1}\left[\begin{matrix}a_2+j_1,\dots,a_p+j_1;\\ b_2+j_1,\dots,b_q+j_1;\end{matrix}\; z\right],\end{aligned} \tag{2.7}$$

provided that $p\leqslant q$ (and $|z|<1$, if $p=q+1$), $\operatorname{Re}(b_1+\beta_1)>0$, and no denominator parameter is zero or a negative integer.

The repeated application of (2.7) to each of its R.H.S. when $a_i=b_i+\beta_i$ $(i=2,\dots,n)$ gives the result

$$\begin{aligned}{}_pF_q&\left[\begin{matrix}b_1+\beta_1,\dots,b_n+\beta_n,a_{n+1},\dots,a_p;\\ b_1,\dots,b_n,b_{n+1},\dots,b_q\quad;\end{matrix}\; z\right]=\\ &=\sum_{j_1=0}^{\infty}\dots\sum_{j_n=0}^{\infty}\binom{\beta_1}{j_1}\dots\binom{\beta_n}{j_n}\frac{(b_2+\beta_2)_{J_1}(b_3+\beta_3)_{J_2}\dots(b_n+\beta_n)_{J_{n-1}}}{(b_1)_{J_1}(b_2)_{J_2}\dots(b_n)_{J_n}}\cdot\\ &\cdot\frac{(a_{n+1})_{J_n}\dots(a_p)_{J_n}}{(b_{n+1})_{J_n}\dots(b_q)_{J_n}}z^{J_n}{}_{p-n}F_{q-n}\left[\begin{matrix}a_{n+1}+J_n,\dots,a_p+J_n;\\ b_{n+1}+J_n,\dots,b_q+J_n;\end{matrix}\; z\right],\end{aligned} \tag{2.8}$$

provided that $p \leqslant q$ (and $|z| < 1$, if $p = q + 1$), n being any positive integer such that $n \leqslant \min(p, q)$, where J_n is given by (1.3), and $\operatorname{Re}(b_i + \beta_i) > 0$ for $i = 1, \ldots, n$.

The result (2.8) immediately reduces to (1.1) if the arbitrary parameters β_i $(i = 1, \ldots, n)$ are replaced by the positive integers m_i $(i = 1, \ldots, n)$. It must be stated here that (2.8) is precisely the same result obtained earlier by Chakrabarti [1, p. 200, eqn. (1.4)] by following a different line of approach employing essentially the Vandermond's convolution theorem [8, p. 243].

3. Another representation of (2.8).

Consider the generalized Leibniz rule due to Osler [5, p. 2, eqn. (3.1)] (see also [3, p. 264])

$$(3.1)\qquad D_z^{\alpha}\{u(z)v(z)\} = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an+\varepsilon} D_z^{\alpha-an-\varepsilon}\{u(z)\}\, D_z^{an+\varepsilon}\{v(z)\},$$

where $0 < a \leqslant 1$, α and ε are arbitrary (real or complex) numbers. Then in view of a known special case of a fractional derivative formula due to Raina and Koul [6, p. 99, eqn. (7)] (see also [4, p. 374]), the following result can easily be arrived at:

$$(3.2)\qquad {}_pF_q\left[\begin{matrix} b_1+\beta_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q\ ; \end{matrix}\ z\right] = $$

$$= \sum_{j_1=-\infty}^{\infty} c_1 \binom{\beta_1}{c_1 j_1 + \varepsilon_1} \frac{1}{(b_1)_{c_1 j_1+\varepsilon_1}\Gamma(1-c_1 j_1-\varepsilon_1)} \cdot {}_pF_q\left[\begin{matrix} 1, a_2, \ldots, a_p\ ; \\ 1-c_1 j_1-\varepsilon_1, b_2, \ldots, b_q; \end{matrix}\ z\right],$$

provided that $p \leqslant q$ $(p = q+1$, if $|z| < 1)$, $0 < c_1 \leqslant 1$, ε_1 is any arbitrary complex number, and none of the denominator parameters is a negative integer or zero.

When $c_1 = 1$, $\varepsilon_1 = 0$, (3.2) does not apparently seem to be reducible to (2.7). However, a slight simplification on the right side of the resulting equation, or else, using the simple identity

$$\frac{1}{\Gamma(1-j)}\,{}_pF_q\left[\begin{matrix}1, a_2, \ldots, a_p\ ;\\ 1-j, b_2, \ldots, b_q;\end{matrix}\; z\right] = \tag{3.3}$$

$$= \frac{(a_2)_j \ldots (a_p)_j}{(b_2)_j \ldots (b_q)_j} z^j\,{}_{p-1}F_{q-1}\left[\begin{matrix}a_2+j, \ldots, a_p+j;\\ b_2+j, \ldots, b_q+j;\end{matrix}\; z\right],$$

makes (3.2) (when $c_1 = 1$ and $\varepsilon_1 = 0$) reducible to (2.7).

The repeated application of (3.2) for each $a_i = b_i + \beta_i$ $(i = 2, \ldots, n)$ yields the result

$${}_pF_q\left[\begin{matrix}b_1+\beta_1, \ldots, b_n+\beta_n, a_{n+1}, \ldots, a_p;\\ b_1, \ldots, b_n, b_{n+1}, \ldots, b_q\ ;\end{matrix}\; z\right] = \tag{3.4}$$

$$= \prod_{i=1}^{n}\left\{\sum_{j_i=-\infty}^{\infty} c_i \binom{\beta_i}{c_i j_i + \varepsilon_i} \frac{1}{(b_i)_{c_i j_i+\varepsilon_i}\Gamma(1-c_i j_i-\varepsilon_i)}\right\}\cdot$$

$$\cdot\,{}_pF_q\left[\begin{matrix}1, \ldots, 1, a_{n+1}, \ldots, a_p\\ 1-c_1 j_1-\varepsilon_1, \ldots, 1-c_n j_n-\varepsilon_n, b_{n+1}, \ldots, b_q;\end{matrix}\; z\right],$$

provided that $p \leqslant q$ (and for $p = q+1$, $|z| < 1$), ε_i $(i = 1, \ldots, n)$ are arbitrary complex numbers, and $0 < c_i \leqslant 1$ $(i = 1, \ldots, n)$.

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