

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

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model for hysteresis**

*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 83 (1990), p. 153-163

[http://www.numdam.org/item?id=RSMUP\\_1990\\_\\_83\\_\\_153\\_0](http://www.numdam.org/item?id=RSMUP_1990__83__153_0)

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## On a Characterization of the Preisach Model for Hysteresis.

MARTIN BROKATE (\*)

### 1. - Introduction.

In [6], Preisach formulated a mathematical model in order to describe hysteresis loops arising in ferromagnetism. It can be viewed as an operator  $W$ , which maps an input function  $u: [0, 1] \rightarrow \mathbf{R}$ , representing the (scalar) magnetic induction, to an output function  $w: [0, 1] \rightarrow \mathbf{R}$ , representing the magnetization. Usually one defines  $W$  by

$$(1) \quad (Wu)(t) = \int_P (W_r u)(t) d\mu(r),$$

where  $\mu$  is a finite Borel measure on the Preisach plane

$$(2) \quad P = \{r: r = (r_1, r_2), r_1 \leq r_2\} \subset \mathbf{R}^2$$

and  $W_r$  denotes an elementary switch with hysteresis, switching to the value 1 when  $u(t)$  increases to the value  $r_2$  and to the value  $-1$  when  $u(t)$  decreases to  $r_1$ . In addition, an initial condition has to be specified for each elementary switch.

Since the action of an ideal switch is instantaneous, this model obviously is rate independent, i.e.

$$W(u \circ \varphi) = (Wu) \circ \varphi$$

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for any (monotone) transformation  $\varphi$  of the time scale. Also, any periodic input  $u(t)$  yields a periodic output  $w(t)$  with the same period. Thus, the map  $t \rightarrow (u(t), w(t))$  generates a hysteresis loop in the  $(u, w)$ -plane. Consider a periodic input (e.g. a sine function) oscillating between the values  $r_1$  and  $r_2$ . The height of the corresponding hysteresis loop (assuming  $\mu$  nonnegative) is given by

$$h(r_1, r_2) = 2\mu(\Delta),$$

$\Delta$  being the triangle  $\{(s_1, s_2): r_1 \leq s_1 \leq s_2 \leq r_2\}$  in the Preisach plane. Using this equality, one may determine the measure  $\mu$  from experiment; on the other hand it shows that the height of the loop does not depend upon the past history. Moreover, the entire shape of the hysteresis loop is fixed by the measure  $\mu$  independent from past history, and any change of input from  $r_1$  to  $r_2$  and back to  $r_1$  erases any memory due the previous input variation in the interval  $[r_1, r_2]$ .

Again, from the behaviour of individual switches it is obvious that the Preisach model has the properties stated above. It was Mayergoyz who pointed out in [4] that the latter two properties, which he calls congruency and wiping out property respectively, are also sufficient for a (nonanticipative and rate independent) operator to be a Preisach operator.

The aim of the present paper is to provide a formal statement and proof of this result. We try to clarify the role of the various assumptions; also, we admit general Borel measures  $\mu$ . For more material on the Preisach operator, we refer to [1, 2, 3, 7].

## 2. – The characterization of the Preisach operator.

Throughout this paper, we set  $T = [0, 1]$  and denote by  $M(T)$  the set of all real valued functions on  $T$ .

DEFINITION 1. Let  $U \subset M(T)$ . An operator  $W: U \rightarrow M(T)$  is called a hysteresis operator if it is rate independent and nonanticipative, i.e. if

$$W(u \circ \varphi) = (Wu) \circ \varphi \quad \forall u \in U$$

for any continuous nondecreasing  $\varphi: T \rightarrow T$  with  $\varphi(0) = 0$ ,  $\varphi(1) = 1$

and  $u \circ \varphi \in U$ ;

$$u_1 = u_2 \quad \text{on } [0, t] \Rightarrow Wu_1 = Wu_2 \quad \text{on } [0, t]$$

for any  $u_1, u_2 \in U$  and any  $t \in T$ .  $\square$

Let  $M_{pm}(T)$  resp.  $C_{pm}(T)$  denote the set of all piecewise monotone continuous functions:

$$M_{pm}(T) = \{u|u: T \rightarrow \mathbb{R}, \exists 0 = t_1 < t_2 < \dots < t_n = 1 \text{ such}$$

$$\text{that } u|[t_i, t_{i+1}] \text{ is monotone for all } 1 \leq i < n\},$$

$$C_{pm}(T) = M_{pm}(T) \cap C(T).$$

The action of a rate independent operator  $W$  on  $u \in C_{pm}(T)$  is essentially specified by the values  $x_i = u(t_i)$  on a monotonicity partition  $\{t_i\}$  of  $u$ . If  $W$  is nonanticipative, the knowledge of the final value  $(Wu)(1)$  for all  $u$  is sufficient. Note also that definition 1 implies that if  $u \in U = C_{pm}(T)$  is constant on some closed interval  $I \subset T$ , then so is  $Wu$ , since we may contract  $I$  to a single point.

We formalize these considerations.

DEFINITION 2. Let

$$X_0 = \{x|x = (x_1, \dots, x_n), n \in \mathbb{N}, x_i \in \mathbb{R}\} \cup \{\emptyset\}$$

be the set of all strings of real numbers including the empty string  $\emptyset$ , set

$$X = \{x|x \in X_0, \text{length}(x) \geq 2\}.$$

Define a concatenation  $x \cup y$  for  $x, y \in X_0$  by

$$x \cup y = (x_1, \dots, x_n, y_1, \dots, y_m)$$

and generate an equivalence relation  $\sim$  on  $X$  from

$$(x_1, x_2, x_3) \sim (x_1, x_3) \quad \text{if } x_1 < x_2 < x_3 \quad \text{or } x_1 \geq x_2 \geq x_3,$$

$$x \sim \bar{x} \Rightarrow y \cup x \cup z \sim y \cup \bar{x} \cup z \quad \forall y, z \in X_0,$$

forming the reflexive, symmetric and transitive hull.  $\square$

DEFINITION 3. We define  $p: M_{pm}(T) \rightarrow X/\sim$  by

$$p(u) = (x_1, \dots, x_n)$$

where  $x_i = u(t_i)$  and  $0 = t_1 < \dots < t_n = 1$  is a monotonicity partition for  $u$  such that  $(u(t_{i+1}) - u(t_i))(u(t_i) - u(t_{i-1})) < 0$  for  $1 < i < n$ .  $U \subset M_{pm}(T)$  is called rich, if  $p|U$  is surjective.  $\square$

For example, the set of all piecewise linear functions on  $T$  is rich.

PROPOSITION 1. Let  $U \subset C_{pm}(T)$  be rich. Then for any hysteresis operator  $W: U \rightarrow M(T)$ ,

$$W_f(x) = (Wu)(1), \quad x = p(u)$$

defines a mapping

$$W_f: X/\sim \rightarrow \mathbb{R}.$$

Conversely, any mapping  $W_f: X/\sim \rightarrow \mathbb{R}$  defines a hysteresis operator  $W: M_{pm}(T) \rightarrow M(T)$  by

$$(Wu)(t) = W_f(p(u_t)),$$

where we set  $u_t = u$  in  $[0, t]$  and  $u_t = u(t)$  on  $[t, 1]$ . Moreover, these correspondences establish a bijection between the set of all hysteresis operators  $W: U \rightarrow M(T)$  and the set of all real valued mappings on  $X/\sim$ .

PROOF. Using the identities

$$(u \circ \varphi)_t = u_{\varphi(t)} \circ \varphi, \quad p(u \circ \varphi) = p(u)$$

for  $\varphi$  as in definition 1, one easily checks that the correspondences  $W \rightarrow W_f$  and  $W_f \rightarrow W$  are well defined and inverse to each other.  $\square$

In this manner, we obtain a canonical prolongation for a hysteresis operator  $W$  defined on  $U \subset C_{pm}(T)$  to  $M_{pm}(T)$ . However, we remark that a general rate independent and nonanticipative operator on  $M_{pm}(T)$  cannot be reduced to an operator on  $X$  resp.  $X/\sim$ , since the discontinuity structure of the inputs yields additional degrees of freedom. We will exclude this from our discussion from now on.

**DEFINITION 4.** A hysteresis operator  $W: M_{pm}(T) \rightarrow M(T)$  is called regular if it coincides with its canonical prolongation from  $C_{pm}(T)$  via proposition 1.  $\square$

The first distinguishing feature of the Preisach model is its memory erasure mechanism. Again, we describe this by an equivalence relation.

**DEFINITION 5.** We generate an equivalence relation  $\approx$  on  $X$  by the monotone reducing rule from definition 2

$$(x_1, x_2, x_3) \approx (x_1, x_3) \quad \text{if } x_2 \in \text{conv} \{x_1, x_3\},$$

the memory erasure rule

$$(E) \quad (x_1, x_2, x_3) \approx (x_2, x_3) \quad \text{if } x_1 \in \text{conv} \{x_2, x_3\}$$

together with

$$x \approx \bar{x} \Rightarrow y \cup x \cup z \approx y \cup \bar{x} \cup z \quad \forall y, z \in X_0,$$

again forming the reflexive, symmetric and transitive hull. We say that a regular hysteresis operator  $W$  has property (E), if  $W$ , factorizes through  $\approx$ , i.e. if  $x \approx y$  implies  $W_r(x) = W_r(y)$ .  $\square$

For a hysteresis operator with property (E) it therefore suffices to consider its action on  $X/\approx$ . We now give the normal form of an element of  $X/\approx$ .

**PROPOSITION 2.** For any  $y \in X$  there exists an  $x \in X$  and an  $m \in \mathbf{R}$  with  $x \approx y$ , having the form

$$x_1 > x_3 > x_5 > \dots > m > \dots > x_6 > x_4 > x_2$$

or the one with inequalities reversed.

**PROOF.** Apply monotone reducing and memory erasure to  $y$  from right to left.  $\square$

We now turn to the second characteristic property of the Preisach model, which states that the height of any hysteresis loop corresponding to inputs of the form

$$x \cup (r_1, r_2, r_1, r_2, r_1, r_2, \dots)$$

only depends on  $r_1$  and  $r_2$  but not on  $x$ . Due to memory erasure, only the first period  $(r_1, r_2, r_1)$  has to be considered. Its first half can be influenced by the past, its second half cannot.

**DEFINITION 6.** We say that a regular hysteresis operator  $W$  has property  $(H)$  if there exists a function  $h: \mathbf{R}^2 \rightarrow \mathbf{R}$  with

$$\begin{aligned} W_f(x \cup (r_1, r_2)) - W_f(x \cup (r_1, r_2, r_1)) &= h(r_1, r_2), \\ h(r_2, r_1) &= -h(r_1, r_2) \end{aligned}$$

for any  $r_1, r_2 \in \mathbf{R}$  and any  $x \in X_0$ .  $\square$

Since for any  $x \in X$  in the normal form of  $X/\approx$  we may view any part  $(x_i, x_{i+1})$  of  $x$  as second half of the first period of a periodic input, with property  $(H)$  we can reduce general inputs to monotone inputs.

**PROPOSITION 3.** Any regular hysteresis operator  $W$  having properties  $(E)$  and  $(H)$  is uniquely determined by the function  $h$  and its values  $W_f(x)$ ,  $x = (x_1, x_2) \in \mathbf{R}^2$ .

**PROOF.** For any  $x \in X$  in the normal form of  $X/\approx$  with  $x = (x_1, \dots, x_n)$ ,  $n \geq 3$ , we have

$$\begin{aligned} (x_1, \dots, x_n) &\approx (x_1, \dots, x_{n-2}, x_n, x_{n-1}, x_n), \\ (x_1, \dots, x_{n-1}) &\approx (x_1, \dots, x_{n-2}, x_n, x_{n-1}) \end{aligned}$$

and therefore  $(H)$  implies

$$W_f(x_1, \dots, x_n) = W_f(x_1, \dots, x_{n-1}) + h(x_{n-1}, x_n). \quad \square$$

The values  $W_f(x)$ ,  $x \in \mathbf{R}^2$ , are, in general, unequal to  $h(x)$  since they also include information from the initial condition, i.e. the initial

configuration of the individual switches. Since it is not easy to translate general configurations of switches into properties of  $W_f(x_1, x_2)$ , and since it would not really contribute to the characterization of the Preisach model, we do not attempt to do this, but restrict ourselves to the following remarks. If we assume that the initial configuration is the result of a previous input, we have the compatibility condition

$$(C) \quad \exists x_0 \in \mathbb{R}: W_f(x) = W_f(x_0 \cup x) \quad \forall x \in X,$$

$x_0$  being the final value of the previous input. If we moreover consider the special situation where  $x_0$  is a lower bound for all threshold values  $r_1$  of existing individual switches, then all switches are on  $-1$  initially and we must have

$$(I) \quad W_f(x_0, x_1) = \begin{cases} h(x_0, x_1) - \mu(P) & \text{if } x_1 \geq x_0, \\ -\mu(P) & \text{if } x_1 < x_0. \end{cases}$$

Together with proposition 2, this yields the following.

**COROLLARY 1.** Any regular hysteresis operator  $W$  having properties (E), (H), (C) and (I) is uniquely determined by the function  $h$ .  $\square$

Now we discuss the function  $h$  describing the height of the hysteresis loops, also called demagnetization function in [3]. Since for  $r_1 < r_2$  we should have

$$(*) \quad h(r_1, r_2) = 2\mu(\Delta(r_1, r_2))$$

with

$$\Delta(r_1, r_2) = \{(s_1, s_2): r_1 \leq s_1 \leq s_2 \leq r_2\},$$

in the case of a nonnegative measure  $\mu$  the function  $h$  has to satisfy the inequalities [3]

$$(N) \quad \begin{cases} h(r_1, r_2) \geq 0, \\ h(r_1, r_2) + h(r_1 + \delta, r_2 - \delta) \geq h(r_1, r_2 - \delta) + h(r_1 + \delta, r_2), \end{cases}$$

for any  $r_1 < r_2$  and any  $0 < \delta \leq r_2 - r_1$ , as can be seen if one draws the corresponding triangles. It was remarked in [3] that (N) plus one sided partial continuity of  $h$  is also sufficient for (\*) to hold for



some measure  $\mu$ . We present a more detailed formulation. Let us denote by  $\partial_1 \partial_2 h$  the mixed second partial derivative of  $h$  in the sense of distributions. If  $h$  is smooth, then one easily sees that (\*) holds with  $2\mu$  having density  $-\partial_1 \partial_2 h$ . In general, one has the following result.

LEMMA 1. Assume that  $h: \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies (N). Let

$$h_0(r_1, r_2) = \begin{cases} h(r_1, r_2), & \text{if } r_1 < r_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following is true:

- (i)  $-\partial_1 \partial_2 h_0 \geq 0$ ,
- (ii)  $h_0$  is nonincreasing w.r.t.  $r_1$  and nondecreasing w.r.t.  $r_2$ ,
- (iii)  $h_0(r_1-, r_2+) = -\partial_1 \partial_2 h_0(\Delta(r_1, r_2))$ ,  $r_1 < r_2$ ,

where

$$h_0(r_1-, r_2+) = \inf_{s_1 < r_1} \inf_{s_2 > r_2} h(s_1, s_2)$$

PROOF. For test functions  $\psi$ , we have

$$\langle -\partial_1 \partial_2 h_0, \psi \rangle = \langle h_0, -\partial_1 \partial_2 \psi \rangle.$$

Apply (N) to difference quotients of  $\psi$  and pass to the limit to obtain (i). Assertion (ii) is a consequence of

$$h_0(r_1, r_2) \geq h_0(r_1, r) + h_0(r, r_2), \quad r_1 < r < r_2.$$

From (i) we know that  $-\partial_1 \partial_2 h_0$  is a regular nonnegative Borel measure. Assertion (iii) is obtained through approximation of the characteristic function of  $\Delta(r_1, r_2)$  from above and below by suitable test functions.  $\square$

If we do not require the measure  $\mu$  to be nonnegative, then  $h$  must be a difference  $h_1 - h_2$  where  $h_1$  and  $h_2$  satisfy (N). This is equivalent to an older notion of a BV function  $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ , elaborated in detail in section 46 of McShane's book [5], not to be confused with

the at present more standard notion that a function  $h$  is  $BV$  if  $\text{grad}(h)$  is a measure—recall that we want  $\partial_1 \partial_2 h$  to be a measure.

We now obtain the main theorem.

**THEOREM 1.** For an operator  $W: U \rightarrow M(T)$  with  $U = C_{pm}(T)$  (or  $U$  a rich subset of  $C_{pm}(T)$ ) the following assertions are equivalent:

(i)  $W$  is a Preisach operator with (signed) finite Borel measure  $\mu$  with compact support in the Preisach plane  $P$  without atoms on the main diagonal, and initial configuration  $\equiv -1$ .

(ii)  $W$  is a regular hysteresis operator satisfying (E), (H), (C) and (I), where the function  $h$  in (H), set to zero in  $\{r_1 \geq r_2\}$ , has bounded variation [5, section 46] and satisfies

$$h(r_1, r_2) = h(r_1-, r_2+)$$

**PROOF.** The implication (i)  $\Rightarrow$  (ii) is discussed in the introduction and easily formalized using well known results [1, 2, 3, 7]. For the converse, consider the Preisach operator  $\tilde{W}$  defined by the measure  $2\mu = -\partial_1 \partial_2 h$  and initial configuration  $-1$ .  $\tilde{W}$  also has the properties stated in (ii), and because of uniqueness obtained in corollary 1,  $\tilde{W}$  has to be equal to  $W$ .  $\square$

In theorem 1, essentially only measures of form  $\mu = \delta_{(r,r)}$  are excluded. Such a measure corresponds to a simple switch at  $u = r$  without hysteresis. However, individual switches with hysteresis as well as a continuous distribution of switches without hysteresis are included.

We finally pose the question: When is a given hysteresis operator  $W: C(T) \rightarrow M(T)$  a Preisach operator?

If  $W: C(T) \rightarrow C(T)$  is continuous, then  $W$  is a Preisach if and only if  $W|_{C_{pm}(T)}$  is, since if  $W$  maps  $C_{pm}(T)$  into  $C(T)$ , the measure  $\mu$  must be zero along horizontal and vertical lines in  $P$ , which is sufficient for a Preisach operator to be continuous on  $C(T)$ . If  $W$  is not continuous on  $C(T)$ , this argument does not work. In this case, consider at first inputs defined as linear interpolate for  $u(t_i) = x_i$ ,  $u(t_*) = u(1) = x_*$ , where  $t_i \uparrow t_* \leq 1$ . Letting  $X_\infty$  be the space of convergent sequences  $(x_n)$ , for a given hysteresis operator  $W$  we may define

$\tilde{W}: X_\infty/\sim \rightarrow X_\infty/\sim$  by

$$(\tilde{W}x)_n = W_f(x_1, \dots, x_n).$$

The uniqueness result of corollary 1 still holds for  $\tilde{W}$  since it holds for all  $(\tilde{W}x)_n$ . Therefore, one only has to formulate a version of the memory erasure property which reduces  $C(T)$  to  $X_\infty$ . To this end, describe the memory by a family  $\sim_t$  of equivalence relations on  $C(T)$  as follows:

$$u \sim_t v \Leftrightarrow W\bar{u} = W\bar{v} \quad \text{on } [t, 1] \quad \forall (\bar{u}, \bar{v}) \in F,$$

$$F = \{(\bar{u}, \bar{v}): \bar{u}|[0, t] = u, \bar{v}|[0, t] = v, \bar{u}|[t, 1] = \bar{v}|[t, 1]\}.$$

This is just the general notion of Nerode equivalence. The memory erasure property ( $E'$ ) consists of two parts. Let  $0 \leq s < t \leq 1$ . We demand that

$$u|[s, t] \subset \text{conv} \{u(s), u(t)\} \Rightarrow u \sim_t v,$$

where we obtain  $v$  from  $u$  replacing  $u|I$  by a straight line interpolating  $u(s)$  and  $u(t)$ ; moreover we demand that

$$u|[0, s] \subset \text{conv} \{u(s), u(t)\} \Rightarrow u \sim_t v$$

with  $v = u(s)$  on  $[0, s]$  and  $v = u$  on  $[s, 1]$ . It is then not difficult to show that for any  $v \in C(T)$  and any  $t \in T$  there exists a  $u \in C(T)$  which can be represented by an  $x \in X_\infty/\sim$  as above, with  $v \sim_t u$  and

$$(Wv)(t) = (Wu)(t) = \lim_{n \rightarrow \infty} (Wx)_n.$$

(One constructs  $v$  resp.  $x$  in the same way as if one wants to define  $(Wv)(t)$  for an arbitrary  $v \in C(T)$ ,  $t \in T$  and a given Preisach operator  $W$ , see e.g. [3].)

With this modification ( $E'$ ) of ( $E$ ), theorem 1 also holds for  $U = C(T)$ .

*Acknowledgements.* The author wants to thank Martin Hilpert, Pavel Krejčí and Augusto Visintin for stimulating discussions.

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Manoscritto pervenuto in redazione il 6 giugno 1989.