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The Cohomology Groups $H^1(\mathbf{P}^3 - \mathbf{P}^1, \mathcal{O}(m))$.

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Introduction.

In my paper: A ring structure on $Z_0(\mathbf{C}^4)$ and an inverse twistor formula (cfr. [C]) are introduced with a sketch of proof, some isomorphism among the cohomology groups $H^1(\mathscr{U}, \mathcal{O}(-n-2))$ and the spaces $\mathscr{I}_s^n(C)$ of holomorphic functions on the cone

$$C = \{z \in \mathbf{C}^4 : z_{00} \cdot z_{11} - z_{01} \cdot z_{10} = 0\}$$

with vanishing order at least n on a plane S of C.

The present article developes the proof using a procedure inspired by a method invented by J. Frenkel (cfr. [F]); the isomorphisms so obtained give new representations of the spaces of holomorphic solutions for the Dirac equations (cf. [C]).

Notations.

Let's fix the following notations:

$$\boldsymbol{H}_j = \{(\omega, \pi) \in \mathbf{C}^4 : \pi_j = 0\} \qquad \boldsymbol{U}_j = \mathbf{C}^4 - H_j \qquad (\text{for } j = 0, 1)$$

$$\boldsymbol{L} = H_0 \cap H_1 , \qquad \boldsymbol{U} = U_0 \cap U_1 , \qquad \boldsymbol{\mathscr{U}} = \{U_0, U_1\}$$

$$\boldsymbol{H}'_j = \pi(H_j) , \quad \boldsymbol{U}'_j = \pi(U_j) , \quad \boldsymbol{L}' = \pi(L) , \quad \boldsymbol{U}' = \pi(U) , \quad \boldsymbol{\mathscr{U}}' = \{U'_0, U'_1\}$$

where: $\boldsymbol{\pi}: \mathbf{C}^4 - \{0\} \to \mathbf{P}^3$ is the natural projection.

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1. – We are going to define a factor F_m of $Z^1(\mathcal{U}, \mathcal{O}(m)) = \mathcal{O}(m)(U_0 \cap U_1)$. Let's consider the two « extension » maps:

$$\boldsymbol{e}_0\colon \mathcal{O}(m)(U) \to \mathcal{O}(m)(U_0) \quad \text{and } \boldsymbol{e}_1\colon \mathcal{O}(m)(U) \to \mathcal{O}(m)(U_1)$$

given by:

$$\boldsymbol{e}_0 \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{\pi}) = \frac{1}{2\pi i}\cdot \int\limits_{|t_1|=r_1} \frac{f(\omega, \pi_0, t_1)}{t_1 - \pi_1}\cdot dt_1 \qquad (\text{for } |\pi_1| < r_1)$$

and

$$\boldsymbol{e}_1 \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{\pi}) = \frac{1}{2\pi i}\cdot \int\limits_{|t_0|=r_0} \frac{f(\omega, t_0, \pi_1)}{t_0 - \pi_0}\cdot dt_0 \qquad (\text{for } |\pi_0| < r_0)\,.$$

Since $e_j f = f$ for $f \in \mathcal{O}(m)(U_j)$ composing the extensions with the restrictions to U we get the projections:

$$\boldsymbol{p}_0, \boldsymbol{p}_1\colon \mathcal{O}(m)(U) \to \mathcal{O}(m)(U) \qquad (p_0^2 = p_1^2 = id)$$

with the properties:

I) $p_0 \circ p_1 = p_1 \circ p_0$,

II) $$(p_0 \circ p_1)(f)(\omega, \pi) = \frac{1}{(2\pi i)^2}\cdot \oint\limits_{|t_j|=r_j}\oint \frac{f(\omega, t_0, t_1)}{(t_0 - \pi_0)\cdot(t_1 - \pi_1)}\cdot dt_0 \cdot dt_1 =$$
$$= f(\omega, \pi) + \frac{1}{2\pi i}\cdot \oint\limits_{|t_0|=R_0} \frac{f(\omega, t_0, \pi_1)}{t_0 - \pi_0}\, dt_0 - \frac{1}{2\pi i}\cdot \oint\limits_{|t_1|=R_1} \frac{f(\omega, \pi_0, t_1)}{t_1 - \pi_1}\cdot dt_1 +$$
$$+ \frac{1}{(2\pi i)^2}\oint\limits_{|t_j|=R_j}\oint \frac{f(\omega, t_0, t_1)}{(t_0 - \pi_0)(t_1 - \pi_1)}\cdot dt$$

(the first equation for $0 < |\pi_j| < r_j$, the second for $|\pi_j| > R_j$).

The new projection

$$\boldsymbol{p} = id - p_0 - p_1 + p_0 \circ p_1\colon \mathcal{O}(m)(U) \to \mathcal{O}(m)(U)$$

defines the subspace

$$F_m = \{f \in \mathcal{O}(m)(U) : pf = f\} .$$

Since

$$p(\omega_0^{p_0}\cdot\omega_1^{p_1}\cdot\pi_0^{q_0}\cdot\pi_1^{q_1}) = \begin{cases} 0 & \text{if } q_0 \geqslant 0 \text{ or } q_1 \geqslant 0 , \\ \omega^p\cdot\pi^q & \text{if } q_0 < 0 \text{ and } q_1 < 0 , \end{cases}$$

the space F_m contains all functions with a Laurent expansion:

$$f(\omega, \pi) = \sum_{j_0, j_1 > 0} a_{k,j}\cdot\frac{\omega_0^{k_0}\cdot\omega_1^{k_1}}{\pi_0^{j_0}\cdot\pi_1^{j_1}} .$$

2. – We are interested in F_m for the following:

THEOREM (2.1). The inclusion $F_m \overset{j}{\hookrightarrow} \mathcal{O}(m)(U) = Z^1(\mathcal{U}, \mathcal{O}(m))$ induces an isomorphism: $\hat{j}: F_m \to H^1(\mathcal{U}, \mathcal{O}(m)) (\simeq H^1(\mathcal{U}', \mathcal{O}(m))$

PROOF. If $f \in B^1(\mathcal{U}, \mathcal{O}(m))$ then $f = f_1 - f_0$ with $f_j \in \mathcal{O}(m)(U_j)$. Since $p(f_1 - f_0) = 0$ then $f = pf = 0$.

For $f \in \mathcal{O}(m)(U)$ let's take $f' = pf$ in F_m, the difference $f - f' = \operatorname{res}_U^{U_1}(e_1 f) - \operatorname{res}_U^{U_0}(e_0 \operatorname{res}_U^{U_1} e_1 f - e_0 f)$ is in $B^1(\mathcal{U}, \mathcal{O}(m))$.

THEOREM (2.2). Are equivalent:

i) f of $\mathcal{O}(m)(U)$ is in F_m;

ii) $\lim_{\pi_j \to \infty} f(\omega_0, \omega_1, \pi_0, \pi_1) = 0$ for every ω and $\pi_{j\pm1} \neq 0$ $(j = 0, 1)$;

iii) there exist homogeneous polynomials $A_j(\omega)$ of degree $j_0 + j_1 + m$ such that:

$$f(\omega, \pi) = \sum_{j_0, j_1 > 0} \frac{A_j(\omega)}{\pi_0^{j_0}\cdot\pi_1^{j_1}} .$$

PROOF. i) $\Rightarrow$ ii) For every f in F_m it holds the inequality:

$$|f(\omega, \pi)| \leqslant \frac{4\cdot R_0\cdot R_1}{|\pi_0|\cdot|\pi_1|}\cdot \max_{|t_j| = R_j} |f(\omega_0, \omega_1, t_0, t_1)| \qquad (|\pi_j| > 2\cdot R_j > 0) .$$

ii) $\Rightarrow$ i) From:

$$f(\omega,\pi)=\frac{1}{2\pi i}\cdot\int\limits_{|t_j|=S_j}\frac{f(\omega,\dots t_j\dots)}{t_j-\pi_j}dt_j-\frac{1}{2\pi_i}\cdot\int\limits_{|t_j|=R_j}\frac{f(\omega,\dots t_j\dots)}{t_j-\pi_j}\cdot dt_j$$

it follows (as $S_j\to\infty$):

$$f(\omega,\pi)=\frac{1}{(2\pi i)^2}\cdot\iint\limits_{|t_j|=R_j}\frac{f(\omega,t_0,t_1)}{(t_0-\pi_0)\cdot(t_1-\pi_1)}\cdot dt\,.$$

i) $\Rightarrow$ iii) $A_j(\omega)=\dfrac{1}{(2\pi i)^2}\cdot\displaystyle\iint\limits_{|t_k|=R_k}t_0^{j_0-1}\cdot t_1^{j_1-1}\cdot f(\omega_0,\omega_1,t_0,t_1)\cdot dt_0\cdot dt_1\,.$

iii) $\Rightarrow$ i) $p\left(\dfrac{\omega_0^{p_0}\cdot\omega_0^{p_1}}{\pi_0^{j_0}\cdot\pi_1^{j_1}}\right)=\dfrac{\omega_0^{p_0}\cdot\omega_1^{p_1}}{\pi_0^{j_0}\cdot\pi_1^{j_1}}\,.$

3. – The map $\sigma\colon F_m\to F_{-m-4}$ defined by:

$$\sigma(f)(\omega,\pi)=\frac{1}{\omega_0\cdot\omega_1\cdot\pi_0\cdot\pi_1}\cdot f\left(\frac{1}{\pi_0},\frac{1}{\pi_1},\frac{1}{\omega_0},\frac{1}{\omega_1}\right)$$

is a well defined isomorphism.

Infact with some computation it is possible to prove:

$$|\sigma(f)(\omega,\pi)|\leqslant\frac{M(r,s)}{r_0\cdot r_1\cdot s_0\cdot s_1}\qquad (M=\max\{|f(u,v)|:|v_j|=1/(2\cdot s_j),\ |u_j|\leqslant 1/r_j\})\,.$$

THEOREM (3.1). For every $f\in F_m$ there exist functions $\{f_p\}$ in F_{-2} (for $p_0+p_1=|m+2|$ and $p_0,p_1\geqslant 0$) such that:

a) $f=\displaystyle\sum_p\frac{1}{\pi_0^{p_0}\cdot\pi_1^{p_1}}\cdot f_{p_0,p_1}$ if $m<-2$,

b) $f=\displaystyle\sum_p\omega_0^{p_0}\cdot\omega_1^{p_1}\cdot f_{p_0,p_1}$ if $m>-2$.

PROOF.

$$a)\quad f = \sum_{h_0+h_1 \geq |m+2|} \frac{1}{\pi_0 \cdot \pi_1} \cdot \frac{A_h(\omega)}{\pi_0^{h_0} \cdot \pi_1^{h_1}} = \sum_p \frac{1}{\pi_0^{p_0} \cdot \pi_1^{p_1}} \cdot \sum_k \frac{A_h(\omega)}{\pi_0^{k_0+1} \cdot \pi_1^{k_1+1}},$$

$$b)\quad f = \sigma_{-m-4}(\sigma_m(f)) = \sigma_{-m-4}\Big(\sum_p \pi_0^{-p_0} \cdot \pi_1^{-p_1} \cdot g_p\Big)$$

4. - THEOREM (4.1). Let C be the cone in $\mathbf{C}^4$ defined by:

$$C = \{z \in \mathbf{C}^4 \colon z_{00} \cdot z_{11} - z_{01} \cdot z_{10} = 0\}$$

and let S, T be:

$$S = \{z \in \mathbf{C}^4 \colon z_{10} = z_{11} = 0\}\,, \qquad T = \{z \in \mathbf{C}^4 \colon z_{01} = z_{11} = 0\}\,.$$

Denoted by $\mathcal{J}_S$, $\mathcal{J}_T$ the ideal sheaves of S and T in C it holds:

$$a)\quad h_n \colon \mathcal{J}_S^n(C) \to F_{-n-2}\,,\ h_n(k) = \frac{1}{\pi_0 \cdot \pi_1} \cdot \frac{1}{\omega_1^n} \cdot k\left(\frac{\omega_0}{\pi_0}, \frac{\omega_0}{\pi_1}, \frac{\omega_1}{\pi_0}, \frac{\omega_1}{\pi_1}\right)$$

is a well defined isomorphism for every $n \geq 0$.

$$b)\quad h_n' \colon \mathcal{J}_T^n(C) \to F_{n-2}\,,\ h_n(k) = \frac{1}{\pi_0 \cdot \pi_1} \cdot \pi_1^n \cdot k\left(\frac{\omega_0}{\pi_0}, \frac{\omega_0}{\pi_1}, \frac{\omega_1}{\pi_0}, \frac{\omega_1}{\pi_1}\right)$$

as a well defined isomorphism for every $n \geq 0$.

PROOF. *a)* $n = 0$ Taken

$$f = \sum_{l_0, l_1} \frac{A_l(\omega)}{\pi_0^{l_0+1} \cdot \pi_1^{l_1+1}} \qquad \text{in } F_{-2}\ (\deg(A_l) = l_0 + l_1)$$

let's consider the function

$$k(z) = z_{00} \cdot z_{01} \cdot f(z_{00} \cdot z_{01}, z_{01} \cdot z_{10}, z_{01}, z_{00}) = \sum_l \left(\frac{z_{01}}{z_{00}}\right)^{l_1} \cdot A_l(z_{00}, z_{10}) =$$

$$= \sum_l \left(\frac{z_{11}}{z_{01}}\right)^{l_1} \cdot A_l(z_{00}, z_{10}) = \sum_l \left(\frac{z_{00}}{z_{01}}\right)^{l_0} \cdot A_l(z_{01}, z_{11}) = \sum_l \left(\frac{z_{10}}{z_{11}}\right)^{l_0} \cdot A_l(z_{01}, z_{11})$$

is holomorphic on $C - \{0\}$ and then on all C (the space C is perfect, cfr. [BS] cor. 3.12 pag. 79); it holds: $h_0(k) = f$.

a) *b)* $n > 0$) follow from the previous case and theorem (3.1).

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