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Pseudo-Closure-Operators.

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In the algebra of R-modules closure operators are used frequently; their common foundation is based as follows: let M be an R-module, $L(M)$ the complete lattice of the sub-modules of M, φ: $L(M) \to L(M)$ a mapping, written as

$$\varphi(N) = N^c \qquad (N \in L(M)) ,$$

such that

(i) $N_1 \subseteq N_2 \Rightarrow N_1^c \subseteq N_2^c$;

(ii) $N \subseteq N^c$;

(iii) $(N^c)^c = N^c$,

then φ is called a *closure operator* on the lattice $L(M)$. The submodule $N \in L(M)$ is *φ-closed* if $N = N^c$.

A majority of the closure operators on $L(M)$ can be defined by means of a Gabriel filter $\mathcal{F}$ on the ring R; the filter $\mathcal{F}$ defines a hereditary torsion functor $\tau = \tau_{\mathcal{F}}$, such that

$$\tau_{\mathcal{F}}(M) = \{m \in M\colon \mathrm{Ann}_R(m) \in \mathcal{F}\}$$

is the $\tau_{\mathcal{F}}$-torsion submodule of M. Any Gabriel filter $\mathcal{F}$ on the ring R induces on the lattice $L(M)$ a *closure operator*, defined by

$$N \mapsto N^c = \{m \in M\colon (N; m) \in \mathcal{F}\} , \qquad N \in L(M) .$$

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Then N^c is a submodule of M, the *$\tau_{\mathcal{F}}$-closure* (or the *$\tau_{\mathcal{F}}$-saturation*) of $N \in L(M)$. The submodule $N \in L(M)$ is *$\tau_{\mathcal{F}}$-closed*, if $N = N^c$, i.e. if and only if $\tau_{\mathcal{F}}(M/N) = 0$.

The family of the $\tau_{\mathcal{F}}$-closed submodules N of M is denoted by

$$\mathrm{Sat}_{\mathcal{F}}(M) = \{N \in L(M) \colon N = N^c\} \ (^1) .$$

We find in the literature (see: Y. Miyashita [1]) a *pseudo-closure operator*, defined on the set $\Sigma(M)$ of all *subsets* of an R-module M, properly containing the element $\{0\} \in M$. For the subset $S \in \Sigma(M)$ we define the *pseudo-closure* S^c of the subset $S \in \Sigma(M)$ by

$$S \mapsto S^c = \{0\} \cup \{m \in M \colon Rm \cap S \neq 0\} , \tag{1}$$

and we set $0^c = 0$. Then we have for subsets $S, T \in \Sigma(M)$:

(i) $S \subseteq T \Rightarrow S^c \subseteq T^c$;

(ii) $S \subseteq S^c$;

(iii) $(S^c)^c = S^c$.

If S is a subset of M, then S^c is called the *pseudo-closure* of S, and S will be called *pseudo-closed*, if $S = S^c$.

If N is a *submodule* of the *R-module* M, then the pseudo-closure N^c of the submodule N is—in general—not a submodule of M.

Example. - Let $R = Z$, $M = Z \oplus Z/2Z$, $N = Z(2; \bar{0})$; then $N_1 = Z(1; \bar{0})$ and $N_2 = Z(1; \bar{1})$ are maximal essential extensions of N in M. We have $N_1 \subseteq N^c$, $N_2 \subseteq N^c$, but N^c is *not* a submodule of N; for $(1; \bar{0}) \in N_1 \subseteq N^c$, $(1; \bar{1}) \in N_2 \subseteq N^c$, but $(0; \bar{1}) = (1; \bar{1}) - (1; \bar{0}) \notin N^c$, since $Z(0; \bar{1}) \cap Z(2; \bar{0}) = (0; \bar{0})$.

Theorem 1. *If $N \in L(M)$ then we have:*

(i) *the pseudo-closure N^c of N in M contains every essential extension of N in M;*

(ii) *if in particular N^c is a submodule of M, then N^c is the unique maximal essential extension of N in M;*

(1) See e.g. B. Stenström [2], p. 207.

(iii) *if* N^c *is the unique maximal essential extension of* N *in* M *for every submodule* N *of* M, *then* M *has the «intersection property» for essentially closed submodules of* M (i.e. *the intersection of any collection of essentially closed submodules of* M *is essentially closed in* M).

PROOF. (i) is a consequence of the definition (1).

(ii) If N^c is a submodule of M, then N^c cannot have a proper essential extension in M: i.e. then N^c is essentially closed in M. That implies that N^c is a complement of some submodule $N' \subseteq M$, and we may choose N' in such a way, that N' contains an element $0 \neq m_0 \in M \setminus N^c$. Then $Rm_0 \cap N^c = 0$ implies that $Rm_0 \cap N = 0$, i.e. $m_0 \notin N^c$, and thus N^c is the unique maximal essential extension of N in M.

(iii) If N^c is a submodule of M for all submodules $N \subseteq M$, then any submodule N of M has a *unique* maximal essential extension $\bar{N} = N^c$ in M, and that implies that M has the «intersection property» for essentially closed submodules of M.

THEOREM 2. *If the* R*-module* M *satisfies the «intersection property» for essentially closed submodules, then for every submodule* $N \subseteq M$ *the pseudo-closure* N^c *is the unique maximal essential extension of* N *in* M.

Indeed, in this situation N^c is the (unique) intersection of all essentially closed submodules of M containing N.

THEOREM 3. *Let* R *be a left Ore domain,* M *a torsionfree* R*-module, and* $N \neq 0$ *a submodule of* M; *then*:

(i) N^c *is the unique maximal essential extension of* N *in* M;

(ii) M *has the «intersection property» for essentially closed submodules.*

PROOF. If $0 \neq m \in N^c$, then for some $0 \neq r \in R$ we have $0 \neq rm \in N$. If $0 \neq r' \in R$, $Rr \cap Rr' \neq 0$ implies that $0 \neq r''r = r'''r'$ for some $0 \neq r'' \in R$, $0 \neq r''' \in R$. Then $0 \neq r''rm = r'''r'm \in N$, and, using the torsion-freeness of M, we have $0 \neq rm \in N^c$.

If m_1, m_2 are in N^c, then $0 \neq r_1 m_1 \in N$, $0 \neq r_2 m_2 \in N$ for some

$0 \neq r_1, r_2 \in R$. Then $Rr_1 \cap Rr_2 \neq 0$ implies that

$$0 \neq \varrho_1 r_1 = \varrho_2 r_2$$

for some $\varrho_1, \varrho_2 \in R$. Therefore

$$0 \neq \varrho_1 r_1(m_1 + m_2) = \varrho_1 r_1 m_1 + \varrho_2 r_2 m_2 \in N;$$

i.e. $m_1 + m_2 \in N^c$. Thus N^c contains with any $m \neq 0$ also rm, and with $m_1, m_2 \in N^c$ also $m_1 + m_2 \in N^c$. Thus N^c is a submodule of M, and (by theorem 1 (ii)), this implies (i) and (ii).

COROLLARY 4. *If R is a commutative integral domain, $N \neq 0$ a submodule of the torsionfree R-module M, then the pseudo-closure N^c of N is a submodule of M, and M has the « intersection property » for essentially closed submodules.*

From the theorems 1 and 2 it follows that the « intersection property » for essentially closed submodules of M is a necessary and sufficient condition therefore that the pseudo-closure N^c of any submodule N of M is a submodule of M.

We will give some other examples of sufficient conditions for R (resp. M) in order that M has the « intersection property » for essentially closed submodules of M. Therefore we define:

DEFINITION (β). Let N be a submodule of M; then the pair $(N; M)$ satisfies the condition (β), if 1_M is the only R-automorphism of M inducing 1_N.

The condition (β) of the pair $(N; M)$ is equivalent with each of the following conditions:

(β') every R-endomorphism f of N has at most one extension f on M;

(β'') $\mathrm{Hom}_R(M/N; M) = 0$.

We give some examples:

1) If N is an essential submodule of the non-singular R-module M, then M/N is singular, i.e. $\mathrm{Hom}(M/N; M) = 0$, and the pair $(N; M)$ satisfies (β).

2) If M is a rational extension of the submodule N, then $\mathrm{Hom}_R(M/N; M) = 0$, i.e. the pair $(N; M)$ satisfies (β).

The following result proves that the property (β)—in a special sense—is a sufficient condition therefore that the pseudo-closure N^c of a submodule N of M is a submodule of M.

THEOREM 5. *Let $N \neq 0$ be a submodule of M, $\hat{M}$ an injective hull of M; if we assume that the pair $(N; \hat{M})$ satisfies the condition (β), then:*

(i) *N has a unique maximal essential extension in M;*

(ii) *M has the «intersection property» for essentially closed submodules of M if the condition (β) holds for any pair $(N; \hat{M})$;*

(iii) *the pseudo-closure N^c of a submodule N is a submodule of M.*

PROOF. Let N_1, N_2 be two maximal essential extensions of N in M, and $\hat{N}_1$, resp. $\hat{N}_2$ the corresponding injective hulls of N in $\hat{M}$. Then $N_i = M \cap \hat{N}_i$ $(i = 1, 2)$. Furthermore there exists an isomorphism $\varphi: \hat{N}_1 \simeq \hat{N}_2$, $\varphi(n) = n$ $(\forall n \in N)$. Let N^* be an injective hull of a complement of N in M. Then $\hat{M} = \hat{N}_1 \oplus N^* \simeq \hat{N}_2 \oplus N^*$. Define: $\alpha \in \mathrm{End}_R(\hat{M})$ by $\alpha(\hat{N}_1) = \varphi(\hat{N}_1) = \hat{N}_2$, $\alpha(n^*) = n^*$ $(\forall n^* \in N^*)$. Then α is an R-automorphism of $\hat{M}$, inducing 1_N. Since the pair $(N; \hat{M})$ satisfies the condition (β), we have $\alpha = 1_{\hat{M}}$, i.e. $\hat{N}_1 = \hat{N}_2$, and therefore $N_1 = \hat{N}_1 \cap M = \hat{N}_2 \cap M = N_2$. Hence any submodule N of M has a unique maximal essential extension in M. Then the proof of (ii) and of (iii) follows from theorem 1.

LITERATURE

[1] Y. MIYASHITA, *On quasi-injective modules*, Journ. of the Fac. of Sc. Hokkado University, Ser. 1, Math., **18** (1965), pp. 158-187.

[2] B. STENSTRÖM, *Rings of Quotients*, Springer, 1975.

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