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Representable Equivalences are Represented by Tilting Modules.

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Let K be a field, let A be a finite-dimensional K-algebra and let ${}_AT$ be a faithful and finite-dimensional left A-module. Under these hypotheses, we will show that, if ${}_AT$ induces an equivalence satisfying the requirements of Menini-Orsatti's Representation Theorem [5], then ${}_AT$ is a tilting module.

The proof of this fact makes use, on the one hand, of the results on torsion theories induced by tilting modules obtained by Hoshino [4], Assem [1] and Smalø [8], and, on the other hand, of the new results obtained by Colpi [2].

In this way, we solve an open problem of [3].

Before we do this, we recall some definitions.

Let ${}_AT$ be a finite dimensional A-module. Then ${}_AT$ is called a tilting module, if the following conditions are satisfied:

(i) The projective dimension of ${}_AT$ is less than or equal to 1.

(ii) $\mathrm{Ext}_A^1({}_AT, {}_AT) = 0$.

(iii) There is an exact sequence of the form

$$0 \to {}_AA \to T' \to T'' \to 0$$

with T' and T'' in add T, where add T denotes the additive category whose objects are direct sums of direct summands of ${}_AT$.

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Finally, let R and S be two rings, let $\mathcal{G}$ be a full subcategory of left R-modules closed under direct sums and factor modules, let $\mathcal{D}$ be a full subcategory of left S-modules containing ${}_SS$ and closed under submodules, and let

$$\mathcal{G} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be an equivalence with F and G additive functors. Then Menini-Orsatti's theorem ([5] Theorem 3.1) asserts that there is a module ${}_RM$, with endomorphism ring S, such that $F \approx \mathrm{Hom}_R({}_RM, -)$ and $\mathcal{G}$ is the category of all R-modules generated by ${}_RM$, while $G \approx M_S \otimes -$ and $\mathcal{D}$ is the category of all S-modules cogenerated by $\mathrm{Hom}_R({}_RM, {}_RQ)$, where ${}_RQ$ is an injective cogenerator of the category of all R-modules.

In the following, according to [2] and [3], such a module ${}_RM$ is called a $*$-module. Using this terminology, we deduce from ([5] Theorem 4.3) that any tilting module is a $*$-module.

The next statement shows that the relationship between $*$-modules and tilting modules is as strong as might be expected. In the proof of the next theorem all modules will be finite-dimensional.

THEOREM 1. Let A be a finite-dimensional K-algebra, let ${}_AT$ be a finite-dimensional faithful $*$-module. Then ${}_AT$ is a tilting module.

PROOF. Let $\mathcal{T}$ be the category of all modules generated by ${}_AT$. We claim that

$$\mathcal{T} = \{{}_AX \mid \mathrm{Ext}^1_A({}_AT, {}_AX) = 0\} .$$

To see this, take any module ${}_AX$ and let ${}_AI$ be an injective module such that ${}_AX \leqslant {}_AI$. Since ${}_AT$ is faithful, $\mathcal{T}$ contains any injective module; hence ${}_AI \in \mathcal{T}$. Moreover, applying $\mathrm{Hom}_A({}_AT, -)$ to the short exact sequence

$$0 \to {}_AX \to {}_AI \xrightarrow{\pi} {}_AI/{}_AX \to 0 ,$$

we get the exact sequence

$$(*) \quad \mathrm{Hom}_A({}_AT, {}_AI) \xrightarrow{\mathrm{Hom}_A({}_AT, \pi)} \mathrm{Hom}_A({}_AT, {}_AI/{}_AX) \to \mathrm{Ext}^1_A({}_AT, {}_AX) \to 0 .$$

Suppose first that ${}_AX \in \mathcal{T}$. Then, by ([2] Corollary 4.2), $\mathrm{Hom}_A({}_AT, \pi)$ is surjective. Hence, by $(*)$, we have $\mathrm{Ext}^1_A({}_AT, {}_AX) = 0$.

Assume now that $\mathrm{Ext}^1_A({}_AT, {}_AX) = 0$. Then we deduce from (*) that $\mathrm{Hom}_A({}_AT, \pi)$ is surjective. Consequently, by ([2] Proposition 4.3), ${}_AX \in \mathcal{T}$ and so $\mathcal{T}$ satisfies our claim. It immediately follows that $\mathcal{T}$ is closed under extensions. Now let ${}_AU$ be the direct sum of a complete set of representatives of the isomorphism classes of the indecomposable modules $U_i \in \mathcal{T}$ which are Ext-projective in $\mathcal{T}$, (see [1] and [8]), that is with the property that $\mathrm{Ext}^1_A(U_i, {}_AX) = 0$ for any ${}_AX \in \mathcal{T}$. Then we know from ([8] Theorem) that ${}_AU$ is a tilting module. Hence, to prove the theorem, it suffices to check that $\mathrm{add}\,{}_AT = \mathrm{add}\,{}_AU$. To this end, we first note that our hypotheses on ${}_AT$ and the above characterization of $\mathcal{T}$ imply that $\mathrm{add}\,{}_AT \subseteq \mathrm{add}\,{}_AU$. Now let ${}_AV$ be an indecomposable summand of ${}_AU$. Then, by ([2] Theorem 4.1), there is an exact sequence in $\mathcal{T}$ of the form

$$0 \to {}_AW \to \bigoplus {}_AT \to {}_AV \to 0\,.$$

Since ${}_AV$ is Ext-projective in $\mathcal{T}$, it follows that ${}_AV \oplus {}_AW \cong \bigoplus {}_AT$. Hence using the Krull-Schmidt theorem we infer that $\mathrm{add}\,{}_AU = \mathrm{add}\,{}_AT$. Therefore ${}_AT$ is a tilting module, and the proof is complete.

As an immediate consequence of Theorem 1 and ([3] Lemma 1), we obtain the following corollary.

COROLLARY 2. Let A be a finite-dimensional K-algebra, let ${}_AM$ be a finite-dimensional module and let $\bar{A} = A/\mathrm{ann}\,{}_AM$. Then ${}_AM$ is a $*$-module if and only if ${}_{\bar{A}}M$ is a tilting module.

The next remark points out another application of Theorem 1.

REMARK 3. Let A be a finite-dimensional K-algebra and let ${}_AT$ be an ω-tilting module in the sense of [5]. Then ${}_AT$ is a tilting module. In fact, the definition of an ω-tilting module implies that ${}_AT$ is faithful and finite-dimensional, while ([5] Theorem 4.3) guarantees that ${}_AT$ is a $*$-module.

The following observation gives a partial answer to the question whether or not $*$-modules over finite-dimensional algebras are actually finitely generated.

REMARK 4. Let A be a finite-dimensional K-algebra, and let ${}_AM$ be a $*$-module. If A is representation finite [6], then ${}_AM$ is finitely generated. Indeed, our hypothesis on A and ([7] Corollary 4.4) guarantee that ${}_AM$ is of the form $\bigoplus_i N_i$, where the N_i's are indecomposable modules of finite dimension over K. On the other hand, by ([2] Theorem 4.1), ${}_AM$ is self-small, i.e. for any morphism $f\colon {}_AM \to \bigoplus_{j\in J} M_j$, with $M_j \cong {}_AM$ for any $j \in J$, there exists a finite subset $F \subseteq J$ such that $f({}_AM) \subseteq \bigoplus_{j\in F} M_j$. Putting things together, we conclude that ${}_AM = \bigoplus_i N_i$ is the direct sum of finitely many N_i's. Therefore ${}_AM$ is finitely generated, as claimed.

REFERENCES

[1] I. Assem, *Torsion theories induced by tilting modules*, Can. J. Math., **36** (1984), pp. 899-913.

[2] R. Colpi, *Some remarks on equivalences between categories of modules*, to appear in Comm. Algebra.

[3] G. D'Este, *Some remarks on representable equivalences*, to appear in vol. 26 of the Banach Center Publications.

[4] M. Hoshino, *Tilting modules and torsion theories*, Bull. London Math. Soc., **14** (1982), pp. 334-336.

[5] C. Menini - A. Orsatti, *Representable equivalences between categories of modules and applications*, Rend. Sem. Mat. Univ. Padova, 82 (1989), pp. 203-231.

[6] C. M. Ringel, *Tame algebras and integral quadratic forms*, Springer LMN 1099 (1984).

[7] C. M. Ringel - H. Tachikawa, *QF-3 rings*, J. Reine Angew. Math., **272** (1974), pp. 49-72.

[8] S. O. Smalø, *Torsion theories and tilting modules*, Bull. London Math. Soc., **16** (1984), pp. 518-522.

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