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On the interior differentiability of weak solutions of parabolic systems with quadratic growth nonlinearities

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On the Interior Differentiability of Weak Solutions of Parabolic Systems with Quadratic Growth Nonlinearities.

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1. Introduction. Statement of the result.

Let $\Omega \subset \mathbb{R}^N$ $(n \geqq 2)$ be a bounded open set, let $0 < T < +\infty$ and $Q = \Omega \times (0, T)$.

By $W_p^m(\Omega)$ $(m = 1, 2, ..., 1 \leqq p < +\infty)$ we denote the usual Sobolev space (i.e. the subspace of those functions in $L^p(\Omega)$ with generalized partial derivatives up to order m in $L^p(\Omega)$). Given $0 < \theta < 1$, let $W_2^\theta(\Omega)$ be the subspace of all functions $u \in L^2(\Omega)$ such that

$$|u|_{\theta,\Omega}^2 := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\theta}}\, dx\, dy < +\infty .$$

Define

$$W_2^{1+\theta}(\Omega) := \{u \in W_2^1(\Omega) \colon u_{x_\alpha} \in W_2^\theta(\Omega) \ (\alpha = 1, ..., n)\} \ (^1) .$$

Next, let $1 \leqq p < +\infty$ and $-\infty < a < b < +\infty$, and let X be

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(1) Throughout $u_{x_\alpha} = \partial u / \partial x_\alpha$ denotes the classical or generalized derivative, respectively, of u with respect to the space variable x_α $(\alpha = 1, ..., n)$. Analogously, $u_t = \partial u / \partial t$.

normed vector space. Then $L^p(a, b; X)$ denotes the vector space of all (classes of equivalent) Bochner-measurable functions $u\colon (a, b) \to X$ such that

$$\|u\|^p_{L^p(a,b;X)} := \int_a^b \|u(t)\|^p_X \, dt < +\infty$$

(cf. e.g. [8]). In all that follows, we shall identify the spaces $L^p(0, T; L^p(\Omega))$ and $L^p(Q)$ (²).

Let denote

$$W_2^{1,0}(Q) := \{u \in L^2(Q)\colon u_{x_\alpha} \in L^2(Q) \ (\alpha = 1, \dots, n)\} .$$

Finally, set

$$W_p^m(\Omega; \mathbb{R}^N) := [W_p^m(\Omega)]^N , \qquad W_2^{1,0}(Q; \mathbb{R}^N) := [W_2^{1,0}(Q)]^N \text{ etc.}$$

We consider the following system of partial differential equations:

$$\frac{\partial u^i}{\partial t} - \frac{\partial}{\partial x_\alpha} A_i^\alpha(x, t, u, \nabla u) = B_i(x, t, u, \nabla u) \text{ in } Q \ (^3) \quad (i = 1, \dots, N) \tag{1.1}$$

where $u = \{u^1, \dots, u^N\}$, $\nabla u = \{u^i_{x_\alpha}\}$ (= matrix of spatial derivatives of u). For the time being, suppose that A_i^α and B_i are Carathéodory functions (⁴) on $Q \times \mathbb{R}^N \times \mathbb{R}^N$ satisfying

$$\left\{ \begin{array}{l} \forall M > 0 \ \exists c = c(M) > 0\colon \\ |A_i^\alpha(x, t, u, \xi)| \leqq c(1 + |\xi|) , \quad |B_i(x, t, u, \xi)| \leqq c(1 + |\xi|^2) \\ \forall (x, t, u, \xi) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN} \qquad \text{with } |u| \leqq M \end{array} \right. \tag{1.2}$$

$(\alpha = 1, \dots, n;\ i = 1, \dots, N)$.

(²) This identification is justified by virtue of the linear isometry

$$L^p(0, T; L^p(\Omega)) \cong L^p(Q) .$$

(³) Without further reference, throughout repeated Greek (resp. Latin) indices imply summation over $1, \dots, n$ (resp. $1, \dots, N$). By $|\ |$ we denote the Euclidean norm in $\mathbb{R}^k$.

(⁴) A real function f on $Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$ is called a Carathéodory function if $(x, t) \mapsto f(x, t, u, \xi)$ is measurable on Q for each $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, and $(u, \xi) \mapsto f(x, t, u, \xi)$ is continuous on $\mathbb{R}^N \times \mathbb{R}^{nN}$ for almost all $(x, t) \in Q$.

Under these hypotheses, a vector function $u \in W_2^{1,2}(Q;\mathbb{R}^N) \cap \cap L^\infty(Q;\mathbb{R}^N)$ *is called a bounded weak solution of* (1.1) (regardless of whether or not u satisfies any boundary or initial condition) *if*

$$(1.3)\qquad -\int_Q u^i \varphi_t^i\,dx\,dt + \int_Q A_i^\alpha(x,t,u,\nabla u)\varphi_{x_\alpha}^i\,dx\,dt = \int_Q B_i(x,t,u,\nabla u)\varphi^i\,dx\,dt$$

for all $\varphi \in W_2^1(Q;\mathbb{R}^N) \cap L^\infty(Q;\mathbb{R}^N)$ *with* $\operatorname{supp}(\varphi) \subset Q$.

The aim of the present paper is to study the interior differentiability of bounded weak solutions of (1.1). Counter examples indicate that bounded weak solutions of elliptic systems need not have locally square integrable second derivatives. It is known, however, that Hölder continuous weak solutions of strongly elliptic systems with quadratic growth nonlinearities possess second derivatives in $L^2_{\mathrm{loc}}(\Omega)$ (cf. [2]). In what follows, we therefore confine ourselves to the study of Hölder continuous weak solutions of (1.1). This is also motivated by the fact that any bounded weak solution of (1.1) is partially Hölder continuous in Q (i.e. Hölder continuous in an open set $Q_0 \subset Q$ with meas $(Q \setminus Q_0) = 0$) provided that the above conditions on A_i^α and B_i are fulfilled, that $\partial A_i^\alpha / \partial \xi_\beta^j$ are uniformly bounded Carathéodory functions on $Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$ satisfying (1.6) below, and that the solution under consideration possesses a certain regularity property with respect to t and meets the well-known smallness condition on its L^∞-norm (cf. [6]). An analogous result for a special case of A_i^α has been presented in [4].

In order to state our main result we sharpen the conditions on A_i^α and B_i as follows:

$$(1.4)\qquad A_i^\alpha \in C^1(Q \times \mathbb{R}^N \times \mathbb{R}^{nN});$$

$$(1.5)\qquad B_i,\ \frac{\partial B_i}{\partial t},\ \frac{\partial B_i}{\partial u^j},\ \frac{\partial B_i}{\partial \xi_\beta^j} \in C(Q \times \mathbb{R}^N \times \mathbb{R}^{nN});$$

$$(1.6)\qquad \begin{cases} \dfrac{\partial A_i^\alpha}{\partial \xi_\beta^j}(x,t,u,\xi)\eta_\alpha^i \eta_\beta^j \geqq \nu|\eta|^2 \\[2ex] \forall (x,t,u,\xi) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN}, \qquad \forall \eta \in \mathbb{R}^{nN} \quad (\nu = \text{const} > 0); \end{cases}$$

for each $M > 0$ there exist constants $c_k = c_k(M) > 0$ $(k = 1, 2)$ such that

$$|A_i^\alpha|, \left|\frac{\partial A_i^\alpha}{\partial x_\beta}\right|, \left|\frac{\partial A_i^\alpha}{\partial t}\right|, \left|\frac{\partial A_i^\alpha}{\partial u^j}\right| \leqq c_1(1+|\xi|); \tag{1.7}$$

$$\left|\frac{\partial A_i^\alpha}{\partial \xi_\beta^j}\right| \leqq c_1; \tag{1.8}$$

$$|B_i|, \left|\frac{\partial B_i}{\partial t}\right|, \left|\frac{\partial B_i}{\partial u^j}\right| \leqq c_2(1+|\xi|^2); \tag{1.9}$$

$$\left|\frac{\partial B_i}{\partial \xi_\beta^j}\right| \leqq c_2(1+|\xi|) \tag{1.10}$$

for all $(x, t, u, \xi) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$ with $|u| \leqq M$ $(\alpha, \beta = 1, \dots, n;\ i, j = 1, \dots, N)$.

Then we have the following

THEOREM. *Let* (1.4)-(1.10) *be satisfied. Let* $u \in W_2^{1,0}(Q; \mathbb{R}^N)$ *be a weak solution of* (1.1) *such that*

$$u \in C^\gamma(\bar{Q}; \mathbb{R}^N)\ (\tfrac{1}{2} < \gamma < 1)\ ({}^5)\,. \tag{1.11}$$

Then:

$$\nabla u \in L^4_{\mathrm{loc}}(Q; \mathbb{R}^{nN})\,. \tag{1.12}$$

COROLLARY. *Let the assumptions of the Theorem be fulfilled. Then*:

$$u_{x_\alpha x_\beta}, u_t \in L^2_{\mathrm{loc}}(Q; \mathbb{R}^N) \qquad (\alpha, \beta = 1, \dots, n)\,, \tag{1.13}$$

$$\nabla u \textit{ is partially Hölder continuous in } Q\,. \tag{1.14}$$

We note that once we have (1.12), statement (1.13) can be easily deduced from (1.3) by the aid of the nowadays classical method of difference quotient (cf. step 2° of the proof below); we therefore dispense with further details. Then (1.14) follows immediately from [5] when taking into account (1.11) and (1.13). ■

(5) That is, there exists a constant $C > 0$ such that

$$|u(x,t) - u(y,s)| \leqq C(|x-y|^\gamma + |t-s|^{\gamma/2})$$

for all $(x, t), (y, s) \in \bar{Q}$.

REMARKS. 1) The above theorem obviously continues to hold when A_i^α, B_i and their derivatives occuring in (1.4)-(1.10), are Carathéodory functions on $Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$. Moreover, (1.12) remains true for any bounded weak solution $u \in W_2^{1,0}(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N)$ such that $u \in C^\gamma(\bar{Q}'; \mathbb{R}^N)$ $(\frac{1}{2} < \gamma < 1)$ for each subcylinder $Q' \subset \bar{Q}' \subset Q$. This can be readily seen by minor technical modifications of our reasoning below.

2) Let (1.4), (1.6)-(1.8) be fulfilled, and let B_i be a Carathéodory function on $Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$ satisfying (1.2) only. Then [3; Theorem 3.III] implies that any Hölder continuous weak solution of (1.1) is in $L^{2(1+\theta)}(t', T; W_p^1(\Omega'; \mathbb{R}^N))$ for all $0 < \theta < 1$, $0 < t' < T$ and $\Omega' \subset \bar{\Omega}' \subset \Omega$, where $2 < p < (2(1+\theta)n)/(n - 2\theta\gamma)$ ($\gamma =$ Hölder exponent of the solution under consideration) (cf. step 1° of the proof below). This result does not, however, give (1.12).

3) In [1], fractional differentiability properties with respect to t for weak solutions of parabolic systems have been established (cf. (2.18) below). The method of proof in that paper rests upon results for evolution equations in Hilbert spaces and seems not to work for (1.1) under our assumptions on A_i^α and B_i (cf. e.g. [7] and the literature therein). ■

2. Proof of the Theorem.

We divide the proof into five steps.

1° Let Ω' be an open set such that $\bar{\Omega}' \subset \Omega$. Let $0 < t' < T$ be arbitrary. From [3] we obtain

$$(2.1) \qquad \begin{cases} u \in L^2(t', T; W_2^{1+\theta}(\Omega'; \mathbb{R}^N)) , \\ \displaystyle\int_{t'}^{T} |\nabla u|^2_{\theta,\Omega'} dt \leq c\Big(1 + \int_Q |\nabla u|^2 dx\, dt\Big) \end{cases}$$

for all $0 < \theta < 1$, where c depends on ν, c_1, c_2 (cf. (1.6)-(1.10)), $\|u\|_{C^\gamma(\bar{Q}; \mathbb{R}^N)}$ as well as on θ, t' and dist $(\Omega', \partial\Omega)$ ($c \to +\infty$ as $\theta \to 1$, $t' \to 0$ or dist $(\Omega', \partial\Omega) \to 0$, respectively).

Let $0 < \theta < 1$, and fix $2 < p < (2(1+\theta)n)/(n - 2\theta\gamma)$. We have:

(2.2) $t \mapsto \nabla u(\cdot, t)$ *is Bochner measurable from* (t', T) *into* $L^p(\Omega'; \mathbf{R}^{nN})$,

$$\int_{t'}^{T} \|\nabla u\|^{2(1+\theta)}_{L^p(\Omega';\mathbf{R}^{nN})} dt \leqq c\Big(1 + \int_Q |\nabla u|^2 dx\, dt\Big) \tag{2.3}$$

with c depending on the same parameters as the constant in (2.1) and additionally on N, meas Ω and T.

To begin with, we fix an open set Ω'' such that $\bar{\Omega}' \subset \Omega'' \subset \bar{\Omega}'' \subset \Omega$. From [2; Theorem 2.I] we get

$$\|\nabla u\|_{L^p(\Omega';\mathbf{R}^n)} \leqq c\big(\|v\|_{W_2^{1+\theta}(\Omega'')} + \|v\|_{C^\gamma(\bar{\Omega}'')}\big)\,. \tag{2.4}$$

$$\|\nabla v\|^{2(1+\theta)}_{L^p(\Omega';\mathbf{R}^n)} \leqq c\{(M^{2\theta} + \|v\|^{2\theta}_{C^\gamma(\bar{\Omega}'')})\|v\|^2_{W_2^{1+\theta}(\Omega'')} + M^{2(1+\theta)}\} \tag{2.5}$$

for all $v \in W_2^{1+\theta}(\Omega'') \cap C^\gamma(\bar{\Omega}'')$. Here c depends on n, θ, p and dist $(\Omega', \partial\Omega)$, while M is a bound for v on $\bar{\Omega}''$: $|v(x)| \leqq M = \text{const}$ for all $x \in \Omega''$ (cf. also [3; p. 756]) ([6]).

Indeed, let K be any open cube in $\mathbf{R}^n$. From [2; Theorem 2.I] one easily derives estimates of the type (2.4) and (2.5) with K in place of Ω' and Ω''. Then we consider a finite number of mutually disjoint, open cubes $\{K_j\}$ $(j = 1, \ldots, m)$ in $\mathbf{R}^n$ such that $\bar{\Omega}' \subset \bigcup_{j=1}^{m} K_j \subset \Omega''$ (m depending on dist $(\Omega', \partial\Omega'')$ and employ the estimates just derived with $K = K_j$ $(j = 1, \ldots, m)$ to obtain (2.4) and (2.5).

We are now going to apply (2.4) and (2.5) to each component of the mollification of u. To this end, let denote

$$\tau(t) := \begin{cases} \exp\Big(-\dfrac{1}{1-t}\Big) & \text{if } 0 \leqq t < 1\,, \\ 0 & \text{if } t \geqq 1\,. \end{cases}$$

Define

$$\sigma(t) := a_1 \tau(t^2)\,, \qquad \omega(x) := a_n \tau(|x|^2)$$

([6]) Note that (2.4) and (2.5) are true for any $0 < \gamma < 1$.

$(t \in \mathbf{R}^1,\ x \in \mathbf{R}^n\ (n \geqq 2))$ where a_1 and a_n are determined by

$$\int_{\mathbf{R}^1} \sigma\, dt = \int_{\mathbf{R}^n} \omega\, dx = 1\,.$$

Next, given $h > 0$, set

$$\sigma_h(t) := \frac{1}{h}\, \sigma\left(\frac{t}{h}\right), \qquad \omega_h(x) := \frac{1}{h^n}\, \omega\left(\frac{x}{h}\right)$$

$(t \in \mathbf{R}^1,\ x \in \mathbf{R}^n)$.

Finally, let Ω^* be an open set such that $\bar{\Omega}'' \subset \Omega^* \subset \bar{\Omega}^* \subset \Omega$.

We extend u by zero onto $(\Omega \times (-\infty, 0)) \cup (\Omega \times (T, +\infty))$ and denote this extension again by u. Consider

$$u_h(x, t) := \int_{\mathbf{R}^1} \sigma_h(t-s)\Big(\int_{\mathbf{R}^n} \omega_h(x-y)\, u(y, s)\, dy\Big)\, ds$$

for any $x \in \Omega''$, $t \in (t', T)$ and $0 < h < \operatorname{dist}(\Omega'', \partial\Omega^*)$. Clearly, $u_h \in C^\infty(\Omega'' \times (t', T))$. From (2.4) we get

$$\Big(\int_{\Omega'} |\nabla u_h(x, t)|^p\, dx\Big)^{2/p} \leqq c\big(\|u_h(t)\|^2_{W_2^{1+\theta}(\Omega'';\mathbf{R}^N)} + \|u_h(t)\|^2_{C^\gamma(\bar{\Omega}'';\mathbf{R}^N)}\big)\,. \tag{2.6}$$

By standard calculations,

$$\int_{\Omega''} |u_h(x, t)|^2\, dx \leqq \int_{\mathbf{R}^1} \sigma_h(t-s)\Big(\int_{\Omega^*} |u(y, s)|^2\, dy\Big)\, ds \leqq$$

$$\leqq \operatorname{meas} \Omega^* \|u\|_{C(\bar{\Omega}^* \times [t', T];\, \mathbf{R}^N)}\,, \tag{2.7}$$

$$\int_{\Omega''} |\nabla u_h(x, t)|^2\, dx \leqq \int_{\mathbf{R}^1} \sigma_h(t-s)\Big(\int_{\Omega^*} |\nabla u(y, s)|^2\, dy\Big)\, ds \tag{2.8}$$

and

$$\int_{\Omega''}\int_{\Omega''} \frac{|\nabla u_h(x, t) - \nabla u_h(y, t)|^2}{|x-y|^{n+2\theta}}\, dx\, dy \leqq$$

$$\leqq \frac{1}{h^n} \int_{\{z:|z|<h\}} \omega\left(\frac{z}{h}\right)\Big\{\int_{\mathbf{R}^1} \sigma_h(t-s)\, \times$$

$$\times \Big(\int_{\Omega''}\int_{\Omega''} \frac{|\nabla u(x-z, s) - \nabla u(y-z, s)|^2}{|x-y|^{n+2\theta}}\, dx\, dy\Big)\, ds\Big\}\, dz \leqq$$

$$\leqq \int_{\mathbf{R}^1} \sigma_h(t-s)\Big(\int_{\Omega^*}\int_{\Omega^*} \frac{|\nabla u(\xi, s) - \nabla u(\xi', s)|^2}{|\xi-\xi'|^{n+2\theta}}\, d\xi\, d\xi'\Big)\, ds \tag{2.9}$$

for all $t \in (t', T)$ and $0 < h < \operatorname{dist}(\Omega'', \partial\Omega^*)$.

Further, it is easily seen that

$$\|u_h(t)\|_{C^\gamma(\bar{\Omega}''; \mathbf{R}^N)} \leqq \|u\|_{C^\gamma(\bar{\Omega}^* \times [t', T]; \mathbf{R}^N)} \tag{2.10}$$

for all $t \in (t', T)$ and $0 < h < \operatorname{dist}(\Omega'', \partial\Omega^*)$.

Thus, inserting (2.7)-(2.10) into (2.6), integrating over (t', T) and using (2.1) (with Ω^* in place of Ω') we obtain

$$\int_{t'}^{T} \Big(\int_{\Omega'} |\nabla u_h(x, t)|^p \, dx \Big)^{2/p} dt \leqq c \Big(\|u\|^2_{C^\gamma(\bar{Q}; \mathbf{R}^N)} + \int_{t'}^{T} \int_{\Omega^*} |\nabla u|^2 \, dx \, dt + \int_{t'}^{T} |\nabla u|^2_{\theta, \Omega^*} dt \Big) \leqq$$
$$\leqq c \Big(1 + \|u\|^2_{C^\gamma(\bar{Q}; \mathbf{R}^N)} + \int_{Q} |\nabla u|^2 \, dx \, dt \Big)$$

for all $0 < h < \operatorname{dist}(\Omega'', \partial\Omega^*)$.

Observing that $u_h \to u$ strongly in $L^p(\Omega'' \times (t', T); \mathbf{R}^N)$ as $h \to 0$, from the latter estimate we infer that

$$\int_{t'}^{T} \Big(\int_{\Omega'} |\nabla u|^p \, dx \Big)^{2/p} dt < +\infty .$$

Whence (2.2) (cf. footnote 2).

Finally, taking into account (2.1) (with Ω'' in place of Ω'), estimate (2.3) is readily deduced from (2.5). ■

2° Let $0 < t_1 < T$. Let $\varphi \in W_2^{1,0}(Q; \mathbf{R}^N) \cap L^\infty(Q; \mathbf{R}^N)$ have its support in $\Omega \times (0, t_1)$. We extent φ by zero onto $(\Omega \times (-\infty, 0)) \cup \cup (\Omega \times (T, +\infty))$ and denote this extension again by φ. Define

$$\varphi_{\bar{\lambda}}(x, t) := \frac{1}{\lambda} \int_{t-\lambda}^{t} \varphi(x, s) \, ds \quad \text{for a.a. } (x, t) \in Q , \quad \forall \lambda > 0 .$$

It is easy to verify that $\varphi_{\bar{\lambda}}$ possesses the generalized derivatives

$$\varphi_{\bar{\lambda} x_\alpha}(x, t) = (\varphi_{x_\alpha})_{\bar{\lambda}}(x, t) ,$$
$$\varphi_{\bar{\lambda} t}(x, t) = \frac{1}{\lambda} (\varphi(x, t) - \varphi(x, t - \lambda))$$

for a.a. $(x, t) \in Q$ $(\alpha = 1, \ldots, n;\ \lambda > 0)$. On the other hand, given $f \in L^p(Q)$ $(1 \leqq p \leqq +\infty)$, we get by a straightforward application of Fubini's theorem

$$\int_Q f(x, t)\varphi_{\bar\lambda}(x, t)\, dx\, dt = \frac{1}{\lambda}\int_0^{t_1}\int_\Omega \left(\int_t^{t+\lambda} f(x, s)\, ds\right)\varphi(x, t)\, dx\, dt$$

for all $0 < \lambda < T - t_1$.

The function $\varphi_{\bar\lambda}$ is admissible in (1.3). Letting denote

$$f_\lambda(x, t) := \frac{1}{\lambda}\int_t^{t+\lambda} f(x, s)\, ds \qquad (\lambda > 0)$$

we obtain

$$\int_0^{t_1}\int_\Omega u^i_{\lambda t}\varphi^i\, dx\, dt + \int_0^{t_1}\int_\Omega (A^\alpha_i)_\lambda \varphi^i_{x_\alpha}\, dx\, dt = \int_0^{t_1}\int_\Omega (B_i)_\lambda \varphi^i\, dx\, dt \tag{2.11}$$

for all $0 < \lambda < T - t_1$.

Next, let Ω' be any open set such that $\bar\Omega' \subset \Omega$. Without loss of generality, we may assume that $\partial\Omega'$ is sufficiently smooth (so that the Sobolev imbedding theorem applies to Ω'). Fix any integer $m > n/2$. Let $\psi \in \mathring{W}^m_2(\Omega'; \mathbb{R}^N)$ [7] be arbitrary. We extend ψ by zero onto $\Omega \setminus \Omega'$ and maintain the notation for the extension. Finally, let $\eta \in C^1((0, t_1))$ with $\operatorname{supp}(\eta) \subset (0, t_1)$. Inserting $\varphi(x, t) := \psi(x)\eta(t)$ $((x, t) \in Q)$ into (2.11) we get by a standard argument

$$\int_{\Omega'} u^i_{\lambda t}(x, t)\psi^i(x)\, dx + \int_{\Omega'} (A^\alpha_i)_\lambda(x, t)\psi^i_{x_\alpha}(x)\, dx = \int_{\Omega'} (B_i)_\lambda(x, t)\psi^i(x)\, dx \tag{2.12}$$

for a.a. $t \in (0, t_1)$ $(0 < \lambda < T - t_1)$, where the exceptional set possibly depends on ψ. However, the space $\mathring{W}^m_2(\Omega'; \mathbb{R}^N)$ being separable, (2.12) is in fact true for any $\psi \in \mathring{W}^m_2(\Omega'; \mathbb{R}^N)$ and independently for a.a. $t \in (0, t_1)$.

[7] $\mathring{W}^m_2(\Omega') := \left\{ v \in W^m_2(\Omega') \colon v = \dfrac{\partial v}{\partial n} = \ldots = \dfrac{\partial^{m-1} v}{\partial n^{m-1}} = 0 \text{ a.e. on } \partial\Omega' \right\}$
here n denotes the unit outward normal along $\partial\Omega'$.

Now, by an approximation argument (via mollification; cf. the preceding step) it is readily seen that (2.12) continues to hold for any $\psi \in W_2^1(\Omega; \mathbf{R}^N) \cap L^\infty(\Omega; \mathbf{R}^N)$ with $\operatorname{supp}(\psi) \subset \Omega'$. ■

3° Let Ω' and Ω'' be open sets such that $\bar{\Omega}' \subset \Omega'' \subset \bar{\Omega}'' \subset \Omega$, and let $\zeta \in C^\infty(\mathbf{R}^n)$ be a cut-off function for Ω'', i.e. $\zeta \equiv 0$ in $\mathbf{R}^n \setminus \Omega''$, $0 \leqq \zeta \leqq 1$ in $\Omega'' \setminus \Omega'$ and $\zeta \equiv 1$ in Ω'. Let $0 < t' < t_1 < T$ be arbitrary, let $|h| < \min\{t', T - t_1\}$ and fix t_2 with $t_1 + |h| < t_2 < T$.

Let denote

$$\Delta_h f(x, t) := f(x, t + h) - f(x, t) .$$

We consider (2.12) with t_2 in place of t_1 $(0 < \lambda < \min\{t_2 - t_1 - |h|, T - t_2\}$, form the difference Δ_h under the integral sign for a.a. $t \in (t', t_1)$ and insert $\psi(x) = (\Delta_h u_\lambda(x, t))\zeta^2(x)$ $((x, t) \in \Omega \times (t', t_1))$ therein. Integration over (t', t_1) gives

$$(2.13) \qquad \frac{1}{2}\int_\Omega |\Delta_h u_\lambda(x, t_1)|^2 \zeta^2(x)\, dx + \int_{t'}^{t_1}\int_\Omega \big(\Delta_h (A_i^\alpha)_\lambda\big)(\Delta_h u^i_{\lambda x_\alpha})\zeta^2\, dx\, dt =$$
$$= -2\int_{t'}^{t_1}\int_\Omega \big(\Delta_h (A_i^\alpha)_\lambda\big)(\Delta_h u^i_\lambda)\zeta\zeta_{x_\alpha}\, dx\, dt +$$
$$+\int_{t'}^{t_1}\int_\Omega \big(\Delta_h (B_i)_\lambda\big)(\Delta_h u^i_\lambda)\zeta^2\, dx\, dt + \frac{1}{2}\int_\Omega |\Delta_h u_\lambda(x, t')|^2\zeta^2(x)\, dx .$$

To proceed, we note that for any $f \in L^p(Q)$ $(1 \leqq p < +\infty)$ there holds $f_\lambda \to f$ strongly in $L^p(\Omega \times (0, t_2))$ as $\lambda \to 0$. On the other hand, a simple calculation shows

$$|\Delta_h u_\lambda(x, t) - \Delta_h u(x, t)| \leqq 2\lambda^{\gamma/2}\|u\|_{C^\gamma(\bar{Q}; \mathbf{R}^N)}$$

for all $(x, t) \in \Omega \times (t', t_1)$ $\big(0 < \lambda < \min\{t_2 - t_1 - |h|, T - t_2\}\big)$. Thus, neglecting the first term on the left of (2.13) and letting tend $\lambda \to 0$ we obtain

$$(2.14) \qquad \nu\int_{t'}^{t_1}\int_\Omega |\Delta_h \nabla u|^2\zeta^2\, dx\, dt \leqq -\int_{t'}^{t_1}\int_\Omega \left(\int_1^0 \frac{\partial A_i^\alpha}{\partial t}(\ldots)\, ds\right)(\Delta_h u^j_{x_\alpha})\zeta^2\, dx\, dt\, h -$$

$$-\int_{t'}^{t_1}\int_{\Omega}\left(\int_0^1 \frac{\partial A_i^\alpha}{\partial u^j}(\ldots)\,ds\right)(\Delta_h u^i_{x_\alpha})(\Delta_h u^j)\zeta^2\,dx\,dt-$$

$$-2\int_{t'}^{t_1}\int_{\Omega}(\Delta_h A_i^\alpha)(\Delta_h u^i)\zeta\zeta_{x_\alpha}\,dx\,dt+\int_{t'}^{t_1}\int_{\Omega}(\Delta_h B_i)(\Delta_h u^i)\zeta^2\,dx\,dt+$$

$$+\frac{1}{2}\int_{\Omega}|\Delta_h u(x,t')|^2\,dx=$$

$$=\sum_{k=1}^{5} I_k$$

for all $|h|<\min\{t', T-t_1\}$; here we have used the ellipticity condition (1.6) as well as the differentiability properties of A_i^α:

$$\Delta_h A_i^\alpha=\int_0^1 \frac{\partial A_i^\alpha}{\partial t}(\ldots)\,ds\,h+\int_0^1 \frac{\partial A_i^\alpha}{\partial u^j}(\ldots)\,ds\,\Delta_h u^j+\int_0^1 \frac{\partial A_i^\alpha}{\partial \xi^j_\beta}(\ldots)\,ds\,\Delta_h u^j_{x_\beta}$$

for a.a. $(x,t)\in\Omega\times(t',t_1)$, where the dots ... represent the variables

$$\Big(x,\,t+sh,\,u(x,t)+s\big(u(x,t+h)-u(x,t)\big),\,\nabla u(x,t)+$$
$$+\,s\big(\nabla u(x,t+h)-\nabla u(x,t)\big)\Big)\qquad (s\in(0,1))\,.$$

We evaluate the integrals on the right hand side of (2.14). By (1.7),

$$I_1\leq\frac{\nu}{5}\int_{t'}^{t_1}\int_{\Omega}|\Delta_h\nabla u|^2\zeta^2\,dx\,dt+c\int_Q\big(1+|\nabla u|^2\big)\,dx\,dt\,h^2$$

($c=$ const depending on ν and c_1 only). To evaluate I_2 we use again (1.7) and employ the Hölder continuity of u to obtain

$$I_2\leq\frac{\nu}{5}\int_{t'}^{t_1}\int_{\Omega}|\Delta_h\nabla u|^2\zeta^2\,dx\,dt+c\int_Q\big(1+|\nabla u|^2\big)\,dx\,dt\,|h|^\gamma\,.$$

where the constant c depends on ν, c_1 and $\|u\|_{C^\gamma(\bar{Q};\mathbf{R}^N)}$. The evaluation of I_3 follows the same lines when taking into account once more the above representation of $\Delta_h A_i^\alpha$. We find

$$I_3 \leqq \frac{\nu}{5} \int_{t'}^{t_1} \int_{\Omega} |\Delta_h \nabla u|^2 \zeta^2 \, dx \, dt + \\ + c\left(\max_{\mathbf{R}^n} |\nabla\zeta|^2 + \max_{\mathbf{R}^n} |\nabla\zeta| \int_Q (1 + |\nabla u|) \, dx \, dt\right) |h|^\gamma .$$

the constant c being dependend on ν, c_1, meas Q and $\|u\|_{C^\gamma(\bar{Q};\mathbf{R}^N)}$ (without loss of generality, we may assume that $|h| \leqq 1$).

Next, making use of the differentiability of B_i we represent $\Delta_h B_i$ in terms of three integrals over the interval $(0, 1)$ (cf. the representation of $\Delta_h A_i^\alpha$ above). Combining (1.9), (1.10) and the Hölder continuity of u one obtains

$$I_4 \leqq \frac{\nu}{5} \int_{t'}^{t_1} \int_{\Omega} |\Delta_h \nabla u|^2 \zeta^2 \, dx \, dt + c \int_{t'}^{t_1} \int_{\Omega} (1 + |\nabla u|^2) \, dx \, dt |h|^\gamma ;$$

here c depends on ν, c_2 and $\|u\|_{C^\gamma(\bar{Q};\mathbf{R}^N)}$ (as above, $|h| \leqq 1$). Finally,

$$I_5 \leqq \tfrac{1}{2} (\text{meas } \Omega) \|u\|^2_{C^\gamma(\bar{Q};\mathbf{R}^N)} |h|^\gamma .$$

Inserting these estimates into (2.14) gives

$$\int_{t'}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \, dx \, dt \leqq c_3\left(1 + \int_Q |\nabla u|^2 \, dx \, dt\right) |h|^\gamma \tag{2.15}$$

for all $|h| < \min\{1, t', T - t_1\}$, where c_3 depends on ν, c_1, c_2, meas Q, $\|u\|_{C^\gamma(\bar{Q};\mathbf{R}^N)}$ and dist $(\Omega', \partial\Omega)$ ($c_3 \to +\infty$ as dist $(\Omega', \partial\Omega) \to 0$).

Let $0 < \varrho < \gamma/2$. Then (2.15) implies

$$\int_{-\delta}^{\delta} \frac{1}{|h|^{1+2\varrho}} \left(\int_{t'}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \, dx \, dt \right) dh \leqq c_4 \left(1 + \int_Q |\nabla u|^2 \, dx \, dt\right) \tag{2.16}$$

for all $0 < \delta \leqq \min\{1, t', T - t_1\}$ $\left(c_4 = \dfrac{2 c_3 \delta^{\gamma - 2\varrho}}{\gamma - 2\varrho}\right)$. ■

4° From (2.16) we deduce the higher integrability of ∇u via its fractional differentiability with respect to t. To this end, let $0 < t_0 < t_1 < T$ be arbitrarily chosen. Define $\delta_0 := \min\{1, t_0, T - t_1\}$. Then there exists an integer $m \geqq 0$ such that

$$t_1 - (m+1)\delta_0 < t_0 \leqq t_1 - m\delta_0 \,. \tag{2.17}$$

Suppose (2.17) is true for $m = 0$, i.e. $t_1 - \delta_0 < t_0$. Obviously, $(t_0, t_1) \subset (t - \delta_0, t + \delta_0)$ for all $t \in (t_0, t_1)$, and $t' = t_0$, $\delta = \delta_0$ are admissible in (2.16). Employing Fubini's theorem and changing variables gives

$$\int_{-\delta_0}^{\delta_0} \frac{1}{|h|^{1+2\varrho}} \left(\int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \, dx \, dt \right) dh \geqq \tag{2.18}$$
$$\geqq \int_{t_0}^{t_1} \int_{t_0}^{t_1} \left(\int_{\Omega'} \frac{|\nabla u(x,s) - \nabla u(x,t)|^2}{|s-t|^{1+2\varrho}} \, dx \right) ds \, dt \,.$$

When (2.17) fails for $m = 0$, we may assume that it is true for $m = 1$, i.e. $t_1 - 2\delta_0 < t_0 \leqq t_1 - \delta_0$. Then $(t_1 - \delta_0, t_1) \subset (t - \delta_0, t + \delta_0)$ for all $t \in (t_1 - \delta_0, t_1)$.

We distinguish two cases. Firstly, if $t_0 = t_1 - \delta_0$ we argue exactly as in the preceding step to obtain (2.18). Secondly, if $t_0 < t_1 - \delta_0$ then $t' = t_1 - \delta_0$ and $\delta = \delta_0 (\leqq \min\{1, t', T - t_1\})$ are admissible in (2.16). Thus, by the same argument as above,

$$\int_{-\delta_0}^{\delta_0} \frac{1}{|h|^{1+2\varrho}} \left(\int_{t_1-\delta_0}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \, dx \, dt \right) dh \geqq$$
$$\geqq \int_{t_1-\delta_0}^{t_1} \int_{t_1-\delta_0}^{t_1} \left(\int_{\Omega'} \frac{|\nabla u(x,s) - \nabla u(x,t)|^2}{|s-t|^{1+2\varrho}} \, dx \right) ds \, dt \,.$$

It remains to derive an estimate over the interval $(t_0, t_1 - \delta_0)$. Set $t' = t_0$ and $\delta_1 = t_1 - t_0 - \delta_0$. We find $(t_0, t_1 - \delta_0) \subset (t - \delta_1, t + \delta_1)$ for all $t \in (t_0, t_0 + \delta_1)$, and $\delta_1 < \delta_0 = \min\{1, t', T - t_1\}$ (for $t_1 - 2\delta_0 < t_0$). Hence t' and δ_1 are admissible in (2.16) and we obtain

$$\int_{-\delta_1}^{\delta_1} \frac{1}{|h|^{1+2\varrho}} \left(\int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \, dx \, dt \right) dh \geqq$$

$$\geqq \int_{t_0}^{t_1-\delta_0}\left(\int_{t-\delta_1}^{t+\delta_1}\int_{\Omega'}\frac{|\nabla u(x,s)-\nabla u(x,t)|^2}{|s-t|^{1+2\varrho}}\,dx\,ds\right)dt \geqq$$

$$\geqq \int_{t_0}^{t_1-\delta_0}\int_{t_0}^{t_1-\delta_0}\left(\int_{\Omega'}\frac{|\nabla u(x,s)-\nabla u(x,t)|^2}{|s-t|^{1+2\varrho}}\,dx\right)ds\,dt\,.$$

Thus, if $t_0 < t_1 - \delta_0$, in place of (2.18) we obtain analogous estimates over the intervals $(t_0, t_1-\delta_0)$ and $(t_1-\delta_0, t_1)$.

The preceding argument can be repeated if (2.17) is true for $m = 2$ etc. More precisely, there exist an integer $1 \leqq k \leqq m+1$ and reals

$$t_0^* = t_0 < t_1^* < t_2^* < \ldots < t_k^* = t_1$$

such that $t_1^* - t_0^* \leqq \delta_0$, $t_{i+1}^* - t_i^* = \delta_0$ $(i = 1, \ldots, m$ if $m \geqq 2)$ and

$$\int_{t_i^*}^{t_{i+1}^*}\int_{t_i^*}^{t_{i+1}^*}\left(\int_{\Omega'}\frac{|\nabla u(x,s)-\nabla u(x,t)|^2}{|s-t|^{1+2\varrho}}\,dx\right)ds\,dt \leqq c_4\left(1+\int_Q|\nabla u|^2\,dx\,dt\right)$$

$(i = 0, 1, \ldots, m)$. Finally, applying the Sobolev imbedding theorem on each interval (t_i^*, t_{i+1}^*) and summing over $i = 0, 1, \ldots, m$ we obtain

$$\left(\int_{t_0}^{t_1}\|\nabla u\|^{2/(1-2\varrho)}_{L^2(\Omega';\mathbf{R}^{nN})}\,dt\right)^{(1-2\varrho)/2} \leqq c\left(1+\int_Q|\nabla u|^2\,dx\,dt\right)^{\frac{1}{2}} \tag{2.19}$$

where the constant c depends on the same parameters as c_4 and additionally on m [8]. ■

5° We fix $4 < p < 4n/(n-2\gamma)$ and $\frac{1}{4} < \varrho < \gamma/2$ [9], and define $\varkappa := (p-4)/(4+2(p-4))$. Then

$$\frac{n(p-2)}{2(n+p\gamma)} < 1\,, \qquad 0 < \varkappa < \frac{1}{2}\,, \qquad \frac{1-4\varkappa\varrho}{1-2\varkappa(1-2\varrho)} < 1\,.$$

[8] Note that $m \to +\infty$ when $t_0 \to 0$ or $t_1 \to T$, respectively.

[9] Recall that $n \geqq 2$ and $\frac{1}{2} < \gamma < 1$.

Next, we choose θ such that

$$\max\left\{\frac{n(p-2)}{2(n+p\gamma)}, \frac{1-4\varkappa\varrho}{1-2\varkappa(1-2\varrho)}\right\} < \theta < 1 .$$

Hence

$$p < \frac{2(1+\theta)n}{n-2\theta\gamma}, \qquad \frac{1-\theta}{2[1-(1+\theta)(1-2\varrho)]} < \varkappa . \tag{2.20}$$

Finally, set

$$\mu := \frac{1-\theta}{2[1-(1+\theta)(1-2\varrho)]}, \qquad q := \frac{2p}{2(1-\mu)+p\mu} .$$

Then, equivalently,

$$\frac{1-\mu}{2(1+\theta)} + \frac{\mu(1-2\varrho)}{2} = \frac{1}{4}, \qquad \frac{1-\mu}{p} + \frac{\mu}{2} = \frac{1}{q} . \tag{2.21}$$

Now, the first inequality in (2.20) guarantees (2.3), while the second one implies $q > 4$. On the other hand, (2.21) permits to apply the classical Riesz-Thorin interpolation theorem to

$$L^{2(1+\theta)}(t_0, t_1; L^p(\Omega'; \mathbf{R}^{nN})) \quad \text{and } L^{2/(1-2\varrho)}(t_0, t_1; L^2(\Omega'; \mathbf{R}^{nN}))$$

at the value μ. Thus, by (2.3) and (2.19),

$$\nabla u \in L^4(t_0, t_1; L^q(\Omega'; \mathbf{R}^{nN})) ,$$

$$\left(\int_{t_0}^{t_1} \|\nabla u\|^4_{L^q(\Omega';\mathbf{R}^{nN})} dt\right)^{\frac14} \leqq$$

$$\leqq c\left(\int_{t_0}^{t_1} \|\nabla u\|^{2(1+\theta)}_{L^p(\Omega';\mathbf{R}^{nN})} dt\right)^{(1-\mu)/(2(1+\theta))} \left(\int_{t_0}^{t_1} \|\nabla u\|^{2/(1-2\varrho)}_{L^2(\Omega';\mathbf{R}^{nN})} dt\right)^{(\mu(1-2\varrho))/2} \leqq$$

$$\leqq c\left(1 + \int_Q |\nabla u|^2 \, dx\, dt\right)^{\frac12}$$

where the latter constant c depends on the same parameters as the constants in (2.3) and (2.19) as well as on μ. ■

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