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# On *-Modules Over Valuation Rings.

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The problem of investigating *-modules over valuation rings was proposed to the author by C. Menini. We recall the definition of *-module, given by D'Este in [3]. Let $R$ be a ring, ${}_RM$ a left $R$-module and ${}_RE$ an injective cogenerator of the category of all $R$-modules; let $S = \mathrm{End}_R(M)$ and $H = \mathrm{Hom}_R({}_RM, {}_RE)$ and denote by $\mathrm{Gen}({}_RM)$ the category of all left $R$-modules generated by ${}_RM$ and by $\mathrm{Cog}({}_SH)$ the category of all left $S$-modules cogenerated by $H$. In this situation, ${}_RM$ is said to be a **-module* if there exists an equivalence of categories

$$\mathrm{Gen}({}_RM) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Cog}({}_SH)$$

such that the functor $F$ is naturally isomorphic to $\mathrm{Hom}_R({}_RM, -)$ and the functor $G$ is naturally isomorphic to $M_S \otimes -$ (we shall write $F \approx \mathrm{Hom}_R({}_RM, -)$, $G \approx M_S \otimes -$).

The main motivation for the study of *-modules is the following result by Menini and Orsatti ([8], Theorem 3.1): let $R$, $S$ be rings; if $\mathcal{G}$ is a full subcategory of $R$-Mod closed under direct sums and factor modules, $\mathcal{D}$ is a full subcategory of $S$-Mod containing ${}_SS$ and closed under submodules, and $\mathcal{G} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is any equivalence with $F$ and $G$ additive functors, then there exists a module ${}_RM$ such that: $S = \mathrm{End}_R({}_RM)$, $\mathcal{G} = \mathrm{Gen}({}_RM)$, $\mathcal{D} = \mathrm{Cog}({}_SH)$ (where ${}_SH$ is as above), $F \approx \mathrm{Hom}_R({}_RM, -)$ and $G \approx M_S \otimes -$.

Recent results on *-modules have been obtained by D'Este [3], D'Este and Happel [4], Colpi [1], Colpi and Menini [2].

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In the present paper we characterize *finitely generated *-modules over a valuation ring R*. Using a theorem by Colpi ([1], Prop. 4.3) and some results in [9] (see also [5], Ch. IX), we prove that a finitely generated module $X$ over a valuation ring $R$ is a *-module if and only if $X \cong (R/A)^n$, for suitable $n \geq 0$ and $A$ ideal of $R$ (Theorem 3). Note that a module of the form $(R/A)^n$ is a *-module for any ring $R$, as a consequence of the above mentioned result by Colpi. Hence our Theorems 3 shows that the class of finitely generated *-modules over a valuation ring is, in a certain sense, as small as possible.

Note that, at present, there are no examples of rings which admit *-modules not finitely generated; Colpi and Menini in [2] proved that *-modules over artinian rings or noetherian domains with Krull dimension one are necessarily finitely generated. The author feels that the same is true for *-modules over valuation rings. Our final Remark 4 gives a contribution in this direction.

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**1.** – In the sequel, $R$ will always denote a *valuation ring*, i.e. a commutative ring, not necessarily a domain, whose ideals are linearly ordered by inclusion; the maximal ideal of $R$ is denoted by $P$. For general terminology and results on modules over valuation rings we refer to the book by Fuchs and Salce [5]; the results we need on finitely generated modules can be found in [9] or in Ch. IX of [5].

In the proof of Theorem 2 we shall need the following facts (see [9] or [5], Ch. IX): let $X$ be a finitely generated $R$-module; then there exists a submodule $B$ of $X$ such that:

i) $B$ is a direct sum of cyclic submodules;

ii) $B$ is pure in $X$;

iii) $B$ is essential in $X$;

such a $B$ is said to be *basic* in $X$; the basic submodules of $X$ are all isomorphic. Moreover, given a basic submodule $B$ of $X$, there exists a *minimal* set of generators $\boldsymbol{x} = \{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$ of $X$ such that:

*a*) $B = \langle x_1, \dots, x_k \rangle = \bigoplus_{i=1}^{k} \langle x_i \rangle$;

b) if $A_j = \mathrm{Ann}\,(x_j + \langle x_1, \dots, x_{j-1}\rangle)$ for all $j > k$, we have $A_{k+1} \leqslant \leqslant A_{k+2} \leqslant \dots \leqslant A_n$;

c) for all $r \in A_{k+1}$ we have the relation

$$(1) \qquad rx_{k+1} = r \sum_{i=1}^{k} a_i^r x_i\,, \qquad \text{for suitable } \textit{units } a_i^r \in R\,.$$

The construction of $\boldsymbol{x}$ needs some explanation: we start with $B = \bigoplus_{i=1}^{k} \langle x_i \rangle$ basic in $X$ and consider $X/B$; if $\{x_{k+1} + B, \dots, x_n + B\}$ is a minimal set of generators of $X/B$, from the purity of $B$ it follows that $\boldsymbol{x} = \{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$ is a minimal set of generators of $X$; in view of Lemma 1.1 of [9], we can permute the indexes $k + 1, \dots, n$ to obtain property b). Since $B$ is pure in $X$, certainly, for all $r \in A_{k+1}$, the relation (1) holds for suitable elements $a_i^r \in R$, not necessarily units. However, if $r \in \mathrm{Ann}\, x_i$ for some $i \leqslant k$, obviously we can replace $a_i^r$ with 1; moreover, if there exist $i \leqslant k$ and $s \in A_{k+1} \setminus \mathrm{Ann}\, x_i$ such that $a_i^s \in P$, then for all $r \in A_{k+1} \setminus \mathrm{Ann}\, x_i$ we have $a_i^r \in P$: in fact, if $r$ divides $s$, from (1) we get $s(a_i^r - a_i^s)x_i = 0$, hence $a_i^s \in P$ implies $a_i^r \in P$; analogously, if $s$ divides $r$, $r(a_i^r - a_i^s)x_i = 0$ implies $a_i^r \in P$. Let now $F = \{i \leqslant k\colon a_i^r \in P \text{ for all } r \in A_{k+1} \setminus \mathrm{Ann}\, x_i\}$; if we replace $x_{k+1}$ with $x'_{k+1} = x_{k+1} + \sum_{i \in F} x_i$, we obtain that

$$\mathrm{Ann}\,(x'_{k+1} + B) = \mathrm{Ann}\,(x_{k+1} + B) = A_{k+1}\,,$$

$$\boldsymbol{x}' = \{x_1, \dots, x_k, x'_{k+1}, \dots, x_n\}$$

is a minimal set of generators of $X$, and (1) becomes

$$(1') \qquad rx'_{k+1} = r \sum_{i=1}^{k} b_i^r x_i \qquad \text{for all } r \in A_{k+1}$$

where $b_i^r = a_i^r$ if $i \notin F$ and $b_i^r = 1 + a_i^r$ if $i \in F$, so that $b_i^r$ is a unit for all $i \leqslant k$ and for all $r \in A_{k+1}$. We conclude that there exists a minimal set of generators $\boldsymbol{x}$ of $X$ which satisfies properties a), b), c), as desired.

Let us now recall Colpi's result (Prop. 4.3 of [1]).

THEOREM 1 (Colpi). *Let $R$ be a ring, ${}_RM$ a left $R$-module. Then ${}_RM$ is a *-module if and only if the following conditions are satisfied:*

i) *${}_RM$ is self-small;*

ii) *for each exact sequence*

$$0 \to L \to N \to N/L \to 0$$

*where $N$ is an object of* Gen $({}_RM)$, *the sequence*

$$0 \to \mathrm{Hom}_R(M, L) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N/L) \to 0$$

*is exact if and only if $L \in \mathrm{Gen}({}_RM)$.* ///

We can now prove our main result.

THEOREM 2. *Let $R$ be a valuation ring, let $X$ be a finitely generated $R$-module and let $\pi\colon X \to X/PX$ be the canonical homomorphism. If the map $\varphi\colon \mathrm{End}\, X \to \mathrm{Hom}_R(X, X/XP)$, $\varphi\colon f \mapsto \pi \circ f$ is surjective, then $X \cong (R/A)^n$ for suitable $n \geqslant 0$ and $A$ ideal of $R$.*

PROOF. In the following we assume $X \neq (R/A)^0 = \{0\}$, otherwise all is trivial. First of all, let us prove that $X$ is a direct sum of cyclic submodules. Let $B$ be a basic submodule of $X$; it is enough to verify that $B = X$. By contradiction, suppose that $B < X$; let

$$\boldsymbol{x} = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$$

be a minimal set of generators of $X$ which satisfies conditions *a*), *b*), *c*) above. Note that, since $B < X$, we have $k < n$, hence condition *c*) and the relation (1) are not trivially satisfied. For all $j \leqslant n$, let $\bar{x}_j = x_j + PX$; we have $X/PX = \bigoplus_{j=1}^{n} \langle \bar{x}_j \rangle$. Let us now consider the homomorphism $g\colon X \to X/PX$ defined extending by linearity the assignments

$$g\colon x_j \mapsto 0 \quad \text{if } j \neq k+1; \qquad g\colon x_{k+1} \mapsto \bar{x}_{k+1}\,.$$

By hypothesis, there exists $f \in \mathrm{End}\, X$ such that $g = \pi \circ f$. Hence, for $j \neq k+1$, we will have

$$f(x_j) = p \sum_{h=1}^{n} a_{hj} x_h\,, \qquad \text{with } p \in P\,, \quad a_{hj} \in R \tag{2}$$

and

(3) $$f(x_{k+1}) = x_{k+1} + q\sum_{h=1}^{n} b_h x_h\,, \qquad \text{with } q \in P\,, \quad b_h \in R\,.$$

From (1), (2), (3), and the linearity of $f$, it follows, for all $r \in A_{k+1}$

(4) $$r\Big(x_{k+1} + q\sum_{h=1}^{n} b_h x_h\Big) = rp\sum_{i=1}^{k} a_i^r\Big(\sum_{h=1}^{n} a_{hi} x_h\Big)\,.$$

Since $A_{k+1} \leqslant A_t$ for all $t > k+1$, and $B$ is pure, we deduce that, for all $r \in A_{k+1}$

(5) $$rq\sum_{h=1}^{n} b_h x_h \in rqB \qquad \text{and } rp\sum_{i=1}^{k} a_i^r\Big(\sum_{h=1}^{n} a_{hi} x_h\Big) \in rpB\,.$$

Let $\bar{p} \in P$ be a common divisor of $p$ and $q$; from (4) and (5) we get $rx_{k+1} \in r\bar{p}B$ for all $r \in A_{k+1}$, i.e.

(6) $$rx_{k+1} = r\bar{p}\sum_{i=1}^{k} c_i^r x_i\,, \qquad \text{with } c_i^r \in R\,.$$

From (1), (6), and the linear independence of $x_1, \ldots, x_k$ we obtain

(7) $$r(a_i^r - \bar{p}c_i^r)x_i = 0 \qquad \text{for } i = 1, \ldots, k;$$

since $a_i^r$ is a unit for all $i$ and $r$, we have that $a_i^r - \bar{p}c_i^r$ is a unit, too, hence (7) implies $r \in \operatorname{Ann} x_i$ for all $r \in A_{k+1}$. But this means that $rx_{k+1} \in B$ implies $rx_{k+1} = 0$, from which $\langle x_{k+1}\rangle \cap B = 0$, and $B$ is not essential, against the definition of basic submodule. We conclude that, necessarily, $X = B$, as desired. It remains to prove that, if $A = \operatorname{Ann} X$, then $X \simeq (R/A)^n$. By contradiction, let us suppose that $X = \bigoplus_{i=1}^{n} \langle x_i \rangle$, where, for a suitable $j \leqslant n$, $\operatorname{Ann} x_j > A$. Let us assume, without loss of generality, that $\operatorname{Ann} x_1 = A$. Let $\eta\colon X \to X/PX$ be the homomorphism which extends by linearity the assignments

$$\eta\colon x_i \mapsto 0 \quad \text{if } i \neq j; \qquad \eta\colon x_j \mapsto \bar{x}_1 = x_1 + PX\,.$$

If $\theta \in \operatorname{End} X$ is such that $\eta = \pi \circ \theta$, then we have

$$\theta(x_j) = x_1 + p \sum_{i=1}^{n} a_i x_i\,, \qquad \text{with } p \in P\,, \quad a_i \in R\,. \tag{8}$$

Choose now $r \in \operatorname{Ann} x_j \setminus A$; from (8) we obtain

$$0 = \theta(rx_j) = r(1 + pa_1)x_1 + rp \sum_{i=2}^{n} a_i x_i\,, \tag{9}$$

from which $r(1 + pa_1)x_1 = 0$, which is impossible, because $r \notin A = \operatorname{Ann} x_i$. This concludes the proof. ////

As an easy consequence of the preceding result we obtain the following

THEOREM 3. *Let $R$ be a valuation ring. A finitely generated $R$-module $X$ is a *-module if and only if $X \simeq (R/A)^n$ for suitable $n \geqslant 0$ and $A$ ideal of $R$.*

PROOF. For any ring $R$, modules of the form $(R/A)^n$ are *-modules as a consequence of Theorem 1, observing that $\operatorname{Gen}\left((R/A)^n\right) = R/A - \operatorname{Mod}$, and $\operatorname{Hom}_R\left((R/A)^n, -\right) \approx \operatorname{Hom}_{R/A}\left((R/A)^n, -\right)$, if $n \geqslant 1$.

Conversely, let us note that $PX \in \operatorname{Gen}(X)$, as it is easy to check. Therefore, if $X$ is a finitely generated *-module, then, by Theorem 1, $X$ must satisfy the condition in the hypothesis of Theorem 2, hence $X$ has the desired form. ///

The problem of finding *-modules which are not finitely generated remains open. We actually think that a *-module over a valuation ring must be finitely generated; this opinion is mainly based on the following remark, derived from discussions with L. Salce.

REMARK 4. The simplest non finitely generated $R$-modules are the *uniserial* ones, i.e. those $R$-modules whose lattice of submodules is linearly ordered. Fuchs and Salce proved that, if $U$ is a *divisible* uniserial module over a valuation *domain* $R$, whose elements have nonzero principal annihilators, then there is an equivalence of categories

$$\operatorname{Gen}(U) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

where $\mathcal{C}$ *is the class of complete torsion-free reduced R-modules*, $F \approx \mathrm{Hom}_R(U, -)$ and $G \approx U \otimes_R -$ (see [6]; this equivalence was inspired by Matlis equivalence in [7]; see also [5], p. 99). Moreover, $U$ is small if and only if it is not countably generated. Nevertheless, for any choice of $R$ we notice that $U$ is not a *-module. This is clear if $U$ is countably generated (see Theorem 1). If $U$ is not countably generated, then also $Q$, the field of fractions of $R$, is not countably generated as an $R$-module; in this case we get that $\mathcal{C}$ *is not closed for submodules*, hence $\mathcal{C}$ cannot be cogenerated by any module. It is worth giving a check of this last fact: let us suppose, by contradiction, that $\mathcal{C}$ is closed for submodules, for a convenient $R$, with $Q$ not countably generated as an $R$-module; with these assumptions, $R$ must be complete, and each *free* $R$-module $F$ is complete, too, in view of Cor. 2.2 of [6]. Let us consider a short exact sequence

$$0 \to K \to F \to Q \to 0$$

with $F$ free; then $F \in \mathcal{C}$ implies $K \in \mathcal{C}$, hence $K$ is closed in $F$ and $Q \cong F/K$ must be Hausdorff in the natural topology, i.e. $\{0\} = \bigcap_{r \in R^*} rQ = Q$, a contradiction.

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