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Overdetermined Hyperbolic Systems on l.e. Convex Sets.

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Introduction.

In a recent paper [7] I discussed at some lenght the Cauchy problem on a half space for general systems of partial differential equations with constant coefficients. Here I turn to the same order of questions for convex sets in $\mathbb{R}^n$ having at least one extremal point. The two situations are closely related only in the case of determined systems. For overdetermined and underdetermined ones they are somehow at variance, as the algebraic invariants of the first are the associated prime ideals of a $\mathfrak{T}$-module M associated to the system, while those of the second are the prime ideals associated to the derived modules $\operatorname{Ext}^j_{\mathfrak{T}}(M, \mathfrak{T})$.

These results are the basis for the computation of local and global cohomology groups of some tangential complexes, giving an extension of some results obtained in [6].

1. Preliminaries.

A. *Spaces of functions and distributions.*

Let G be a locally closed set in $\mathbb{R}^n$. If Ω is any open neighborhood of G in $\mathbb{R}^n$ such that G is a closed subspace of Ω, then the space of

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complex valued, smooth functions on Ω with support contained in G turns out to be independent of the choice of Ω. It will be denoted by $\mathcal{E}_G$. In the same way we define $\mathcal{D}'_G$ as the space of distributions in Ω with support contained in G.

Let $\mathcal{F}_G(\Omega)$ denote the space of smooth functions in Ω that vanish on G with all derivatives. Then we define the Whitney functions on G as the elements of the quotient space W_G described by the exact sequence

$$0 \to \mathcal{F}_G(\Omega) \to \mathcal{E}_\Omega \to W_G \to 0 .$$

The space W_G is also independent of Ω. Note that $\mathcal{F}_G(\Omega)$ equals $\mathcal{E}_{\check{G}}$, for $\check{G}$ denoting the closure in Ω of $\Omega - G$, when G is the closure in Ω of its interior.

Under this last assumption, we define the space of extendible distributions on G (more precisely: distributions in the interior of G that extend to Ω) as the quotient $\check{\mathcal{D}}'_G$ described by the exact sequence:

$$0 \to \mathcal{D}'_{\check{G}} \to \mathcal{D}'_\Omega \to \check{\mathcal{D}}'_G \to 0 .$$

Again the space $\check{\mathcal{D}}'_G$ turns out to be independent of the choice of Ω. We say that a space $\mathcal{F}$ of functions or distributions on an open subset of $\mathbf{R}^n$ is a differential module if we have

$$\frac{\partial}{\partial x_j} f \in \mathcal{F} \qquad \forall f \in \mathcal{F} \text{ and } j = 1, \ldots, n .$$

We consider then $\mathcal{F}$ as a right and left module over the ring $\mathcal{P} = \mathbf{C}[\xi_1, \ldots, \xi_n]$ of polynomials in n indeterminates by the action

$$\xi_j \cdot f = f \cdot \xi_j = \frac{1}{i} \frac{\partial}{\partial x_j} f \qquad \text{for } f \in \mathcal{F} \text{ and } j = 1, \ldots, n .$$

B. *Boundary value problems for overdetermined systems.*

Let $A(D) = \big(A_{ij}(D)\big)_{1 \leq i \leq q, 1 \leq j \leq p}$ be a $q \times p$ matrix of linear partial differential operators with constant coefficients.

Let Ω be an open domain in $\mathbf{R}^n$ and let Σ be a smooth $(n-1)$-dimensional submanifold of $\mathbf{R}^n$ contained in the boundary $\partial\Omega$ of Ω.

Let F be a smooth complex-linear fiber bundle over Σ. A trace operator on $\mathbf{C}^p$-valued functions on Ω with values in the space $\Gamma(\Sigma, F)$ of smooth sections of F over Σ is a linear and continuous map

$$\beta\colon W^p_\Sigma \to \Gamma(\Sigma, F)$$

preserving the supports. In any local trivialization, it can be expressed by the composition of the action of a matrix of linear partial differential operators with smooth coefficients on $\mathbf{C}^p$-valued smooth functions and of the restriction to Σ of a $\mathbf{C}^d$-valued smooth function ($d = \dim_{\mathbf{C}}$ of the fibers of F).

We can consider then a very general boundary value problem for the operator $A(D)$ and a trace operator β by giving:

$$f \in W^q_{\Sigma\cup\Omega} \quad \text{and} \quad u_0 \in \Gamma(\Sigma, F)$$

and searching for a

$$u \in W^p_{\Sigma\cup\Omega} \text{ satisfying}$$

$$A(D)u = f \quad \text{on } \Omega \tag{1}$$

$$\beta(u) = u_0 \quad \text{on } \Sigma\,. \tag{2}$$

To solve such a problem, it is necessary that f and u^0 satisfy suitable compatibility conditions.

It is well known (cf. [2]) that necessary and sufficient conditions for (1) being locally solvable in Ω can be expressed in the form $B(D)f = 0$ by a new matrix of constant coefficients differential operators $B(D) = (B_{ij}(D))_{1\leq i\leq r,\, 1\leq j\leq q}$, that is determined in a purely algebraic way by the requirement that

$$\mathfrak{P}^r \xrightarrow{{}^tB(\xi)} \mathfrak{P}^q \xrightarrow{{}^tA(\xi)} \mathfrak{P}^p \tag{3}$$

be an exact sequence of $\mathfrak{P}$-modules and $\mathfrak{P}$-homomorphisms.

It is also an experimental truth that it is advantageous to insert the exact sequence (3) into a Hilbert resolution:

$$0 \to \mathfrak{P}^{p_d} \xrightarrow{{}^tA_{d-1}(\xi)} \mathfrak{P}^{p_{d-1}} \to \dots \to \mathfrak{P}^{p_2} \xrightarrow{{}^tA_1(\xi)} \mathfrak{P}^{p_1} \xrightarrow{{}^tA_0(\xi)} \mathfrak{P}^{p_0} \to M \to 0 \tag{4}$$

where either $M = 0$, or the torsion of M is different from zero (cf. [2]), in such a way that $A = A_j$ and $B = A_{j+1}$ for some j with $0 \leq j < d \leq n+1$ and j is as large as possible.

Then, if for instance β is the zero operator and $u^0 = 0$, the obstruction to solve the equation (1) for all f satisfying the compatibility condition

$$B(D)f = 0 \tag{5}$$

is expressed by the group

$$\operatorname{Ext}^j(M, W_{\Sigma\cup\Omega}) \simeq \frac{\operatorname{Ker}\left(B(D)\colon W^q_{\Sigma\cup\Omega} \to W^t_{\Sigma\cup\Omega}\right)}{\operatorname{Im}\left(A(D)\colon W^p_{\Sigma\cup\Omega} \to W^q_{\Sigma\cup\Omega}\right)}, \tag{6}$$

and this isomorphism shows that all invariants of the problem should be expressed only in terms of the module M and the space $W_{\Sigma\cup\Omega}$.

C. *Complex characteristic variety and non-characteristic hypersurfaces.*

Given a point $\xi_0 \in \mathbb{C}^n$, we denote by $\mathfrak{w}_{\xi_0}$ the maximal ideal in $\mathfrak{T}$ of all polynomials vanishing at ξ_0. The quotient $\mathfrak{T}/\mathfrak{w}_{\xi_0}$ is isomorphic to $\mathbb{C}$, so that by tensoring the Hilbert resolution (4) by $\mathfrak{T}/\mathfrak{w}_{\xi_0}$ we obtain a complex of finite dimensional vector spaces and linear maps

$$\begin{aligned} (7)\qquad & 0 \to \mathbb{C}^{p_d} \xrightarrow{{}^tA_{d-1}(\xi_0)} \mathbb{C}^{p_{d-1}} \to \dots \to \\ & \qquad \to \mathbb{C}^{p_2} \xrightarrow{{}^tA_1(\xi_0)} \mathbb{C}^{p_1} \xrightarrow{{}^tA_0(\xi_0)} \mathbb{C}^{p_0} \to M \otimes \mathfrak{T}/\mathfrak{w}_{\xi_0} \to 0\,. \end{aligned}$$

The cohomology groups of this complex are the modules

$$\operatorname{Tor}_j^{\mathfrak{T}}(M, \mathfrak{T}/\mathfrak{w}_{\xi_0}) \qquad (j = 0, 1, \dots, d)\,.$$

If $\operatorname{Tor}_j^{\mathfrak{T}}(M, \mathfrak{T}/\mathfrak{w}_{\xi_0})$ is zero for some j_0, then it is also equal to zero for all larger j. Indeed the largest integer j, for which this group is different from zero, is the homological dimension of the localization of M with respect to $\mathfrak{w}_{\xi_0}$. If we set

$$V_j = \{\xi \in \mathbb{C}^n \colon \operatorname{Tor}_j^{\mathfrak{T}}(M, \mathfrak{T}/\mathfrak{w}_{\xi}) \neq 0\}$$

we obtain a sequence of affine algebraic varieties with

$$\{0\} \subset V_d \subset V_{d-1} \subset \dots \subset V_1 \subset V_0\,.$$

One usually defines V_0 as the complex characteristic variety of M, although sometimes it is necessary to consider instead the disjoint union of the affine algebraic varieties of the prime ideals associated to M. Let us set $V_0 = V(M)$.

Notice that, as

$$\operatorname{Ker}\left(A_j\colon \mathfrak{T}^{p_j} \to \mathfrak{T}^{p_{j+1}}\right) \otimes \mathfrak{T}/\mathfrak{w}_\xi \subset \operatorname{Ker}\left(A_j(\xi)\colon \mathbf{C}^{p_j} \to \mathbf{C}^{p_{j+1}}\right)$$

and the dimension over $\mathbf{C}$ of

$$\frac{\operatorname{Ker} A_j(\xi)}{\operatorname{Im} A_{j-1}(\xi)} \quad \text{and of} \quad \frac{\operatorname{Ker} {}^tA_{j-1}(\xi)}{\operatorname{Im} {}^tA_j(\xi)}$$

are the same, V_j is always included in the characteristic variety of $\operatorname{Ext}^j_{\mathfrak{T}}(M, \mathfrak{T})$.

To any affine algebraic variety V in $\mathbf{C}^n$ we associate a homogeneous affine variety $\tilde{V}$ in $\mathbf{C}^n$, that we call the asymptotic cone of V, defined as the set of all ξ in $\mathbf{C}^n$ that are limits of sequences $\{\varepsilon_m \xi_m\}$ with $\{\xi_m\} \subset V$ and $\{\varepsilon_m\} \subset \mathbf{C}$, $\varepsilon_m \to 0$.

Let ν_x denote, for all x in Σ, the exterior normal to Ω (we assume that Ω is, for each x in Σ, on one side with respect to Σ, i.e. that there is an open ball B around x such that $\Omega \cap B$ is one of the two connected components of $B - \Sigma$).

We say that Σ *is non-characteristic for* M if $\nu_x \notin \tilde{V}(M)$ for $x \in \Sigma$.

REMARK. If $\nu_x \in \tilde{V}(M)$, but $\nu_x \notin \tilde{V}_j$ for some j, one can « shorten » the Hilbert resolution (4), essentially by delating a certain numbers of rows of ${}^tA_j$ to obtain a matrix of maximal rank over the field of rational functions and having maximal rank as a $\mathbf{C}$-matrix at ν_x. We obtain in this way a new module M_j for which Σ is non-characteristic near x.

When Σ is non-characteristic for M, it was proved in [3] that we can construct a complex of smooth linear partial differential operators on fiber bundles over Σ (the *tangential* or *boundary complex*):

$$(8) \qquad \Gamma(\Sigma, F_0) \xrightarrow{\alpha_0} \Gamma(\Sigma, F_1) \xrightarrow{\alpha_1} \Gamma(\Sigma, F_2) \to \dots \to \Gamma(\Sigma, F_d) \to 0$$

and trace operators

$$\beta_j\colon W^{p_j}_{\Sigma} \to \Gamma(\Sigma, F_j) \qquad (\text{for } j = 0, 1, \dots)$$

with the property that

$$\beta_{j+1} \circ A_j = \alpha_j \circ \beta_j\,.$$

They induce therefore natural maps

$$\beta_*\colon \mathrm{Ext}^j_{\mathcal{T}}(M, W_\Sigma) \to H^j(\Gamma(\Sigma, F_*), \alpha_*) \tag{9}$$

and the canonical construction described in [3] makes β_* an isomorphism (formal Cauchy Kowalewska theorem).

One can also consider the tangential complex on distributions

$$\mathfrak{D}'(\Sigma, F_0) \xrightarrow{\alpha_0} \mathfrak{D}'(\Sigma, F_1) \xrightarrow{\alpha_1} \mathfrak{D}'(\Sigma, F_2) \to \dots \to \mathfrak{D}'(\Sigma, F_d) \to 0 \tag{8'}$$

and under the same assumptions one finds natural maps

$$\beta^*\colon H^j(\mathfrak{D}'(\Sigma, F_*), \alpha_*) \to \mathrm{Ext}^{j+1}_{\mathcal{T}}(M, \mathfrak{D}'_\Sigma) \tag{10}$$

that turn out to be isomorphisms (cf. [8]).

Tangential complexes give very interesting examples of complexes of linear partial differential operators with smooth coefficients and are important in some geometrical problems, as the tangential Cauchy-Riemann complex considered in [8].

However, the isomorphisms given by (9) and (10) allow us to disregard completely the boundary complex and to work with the groups $\mathrm{Ext}^j_{\mathcal{T}}(M, W_\Sigma)$ and $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathfrak{D}'_\Sigma(\Omega))$ that are well defined even if we drop all assumptions on Σ, only requiring that it is relatively open in $\partial\Omega$.

Before ending this subsection, I want to state in the smooth non-characteristic case the Cauchy problem for functions and distributions.

Cauchy Problem for functions: at the step j, $j \geqq 1$:

Given $f \in W^{p_j}_{\Sigma \cup \Omega}$ *and* $u^0 \in \Gamma(\Sigma, F_{j-1})$, *satisfying the compatibility conditions*:

$$A_j(D) f = 0 \qquad \text{on } \Omega \tag{11}$$

$$\alpha_{j-1}(u^0) = \beta_j(f) \qquad \text{on } \Sigma \tag{12}$$

find $u \in W^{p_{j-1}}_{\Sigma \cup \Omega}$ *such that*

$$A_{j-1}(D) u = f \qquad \text{on } \Omega \tag{13}$$

$$\beta_{j-1}(u) = u_0 \qquad \text{on } \Sigma\,. \tag{14}$$

Note that β^* is induced by maps $\beta'_j\colon \mathfrak{D}'(\Sigma, F_j) \to \mathfrak{D}'^{p_{j+1}}_{\Sigma}$, with

$$A_{j+1}\circ\beta'_j = \beta'_{j+1}\circ\alpha_j\,.$$

Cauchy problem for distributions:

Given $f \in \mathfrak{D}'^{p_j}_{\Sigma\cup\Omega}$ *and* $u_0 \in \mathfrak{D}'(\Sigma, F_{j-1})$ *satisfying the compatibility condition:*

$$A_j(D)f + \beta'_j(\alpha_{j-1}(u_0)) = 0 \tag{15}$$

find $u \in \mathfrak{D}'^{p_{j+1}}_{\Sigma\cup\Omega}$ *such that*

$$A_{j-1}(D)u = f + \beta'_{j-1}(u^0)\,. \tag{16}$$

Let us consider the exact sequences

$$0 \to \mathcal{E}_{\Sigma\cup\Omega} \to W_{\Sigma\cup\Omega} \to W_\Sigma \to 0 \tag{17}$$

and

$$0 \to \mathfrak{D}'_\Sigma \to \mathfrak{D}'_{\Sigma\cup\Omega} \to \check{\mathfrak{D}}'_{\Sigma\cup\Omega} \to 0\,, \tag{18}$$

and the corresponding long exact sequences for the Ext functor:

$$\begin{aligned}0 &\to \mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_{\Sigma\cup\Omega}) \to \mathrm{Ext}^0_{\mathcal{T}}(M, W_{\Sigma\cup\Omega}) \to \mathrm{Ext}^0_{\mathcal{T}}(M, W_\Sigma) \to\\ &\to \mathrm{Ext}^1_{\mathcal{T}}(M, \mathcal{E}_{\Sigma\cup\Omega}) \to \mathrm{Ext}^1_{\mathcal{T}}(M, W_{\Sigma\cup\Omega}) \to \mathrm{Ext}^1_{\mathcal{T}}(M, W_\Sigma) \to\\ &\to \mathrm{Ext}^2_{\mathcal{T}}(M, \mathcal{E}_{\Sigma\cup\Omega}) \to \dots\end{aligned} \tag{19}$$

and

$$\begin{aligned}0 &\to \mathrm{Ext}^0_{\mathcal{T}}(M, \mathfrak{D}'_\Sigma) \to \mathrm{Ext}^0_{\mathcal{T}}(M, \mathfrak{D}'_{\Sigma\cup\Omega}) \to \mathrm{Ext}^0_{\mathcal{T}}(M, \check{\mathfrak{D}}'_{\Sigma\cup\Omega}) \to\\ &\to \mathrm{Ext}^1_{\mathcal{T}}(M, \mathfrak{D}'_\Sigma) \to \mathrm{Ext}^1_{\mathcal{T}}(M, \mathfrak{D}'_{\Sigma\cup\Omega}) \to \mathrm{Ext}^1_{\mathcal{T}}(M, \check{\mathfrak{D}}'_{\Sigma\cup\Omega}) \to\\ &\to \mathrm{Ext}^2_{\mathcal{T}}(M, \mathfrak{D}'_\Sigma) \to \dots\,.\end{aligned} \tag{20}$$

Then one realizes that the compatibility conditions (11)-(12) state, due to the formal Cauchy-Kowalewska theorem, only that f defines a cohomology class in $\mathrm{Ext}^j_{\mathcal{T}}(M, W_{\Sigma\cup\Omega})$ that is mapped into the zero class in $\mathrm{Ext}^j_{\mathcal{T}}(M, W_\Sigma)$.

To solve (13), it is then necessary that f defines the zero cohomology class also in $\mathrm{Ext}^j_{\mathcal{T}}(M, W_{\Sigma\cup\Omega})$. In this case we can find

$w \in W^{p_{j-1}}_{\Sigma \cup \Omega}$ such that $A_{j-1}(D)w = f$ on Ω and it follows that $\alpha_{j-1}(u^0 - \beta_{j-1}(w)) = 0$ on Σ.

A necessary and sufficient condition in order that problem (13)-(14) could be solved is then the fact that the cohomology class of $\mathrm{Ext}^{j-1}_{\mathfrak{T}}(M, W_\Sigma)$ defined by $u^0 - \beta_{j-1}(w)$ be the image of an element of $\mathrm{Ext}^{j-1}_{\mathfrak{T}}(M, W_{\Sigma \cup \Omega})$.

In conclusion we have, under all the assumptions made above:

PROPOSITION 1. *A necessary and sufficient condition in order that the Cauchy problem* (13)-(14) *be solvable for all* $f \in W^{p_j}_{\Sigma \cup \Omega}$ *and* $u^0 \in \Gamma(\Sigma, F_{j-1})$ *satisfying the compatibility conditions* (11)-(12), *is that*

$$\mathrm{Ext}^{j}_{\mathfrak{T}}(M, \mathcal{E}_{\Sigma \cup \Omega}) = 0 .$$

Analogously, arguing in the same way for the case of distributions, we have

PROPOSITION 2. *A necessary and sufficient condition in order that the Cauchy problem* (16) *be solvable for all* $f \in \mathcal{D}'^{p_j}_{\Sigma \cup \Omega}$ *and* $u^0 \in \mathcal{D}'(\Sigma, F_{j-1})$ *satisfying the compatibility condition* (15), *is that*

$$\mathrm{Ext}^{j}_{\mathfrak{T}}(M, \mathcal{D}'_{\Sigma \cup \Omega}) = 0 .$$

REMARK. Under all assumptions above, if the condition

$$\mathrm{Ext}^{j}_{\mathfrak{T}}(M, \mathcal{E}_{\Sigma \cup \Omega}) = 0$$

holds, then problem (1)-(2) will reduce to a problem on the boundary: one has to find $w \in W^{p_j}_{\Sigma}$ such that:

$$\begin{cases} \alpha_j(\beta_j(w)) = \beta_{j+1}(f) \\ \beta(w) = u^0 . \end{cases}$$

2. A theorem of flatness.

Let H be a closed convex subset of $\mathbf{R}^n$. We say that H is *linearly exhausted*, or in short that is l.e., if we can find $\xi \in \mathbf{R}^n$ such that, for every $c \in \mathbb{R}$, the set

$$H_c = \{x \in H : \langle x, \xi \rangle \leqq c\} \tag{1}$$

is compact.

We recall that a point $x_0 \in H$ is said to be an extremal point of H if it is not an interior point of a segment with end-points in H.

We have

LEMMA 1. *A necessary and sufficient condition in order that a closed convex set H be l.c. is that it contains at least an extremal point.*

PROOF. – Assume that x_0 be an extremal point of H and let B be the open unit ball about x_0. If $H - B$ is empty, then H is compact and hence trivially l.c. Assume therefore that $H - B$ is not empty and let K be its convex envelope. I claim that K is closed and does not contain x_0. Indeed, if x is in $K \cap B$, then we write x as a linear combination

$$x = \lambda_1 x_1 + \ldots + \lambda_k x_k \tag{2}$$
$$\text{with } 0 < \lambda_i < 1\,, \ \lambda_1 + \ldots + \lambda_k = 1\,, \ x_1, \ldots, x_k \in H - B\,.$$

We can assume that the number k is minimal. As x belongs to a simplex with vertices in $H - B$, we have $k \leqq n + 1$.

Let us set now

$$y = (\lambda_1 + \lambda_2)^{-1}(\lambda_1 x_1 + \lambda_2 x_2)\,.$$

It cannot be a point in $H - B$, because otherwise we could express x as a linear combination of $k - 1$ vectors. Hence y is in B and therefore it can be written as a linear combination

$$y = \mu_1 \tilde{x}_1 + \mu_2 \tilde{x}_2, \qquad 0 < \mu_1, \mu_2 < 1\,, \qquad \mu_1 + \mu_2 = 1\,,$$

of the intersections $\tilde{x}_1, \tilde{x}_2$ of the line through x_1, x_2 with the boundary ∂B of B. It follows then that

$$x = [(\lambda_1 + \lambda_2)\mu_1]\tilde{x}_1 + [(\lambda_1 + \lambda_2)\mu_2]\tilde{x}_2 + \lambda_3 x_3 + \ldots + \lambda_k x_k$$

is a linear combination of the form (2) but with the first two vectors belonging to $\partial B \cap H$.

Iterating this argument, we prove that every vector x in $K \cap B$ can be expressed as a linear combination

$$x = \lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1}$$

with $0 \leqq \lambda_i \leqq 1$, $\lambda_1 + \ldots + \lambda_{n+1} = 1$, $x_1, \ldots, x_{n+1} \in H \cap \partial B$.

Being the image under a continuous map of the topological product of the n-dimensional standard simplex

$$\{(\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{R}^{n+1} \colon 0 \leqq \lambda_i \leqq 1 \text{ and } \lambda_1 + \ldots + \lambda_{n+1} = 1\}$$

and of $(n+1)$ copies of $H \cap \partial B$, the set $K \cap \bar{B}$ is compact and hence $K = (H - B) \cup (K \cap \bar{B})$ is closed. Clearly $x_0 \notin K$ because x_0 does not belong to any simplex in H unless as a vertex.

By Hahn-Banach theorem, we can find then $\xi \in \mathbb{R}^n$ such that

$$\langle x_0, \xi \rangle < m = \inf_K \langle x, \xi \rangle .$$

It is obvious that H_c is compact for $\langle x_0, \xi \rangle \leqq c < m$, as H_c is contained in B and is closed. Then it follows that H_c is compact for every c.

Viceversa, if $\xi \in \mathbb{R}^n$ gives a linear exhaustion of H, let m be the minimum of $x \to \langle x, \xi \rangle$ in H. Then H_m is compact and by the Krein-Milman's theorem has an extremal point, that turns out to be also an extremal point of H.

From the proof of the first implication in the above lemma, taking instead of the unit ball about x_0 balls of arbitrarily small radii $\varepsilon > 0$, we obtain also

LEMMA 2. *A necessary and sufficient condition in order that a point $x_0 \in \partial H$ has a fundamental system of open neighborhoods in H (resp. in ∂H) of the form*

$$\{x \in H \colon \langle x, \xi \rangle < R\} \quad (\text{resp. } \{x \in \partial H \colon \langle x, \xi \rangle < R\}) \qquad (R \in \mathbb{R})$$

is that x_0 be an extremal point of H.

For $\xi \in \mathbb{R}^n$ and $R \in \mathbb{R} \cup \{+\infty\}$, we set

$$H_\xi(R) = H(R) = \{x \in H \colon \langle x, \xi \rangle < R\} .$$

Then we have, for a l.e. convex set H:

THEOREM 1. *If $\xi \in \mathbb{R}^n$ gives a l.e. of H, then $\mathcal{E}_{H_\xi(R)}$ is a flat differential module for every R with $-\infty < R \leqq +\infty$. In particular $\mathcal{E}_H$ is flat if H is l.e.*

PROOF. We have to prove that, if M is a unitary $\mathcal{P}$-module of finite type, then

$$\operatorname{Tor}_j^{\mathcal{P}}(M, \mathcal{E}_{H(R)}) = 0 \qquad \text{for } j \geqq 1 .$$

Having fixed M, let (1.4) be a Hilbert resolution of M. Having taken $j \geqq 1$, we have to show that, if $f \in \mathcal{E}_{H(R)}^{p_j}$ satisfies the integrability condition

$$ {}^tA_{j-1}(D) f = 0 , \tag{3}$$

then we can find $u \in \mathcal{E}_{H(R)}^{p_{j+1}}$ such that

$$ {}^tA_j(D) u = f . \tag{4}$$

For fixed r, s with

$$m = \min_H \langle x, \xi \rangle < r < s < R ,$$

let $\chi = \chi_{r,s}(t)$ be a smooth function of one real variable with

$$\chi(t) = 1 \quad \text{for } t < r , \qquad \chi(t) = 0 \quad \text{for } t > s .$$

The function

$$\tilde{f} = f_{r,s} = \chi(\langle x, \xi \rangle) \cdot f(x)$$

has support contained in H_s and hence compact and

$$ {}^tA_{j-1}(D) \tilde{f}$$

has support in $H_s - H(r)$.

Because, by proposition 2, p. 220 in [5] we have

$$({}^tA_{j-1}(D) \mathcal{D}(\mathbb{R}^n)) \cap \mathcal{E}_{H_s - H(r)}^{p_{j-1}} = {}^tA_{j-1}(D)\, \mathcal{E}_{H_s - H(r)}^{p_j} ,$$

we can find then

$$\tilde{g} = g_{rs} \in \mathcal{E}^{p_j}_{H_s - H(r)}$$

such that

$${}^tA_{j-1}(D)\tilde{f} = {}^tA_{j-1}(D)\tilde{g} \,.$$

The difference $\tilde{f} - \tilde{g}$ is therefore a function in $\mathcal{E}^{p_j}_{H_s}$ satisfying

$${}^tA_{j-1}(D)(\tilde{f} - \tilde{g}) = 0 \,.$$

Because $\mathcal{E}^{p_j}_{H_s}$ is flat (cf. [5]), we can find $\tilde{u} = u_{rs} \in \mathcal{E}^{p_{j+1}}_{H_s}$ such that

$$\tilde{f} - \tilde{g} = {}^tA_j(D)\tilde{u} \,.$$

As a consequence, we have found that for every r, s with $m < r < < s < R$ there is $u_{rs} \in \mathcal{E}^{p_{j+1}}_{H_s}$ such that

$${}^tA_j(D)\, u_{rs} = f \quad \text{ on } H(r) \,.$$

Let us take now an increasing sequence $\{r_k\}$ with

$$m < r_1 < r_2 < \ldots < r_k < r_{k+1} < \sup_k r_k = R \,.$$

We can construct by induction a sequence $\{u_k\} \subset \mathcal{E}^{p_{j+1}}_{H(R)}$ with the properties:

$$\operatorname{supp} u_k \subset H_{r_{k+1}}$$

$$u_{k+1} = u_k \quad \text{ on } H(r_k)$$

$${}^tA_j(D)\, u_k = f \quad \text{ on } H(r_k) \,.$$

Indeed we can take

$$u_1 = u_{r_1 r_2} \,.$$

Assuming we have found $u_1, \ldots, u_k$, we consider

$$u_{r_{k+1}, r_{k+2}} - u_k \,.$$

Its support is contained in $H_{r_{k+2}}$ and

$${}^tA_j(D)(u_{r_{k+1}, r_{k+2}} - u_k)$$

has support contained in $H_{r_{k+1}} - H(r_k)$.

Hence we can find

$$v \in \mathcal{E}^{p_j}_{H_{r_{k+2}}-H(r_k)}$$

such that

$${}^tA_j(D)(u_{r_{k+1},r_{k+2}} - u_k) = {}^tA_j(D)v\,.$$

We set then

$$u_{k+1} = u_k + v\,.$$

Clearly

$$\operatorname{supp} u_{k+1} \subset H_{r_{k+2}}$$

$${}^tA_j(D)u_{k+1} = {}^tA_j(D)u_{r_{k+1},r_{k+2}} = f \qquad \text{on } H(r_{k+1})$$

$$u_{k+1} = u_k \qquad \text{on } H(r_k)\,.$$

Having obtained the sequence $\{u_k\}$, as it is locally constant we can compute its limit

$$u = \lim_{k\to\infty} u_k \in \mathcal{E}^{p_{j+1}}_{H(R)}$$

and observe that

$${}^tA_j(D)u = f \qquad \text{on } H(R)\,.$$

The proof is complete.

The same proof applies with only minor changes to yield:

THEOREM 1′. *If $\xi \in \mathbb{R}^n$ gives a l.e. of H, then $\mathcal{D}'_{H_\xi(R)}$ is a flat differential module for every R in* $\mathbb{R}$.
In particular $\mathcal{D}'_H$ is flat when H is l.e.

From the theorem above, we have

COROLLARY 1. *With the assumption of Theorem* 1: *for every unitary $\mathcal{T}$-module of finite type M and $\min\limits_H \langle x, \xi\rangle < R \leqq +\infty$ we have*

$$\operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_{H(R)}) \cong \operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{T}) \otimes \mathcal{E}_{H(R)}\,,$$

$$\operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_{H(R)}) \cong \operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{T}) \otimes \mathcal{D}'_{H(R)}\,.$$

For the study of the vanishing of these groups, we can reduce to prime ideals by means of an algebraic theorem:

PROPOSITION 3. *Let M be a unitary $\mathfrak{I}$-module of finite type and let N be a flat $\mathfrak{I}$-module. Then a necessary and sufficient condition in order that*

$$M \otimes N = 0$$

is that

$$(\mathfrak{I}/\mathscr{P}) \otimes N = 0 \qquad \textit{for every } \mathscr{P} \in \operatorname{Ass}(M) .$$

PROOF. – If $\mathscr{P}$ is a prime ideal in Ass (M), then we can find a submodule M_1 of M isomorphic to $\mathfrak{I}/\mathscr{P}$. From the exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

we deduce, because N is flat, an exact sequence

$$0 \to M_1 \otimes N \to M \otimes N \to (M/M_1) \otimes N \to 0 .$$

Clearly if the term in the middle is zero, also

$$M_1 \otimes N \cong (\mathfrak{I}/\mathscr{P}) \otimes N \text{ is zero} .$$

Hence the condition is necessary.

Let us prove the sufficiency. First we consider the case of a $\mathscr{P}$-comprimary module M. We argue by induction on the smallest positive integer k such that

$$\mathscr{P}^k M = 0 .$$

If $k = 1$, then M is torsion free as a $\mathfrak{I}/\mathscr{P}$-module and we can build up an exact sequence

$$0 \to M \to (\mathfrak{I}/\mathscr{P})^h \to Q \to 0 .$$

Tensoring by N we have the exact sequence

$$0 \to M \otimes N \to (\mathfrak{I}/\mathscr{P})^h \otimes N \to Q \otimes N \to 0$$

and then $M \otimes N = 0$ because $(\mathfrak{I}/\mathscr{P})^h \otimes N \cong [(\mathfrak{I}/\mathscr{P}) \otimes N]^h = 0$.

Assume now that $k \geqq 2$ and that $M' \otimes N = 0$ for all $\mathscr{P}$-coprimary modules M' of finite type such that $\mathscr{P}^l M' = 0$ for some l with $0 < l < k$.

Then we set

$$M_0 = \{m \in M | \mathscr{P} m = 0\} .$$

We note that M_0 and M/M_0 are both $\mathscr{P}$-coprimary of finite type and that

$$\mathscr{P} M_0 = 0 , \qquad \mathscr{P}^{k-1}(M/M_0) = 0 .$$

From the exact sequence

$$0 \to M_0 \to M \to M/M_0 \to 0$$

we have then an exact sequence

$$0 \to M_0 \otimes N \to M \otimes N \to (M/M_0) \otimes N \to 0$$

in which the first and third terms are zero, so that also the second one is zero.

We can argue now, having dropped the assumption that M be $\mathscr{P}$-coprimary, by induction on the number of prime ideals in Ass (M). Then we note that, if Φ is any subset of Ass (M), we can find a submodule M_1 of M such that

$$\text{Ass}\,(M_1) = \Phi \quad \text{ and } \text{Ass}\,(M/M_1) = \text{Ass}\,(M) - \Phi .$$

From the exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

we deduce an exact sequence

$$0 \to M_1 \otimes N \to M \otimes N \to (M/M_1) \otimes N \to 0$$

in which the terms $M_1 \otimes N$ and $(M/M_1) \otimes N$ are zero if we assume that $\phi \neq \Phi \neq \text{Ass}\,(M)$ and that the statement is true for modules having a smaller number of associated prime ideals than M. The proof is complete.

3. Propagation cones and fundamental solutions for hyperbolic systems.

We need to rehearse some notations and results of [7] about hyperbolic $\mathfrak{T}$-modules. We restrict here to the case of $\mathfrak{T}$-modules of the form $\mathfrak{T}/\mathscr{P}$, with $\mathscr{P}$ a prime ideal.

Let $V(\mathscr{P})$ be the affine algebraic variety of common zeroes of polynomials in $\mathscr{P}$. We consider the semi-algebraic cone $\tilde{V}^{\mathbf{R}}(\check{\mathscr{P}})$ in $\mathbf{R}^n$, defined as the set of limits of sequences $\{\varepsilon_m \operatorname{Re}(i\zeta_m)\}$ with $\{-\zeta_m\} \subset \subset V(\mathscr{P})$ and $\varepsilon_m > 0$, $\varepsilon_m \to 0$. The direction ν in $\mathbf{R}^n - \{0\}$ is said to be hyperbolic for $\mathfrak{T}/\mathscr{P}$ if ν does not belong to $\tilde{V}^{\mathbf{R}}(\check{\mathscr{P}})$.

When $\mathscr{P}$ is a principal ideal, $\tilde{V}^{\mathbf{R}}(\check{\mathscr{P}})$ is the set of all imaginary parts of elements of $\tilde{V}(\mathscr{P})$ and $\mathbf{R}^n - \tilde{V}^{\mathbf{R}}(\check{\mathscr{P}})$ is either empty or the reunion of disjoint open convex cones. If $\mathscr{P} = (P)$, then, assuming $\mathbf{R}^n - - V^{\mathbf{R}}(\check{\mathscr{P}}) \neq \emptyset$, for each connected component Γ of $\mathbf{R}^n - \tilde{V}^{\mathbf{R}}(\check{\mathscr{P}})$ there s a fundamental solution $E \in \mathfrak{D}'_{\Gamma^0}$ for P.

No one of these statements remains true in general, as has been shown in [7]. However, we have the following

THEOREM 2. *Let $\mathscr{P}$ be a prime ideal in $\mathfrak{T}$, generated by polynomials $P_1, \ldots, P_s$.*

Let Γ be an open convex cone in $\mathbf{R}^n$ such that

$$\bar{\Gamma} \cap \tilde{V}^{\mathbf{R}}(\check{\mathscr{P}}) \subset \{0\} . \tag{1}$$

Then we can find distributions $E_1, \ldots, E_s \in \mathfrak{D}'_{\Gamma^0}$ such that

$$P_1(D)E_1 + \ldots + P_s(D)E_s = \delta . \tag{2}$$

PROOF. For $\nu \in \Gamma$, we set for every real $R > 0$,

$$\Gamma^0_R = \{x \in \Gamma^0 : \langle x, \nu\rangle \leq R\}$$

$$\Gamma^0(R) = \{x \in \Gamma^0 : \langle x, \nu\rangle < R\}$$

$$K_R = \{x \in \Gamma^0 : \langle x, \nu\rangle = R\} .$$

Let h_{K_R} be the support function of K_R:

$$h_{K_R}(\xi) = \sup\{\langle x, \xi\rangle : x \in K_R\} . \tag{3}$$

By the assumption (1), we have

$$h_{K_R}(-\theta) > 0 \quad \text{for } \theta \in \tilde{V}^{\mathbf{R}}(\check{\mathscr{P}}) - \{0\}\,. \tag{4}$$

Indeed, as $\bar{\Gamma} = (\Gamma^0)^0$, from $0 \notin \bar{\Gamma}$ it follows that we can find x in Γ^0 such that $\langle x, \theta\rangle < 0$. Noting that $\langle x, \nu\rangle > 0$, we have for $x_0 = R\langle x, \nu\rangle^{-1}x$, that $x_0 \in K_R$ and $\langle x_0, \theta\rangle < 0$; this implies (4). By the definition of $\tilde{V}^{\mathbf{R}}(\check{\mathscr{P}})$, property (4) implies that $h_{K_R}(\operatorname{Im}\zeta)$ is bounded from below on $V(\mathscr{P})$.

Then the function 1 on $V(\mathscr{P})$ can be extended to an entire function $F(\zeta)$ in $\mathbf{C}^n$ satisfying an inequality of the form

$$|F(\zeta)| \leqq C(1 + |\zeta|)^N \exp H_{K_R}(\operatorname{Im}\zeta) \qquad \text{for } \zeta \in \mathbf{C}^n\,. \tag{5}$$

Then $F(\zeta)$ is the Fourier-Laplace transform of a distribution w_R in $\mathbf{R}^n$ having compact support in K_R and we can find distributions $E_1^R, \ldots, E_s^R \in \mathfrak{D}'_{\Gamma_R^0}$ such that

$$\delta - w_R = P_1(D)E_1^R + \ldots + P_s(D)E_s^R\,.$$

I claim that it is possible to construct a sequence $\{(\tilde{E}_1^k, \ldots, \tilde{E}_s^k)\} = \{\tilde{E}^k\}$ with the properties

$$\tilde{E}_j^k \in \mathfrak{D}'_{\Gamma_k^0} \quad \text{for } j = 1, \ldots, s \tag{6}$$

$$\tilde{E}_j^{k+1} = \tilde{E}_j^k \quad \text{on } \Gamma^0(k) \quad \text{for } j = 1, \ldots, s \tag{7}$$

$$P_1(D)\tilde{E}_1^k + \ldots + P_s(D)\tilde{E}_s^k = \delta \quad \text{on } \Gamma^0(k)\,. \tag{8}$$

We prove the existence of the sequence $\{\tilde{E}^k\}$ by induction. We set first

$$\tilde{E}_1 = (E_1^1, \ldots, E_s^1)\,.$$

Then (6) and (8) are satisfied for $k = 1$, while (7) is empty for $k = 0$.

Assume we have found $\tilde{E}^1, \ldots, \tilde{E}^m$ so that (6) and (8) hold for $1 \leqq k \leqq m$ and (7) holds for $1 \leqq k \leqq m - 1$. Then we consider

$$E^{m+1} - \tilde{E}^m\,.$$

We have $E^{m+1}-\tilde{E}^m \in \mathfrak{D}'^s_{\Gamma^0_{m+1}}$ and

$$v = P_1(D)(E_1^{m+1}-\tilde{E}_1^m)+\ldots+P_s(D)(E_s^{m+1}-\tilde{E}_s^m) \in \mathfrak{D}'_{\Gamma^0_{m+1}-\Gamma^0(m)}\,.$$

Then we can find $g \in \mathfrak{D}'^s_{\Gamma^0_{m+1}-\Gamma^0(m)}$ such that

$$P_1(D)g_1+\ldots+P_s(D)g_s = v \tag{10}$$

and we can set

$$\tilde{E}^{m+1} = \tilde{E}^m + g\,. \tag{11}$$

With this choice (6) and (8) are now satisfied for $1 \leqq k \leqq m+1$ and (7) for $1 \leqq k \leqq m$, so the inductive statement is proved.

Being locally constant, the sequence $\{\tilde{E}^k\}$ converges to $E \in \mathfrak{D}'^s_{\Gamma^0}$, for which clearly we have

$$P_1(D)E_1+\ldots+P_s(D)E_s = \delta\,.$$

Such an $E \in \mathfrak{D}'^s_{\Gamma^0}$ is called a *fundamental solution* for $\mathcal{T}/\mathscr{P}$.

REMARK. Let $n=3$ and let the ideal $\mathscr{P}$ be generated by $\zeta_1^2+i\zeta_2$ and $\zeta_2^2+\zeta_3^2$. Then $\tilde{V}^{\mathbf{R}}(\check{\mathscr{P}})$ is the plane $\{\theta \in \mathbf{R}^3\colon \theta_1 = 0\}$, so that $\Gamma = \{\theta_1 > 0\}$ is an open cone in $\mathbf{R}^3 - \tilde{V}^{\mathbf{R}}(\check{\mathscr{P}})$, but there is no fundamental solution for $\mathcal{T}/\mathscr{P}$ with support contained in

$$\Gamma^0 = \{x \in \mathbf{R}^3\colon x_1 \geqq 0,\, x_2 = x_3 = 0\}\,,$$

although there are fundamental solutions with support in any closed convex cone with vertex at 0 containing $(1, 0, 0)$ as an interior point.

4. Hyperbolicity with respect to a l.c. convex domain.

Let H be a proper closed convex set in $\mathbf{R}^n$.

To each point x_0 of the boundary ∂H of H in $\mathbf{R}^n$ we associate a proper closed convex cone $\Gamma(x_0, H)$, defined as the set of all $\xi \in \mathbf{R}^n$ such that

$$\langle x_0, \xi\rangle \leqq \langle x, \xi\rangle \quad \forall x \in H\,. \tag{1}$$

In convex analysis, $\Gamma(x_0, H)$ is called the supporting cone of H at x_0 and is defined as the sub-gradient of the convex function that equals 0 on H and $+\infty$ on $\mathbb{R}^n - H$. Note that we have a monotonicity property:

$$\langle x_1 - x_0, \xi_1 - \xi_0 \rangle \geqq 0 \tag{2}$$

if $x_0, x_1 \in \partial H$ and $\xi_1 \in \Gamma(x_1, H)$, $\xi_0 \in \Gamma(x_0, H)$.

Let us also define $\Gamma_\infty(H) = \{\xi \in \mathbb{R}^n : \inf_{x\in H} \langle x, \xi \rangle > -\infty\}$. Clearly this is a convex set. We have:

LEMMA 3. *A necessary and sufficient condition in order that $\Gamma_\infty(H)$ have an interior point, is that H be l.e.*

PROOF. Assume that $\Gamma_\infty(H)$ contains an open set. Then it contains a basis $\xi_1, \ldots, \xi_n$ of $\mathbb{R}^n$ and $\xi = \xi_1 + \ldots + \xi_n$ gives a linear exhaustion of H. Vice versa, if ξ gives a l.e. of H, if $x_0 \in H$ is such that $\langle x_0, \xi \rangle = m = \min_H \langle x, \xi \rangle$, then the polar cone of the proper closed cone with vertex at 0 generated by the vectors $x - x_0$ with $\langle x, \xi \rangle = m + 1$ and $x \in H$, has a non empty interior (containing ξ) and is contained in $\Gamma_\infty(H)$.

Note that interior points of $\Gamma_\infty(H)$ belong to $\Gamma(x, H)$ for some x in ∂H.

Then, if we set $\Gamma(H) = \bigcup_{x\in\partial H} \Gamma(x, H)$ we obtain:

LEMMA 4. *Assume that H is a l.e. closed convex set. Then $\Gamma(H) \subset \Gamma_\infty(H)$ and these two sets are convex and with the same interior.*

Let now M be a unitary $\mathcal{T}$-module of finite type and let H be a l.e. closed convex set in $\mathbb{R}^n$.

We say that *M is hyperbolic in H in dimensions j (with $j \geqq 1$) if either* $\operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{T}) = 0$, *or* $\operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{T}) \neq 0$ *and for every* $\mathscr{P} \in \operatorname{Ass}\cdot(\operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{T}))$ *we have* $\Gamma(H) \cap \tilde{V}^{\mathbb{R}}(\check{\mathscr{P}}) \subset \{0\}$.

Then we have the following

THEOREM 3. *Let M be a unitary left $\mathcal{T}$-module of finite type and let H be a l.e. closed convex set in $\mathbb{R}^n$. If M is hyperbolic in H in dimension j ($j \geqq 1$), then*

$$\operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_H) = \operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_H) = 0\,. \tag{3}$$

PROOF. If $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{T}) = 0$, then the statement follows from Theorem 1 and Theorem 1′. Therefore we assume that $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{T}) \neq 0$. By Corollary 1, we have $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_H) \simeq \mathrm{Ext}^j_{\mathcal{T}}(M, P) \otimes \mathcal{E}_H$. By proposition 3, a necessary and sufficient condition for having $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_H) = 0$ is then that $(\mathcal{T}/\mathscr{P}) \otimes \mathcal{E}_H = 0$ for every $\mathcal{T} \in \mathrm{Ass}\cdot$ $\cdot(\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{T}))$. Let $\mathscr{P}$ be such a prime ideal and let $P_1, \ldots, P_s$ be a set of generators of $\mathscr{P}$. Then we have an isomorphism:

$$(\mathcal{T}/\mathscr{P}) \otimes \mathcal{E}_H \simeq \mathcal{E}_H/(P_1(D)\mathcal{E}_H + \ldots + P_s(D)\mathcal{E}_H)\,. \tag{4}$$

Therefore we have to show that, for every $f \in \mathcal{E}_H$, we can find $u_1, \ldots, u_s \in \mathcal{E}_H$ such that

$$P_1(D)u_1 + \ldots + P_s(D)u_s = f\,. \tag{5}$$

Let ξ give a linear exhaustion of H.

For any given R with $R \in \mathbb{R}$, $R > \min\limits_H \langle x, \xi\rangle$, let us consider the cone Γ^0 generated by all points of the form $x - x_0$ with $x \in H$, $\langle x, \xi\rangle = R$ and x_0 a fixed point of ∂H with $\langle x_0, \xi\rangle = \min\limits_H \langle x, \xi\rangle$. It is a proper convex cone and its polar cone Γ is closed.

I claim that $\Gamma \subset \Gamma(H)$. Indeed we have $\{x\colon \langle x, \xi\rangle \geqq R\} \cap H \subset$ $\subset \Gamma^0 + x_0$ and hence $\langle x, \eta\rangle \geqq \langle x_0, \eta\rangle$ for all $\eta \in \Gamma$. Therefore $x \to \langle x, \eta\rangle$ on H has a minimum in $H_R = \{x \in H\colon \langle x, \xi\rangle \leqq R\}$ and $\eta \in \Gamma(H)$.

Therefore we can find an open convex cone Γ_1 in $\mathbb{R}^n$ with

$$\Gamma \subset \Gamma_1 \cup \{0\} \quad \text{and } \bar{\Gamma}_1 - \{0\} \subset \mathbb{R}^n - \tilde{V}^{\mathbb{R}}(\check{\mathscr{P}})\,.$$

By Theorem 2 we can find then a fundamental solution

$$E = (E_1, \ldots, E_s) \in \mathcal{D}'^s_{\Gamma_1^0} \quad \text{of } P_1(D)E_1 + \ldots + P_s(D)E_s = \delta\,.$$

We note now that, for every $x \in H(R)$, we have

$$H \cap (x - \Gamma_1^0) \text{ compact}$$

and

$$x + \Gamma_1^0 \subset H\,.$$

Therefore, if we define, for $x \in H(R)$,

$$v_j(x) = E_j * f(x)\,,$$

we obtain $v_j \in \mathcal{E}_{H(R)}$ and

$$P_1(D)v_1 + \dots + P_s(D)v_s = f \quad \text{on } H(R)\,.$$

Arguing as in the proof of Theorem 1, we can construct then $u_1, \dots, u_s \in \mathcal{E}_H$ such that (5) holds.

The proof for $\mathcal{D}'_H$ is analogous and therefore is omitted. We also have the following

THEOREM 4. *Let $\xi_0 \in \mathbb{R}^n$ give a l.e. of H and let, for a fixed $x_0 \in H$,*

$$\langle x_0, \xi_0 \rangle \leqq \langle x, \xi_0 \rangle \qquad \forall x \in H\,.$$

Let us fix R with $\langle x_0, \xi_0 \rangle < R < +\infty$ and set, for $\langle x_0, \xi_0 \rangle < r < R$,

$$\Gamma_r^0 = \{t(x - x_0) : t \geqq 0,\ \langle x, \xi_0 \rangle = r,\ x \in H\}\,.$$

Let Γ_r be the polar cone of Γ_r^0:

$$\Gamma_r = \{\xi \in \mathbb{R}^n : \langle x, \xi \rangle \geqq 0\ \forall x \in \Gamma_r^0\}\,.$$

If, for a unitary $\mathcal{T}$-module M of finite type we have for some $j \geqq 1$ either $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{T}) = 0$ or $\Gamma(H) \cap \tilde{V}^{\mathbb{R}}(\check{\mathscr{P}}) \subset \{0\}$ for every $\langle x_0, \xi_0 \rangle < r < R$ and every $\mathscr{P} \in \mathrm{Ass}\,(\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{T}))$, then we have

$$\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_{H_{\xi_0}(R)}) = \mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_{H_{\xi_0}(R)}) = 0\,.$$

The proof of this theorem is indeed the first part of the proof of theorem 3.

Note now that any extremal point of H has a fundamental system of open neighborhoods in H that are of the form $U = H_\xi(R)$ for $\xi \in \Gamma(H)$, $R \in \mathbb{R}$ suitably chosen.

We obtain therefore a local statement:

THEOREM 5. *Let M be a unitary left $\mathcal{T}$-module.*

Let H be a l.e. closed convex set in $\mathbb{R}^n$ and let x_0 be an extremal point of H. Let $j \geqq 1$ be fixed.

If either $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{T}) = 0$ or $\Gamma(x_0, H) \cap \tilde{V}^{\mathbb{R}}(\check{\mathscr{P}}) \subset \{0\}$ for every $\mathscr{P} \in \mathrm{Ass}\,(\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{T}))$, then $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_{H \cap U}) = \mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_{H \cap U}) = 0$ for *a fundamental sequence of convex open neighborhoods* U *of* x_0 *in* $\mathbf{R}^n$.

A slight improvement of a result of [6] tells us that the hyperbolicity condition introduced above is close to be also a necessary condition. Indeed we have:

THEOREM 6. *Assume that the linear functional* $x \to \langle x, \xi \rangle$ *attains a minimum on* H *on a unique point* x_0 *in* ∂H *and let* $\mathscr{P}$ *be a prime ideal in* $\mathfrak{T}$ *such that* $\xi \in \check{V}^{\mathbf{R}}(\check{\mathscr{P}})$.

Then the natural restriction map

$$(\mathfrak{T}/\mathscr{P}) \otimes \mathcal{E}_H \to (\mathfrak{T}/\mathscr{P}) \otimes \mathfrak{D}'_{H(R)}$$

has not zero image for $\langle x_0, \xi \rangle < R \leqq +\infty$.

The proof of this theorem is the same as that of Proposition 19, p. 229 in [6]. We only need to use the estimates for semi-algebraic sets to control the growth of $|\zeta_m|$ as, with a sequence $\{\zeta_m\} \subset V(\mathscr{P})$, we approximate ξ by $m^{-1}\,\mathrm{Re}\,(i\zeta_m)$. (Cf. [5], Appendix to vol. II.)

5. An extension theorem of Hartogs.

A classical theorem of Hartogs says that every complex valued function of n complex variables $z_1, \ldots, z_n$, that is holomorphic in a neighborhood of the disc $|z_1| \leqq 1$, $z_2 = \ldots = z_n = 0$ and in a neighborhood of the circles $|z_1| = 1$, $|z_2| \leqq 1, \ldots, |z_3| \leqq 1$, can be uniquely extended to a holomorphic function in the polycilinder $|z_1| \leqq 1$, $|z_2| \leqq 1, \ldots, |z_n| \leqq 1$.

The results of the previous sections allow us to explain this result in the general framework of the theory of overdetermined systems with constant coefficients.

Let $\xi \in \mathbf{R}$ give a l.e. of a closed convex set H in $\mathbf{R}^n$.

For some fixed real R, with $\min\limits_H \langle x, \xi \rangle < R \leqq +\infty$, let G be any open set in $\mathbf{R}^n$ with $G \cap H = H(R) = \{\langle x, \xi \rangle < R,\ x \in H\}$. Let F denote the closure in G of $G - H$. From the exact sequences

$$0 \to \mathcal{E}_{H(R)} \to \mathcal{E}_G \to W_F \to 0 \tag{1}$$

$$0 \to \mathfrak{D}'_{H(R)} \to \mathfrak{D}'_G \to \check{\mathfrak{D}}'_F \to 0 \tag{2}$$

we deduce for every unitary $\mathcal{T}$-module M of finite type long exact sequences for the Ext functor:

$$(3) \qquad 0 \to \mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_{H(R)}) \to \dots \to \mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_G) \to \mathrm{Ext}^j_{\mathcal{T}}(M, W_F) \to$$
$$\to \mathrm{Ext}^{j+1}_{\mathcal{T}}(M, \mathcal{E}_{H(R)}) \to \dots$$

$$(4) \qquad 0 \to \mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'_{H(R)}) \to \dots \to \mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_G) \to \mathrm{Ext}^j_{\mathcal{T}}(M, \check{\mathcal{D}}'_F) \to$$
$$\to \mathrm{Ext}^{j+1}_{\mathcal{T}}(M, \mathcal{D}'_{H(R)}) \to \dots .$$

We realize then that the following holds:

THEOREM 7. *Assume that M satisfies the conditions of Theorem 4. Then the natural maps*

$$(5) \qquad \mathrm{Ext}^j_G(M, \mathcal{E}_G) \to \mathrm{Ext}^j_{\mathcal{T}}(W_F)$$

and

$$(6) \qquad \mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_G) \to \mathrm{Ext}^j_{\mathcal{T}}(\check{\mathcal{D}}'_F)$$

are onto.

We can read this theorem as an extension theorem for forms, using the Hilbert resolution (1.4). Let us give the argument for functions, as the same holds also for distributions.

Let $f \in W_F^{p_j}$ satisfy the homogeneous equation $A_j(D)f = 0$. Then we can extend f to $\tilde{f} \in \mathcal{E}_G^{p_j}$ and clearly $A_j(D)\tilde{f} \in \mathcal{E}^{p_{j+1}}_{H(R)}$ and satisfies the integrability condition $A_{j+1}(D)(A_j(D)\tilde{f}) = 0$. By the assumption that $\mathrm{Ext}^{j+1}_{\mathcal{T}}(M, \mathcal{E}_{H(R)})$ is zero, we deduce that we can find $g \in \mathcal{E}^{p_j}_{H(R)}$ such that $A_j(D)\tilde{f} = A_j(D)g$ on G.

Hence the function $\tilde{f} - g$ gives an extension of f satisfying the homogeneous equation $A_j(D)(\tilde{f} - g) = 0$ on G.

EXAMPLE. If M is the unitary $\mathcal{T}$-module

$$\mathbf{C}[\zeta_1, \dots, \zeta_n, \zeta_{n+1}, \dots, \zeta_{2n}]/(\zeta_1 + i\zeta_2, \dots, \zeta_{2n-1} + i\zeta_{2n})$$

associated to the Dolbeault complex in $\mathbf{C}^n$, we obtain Hartogs extension theorem for $j < n - 1$, while for $j = n - 1$ we realize that M is not hyperbolic in dimension n.

If we consider instead the De Rham complex in $\mathbb{R}^n$, associated to $M = \mathbb{C}[\zeta_1, \ldots, \zeta_n]/(\zeta_1, \ldots, \zeta_n)$, we have hyperbolicity in all dimensions j provided that the convex set H has a l.e. ξ such that $\langle x, \xi \rangle$ is not bounded from above on H.

For elliptic complexes, i.e. if we have $\tilde{V}(\mathscr{P}) \cap \mathbb{R}^n \subset \{0\}$ for all $\mathscr{P} \in \mathrm{Ass}\,(M)$, we can combine theorem 6 with the unique continuation theorem to obtain:

THEOREM 8. *Let M be an elliptic unitary $\mathfrak{I}$-module of finite type and let B be an open neighborhood of $H(R)$ in G such that one of the following two conditions holds*:

(7) $\quad G - B$ *has no connected component relatively compact in* $G \cup \{\langle x, \xi \rangle \leqq R\}$,

(8) $\quad \mathrm{Ext}^1_{\mathfrak{I}}\,(M, \mathfrak{I}) = 0$.

Then, if M satisfies the hypothesis of theorem 4, every solution $u \in W^{p_0}_{G-B}$ of the homogeneous equation $A_0(D)u = 0$ in $G - B$, extends uniquely to a solution $\tilde{u} \in \mathcal{E}^{p_0}_G$ of $A_0(D)\tilde{u} = 0$ on G.

REMARK. In the case of a compact convex set H, we recover in theorems 7 and 8 some results obtained by other authors (cf. [4], [10]).

6. Applications to the tangential complex.

Let M be a unitary $\mathfrak{I}$-module of finite type.

We fix a l.e. convex set H in $\mathbb{R}^n$ and we assume that, for some $\xi \in \mathbb{R}^n$ giving a l.e. of H and some R with $\min \langle x, \xi \rangle < R \leqq +\infty$, the set $\partial H \cap \{x \in \mathbb{R}^n \colon \langle x, \xi \rangle < R\}$ be a smooth hypersurface, non-characteristic for M.

Then, as we noted in § 1, C, we can define the tangential complex (1.8) on $\Sigma(R) = \partial H \cap \{x \in \mathbb{R}^n \colon \langle x, \xi \rangle < R\}$.

Using the exact sequences (1.19) and (1.20) we can deduce then some results related to the complex (1.8).

Indeed we have:

THEOREM 9. *Under the assumptions of theorem* 4:

if $j \geqq 2$, *then* $H^{j-1}\big(\Gamma(\Sigma(R), F_*), \alpha_*\big) = H^{j-1}\big(\mathfrak{D}'(\Sigma(R), F_*), \alpha_*\big) = 0,$

if $j=1$, *then every element* $u \in H^0\big(\Gamma(\Sigma(R), F_*), \alpha_*\big)$ $\big($*resp.* $u \in H^0 \cdot$ $\cdot\big(\mathfrak{D}'(\Sigma(R), F_*), \alpha_*\big)\big)$ *is the trace of a solution* $\tilde{u} \in W^{p_0}_{H(R)}$ (*resp.* $\tilde{u} \in \mathfrak{D}'^{p_0}_{H(R)}$) *of the equation* $A_0(D)\tilde{u} = 0$.

Under the assumptions of theorem 5:

if $j \geqq 2$, *then the point* x_0 *has a fundamental system of open neighbourhoods* $U \cap \partial H$ *such that* $H^{j-1}(\Gamma(U \cap \partial H, F_*), \alpha_*) = H^{j-1}(\mathfrak{D}'(U \cap \partial H, F_*), \alpha_*) = 0$;

if $j=1$, *then the point* x_0 *has a fundamental system of open neighborhoods* U *with the property that the trace map*

$$\mathrm{Ext}^0_{\mathfrak{T}}(M, W_{H \cap U}) \to H^0(\Gamma(\Sigma \cap U, F_*), \alpha_*)$$

is onto.

Under the assumption of theorem 6 *for some* $\mathscr{P} \in \mathrm{Ass}\,\big(\mathrm{Ext}^{j+1}_{\mathfrak{T}}(M, \mathfrak{T})\big)$, *with* $j \geqq 1$:

$$\text{Image } H^j\Big(\big(\Gamma(\Sigma(R), F_*), \alpha_*\big) \to H^j\big(\mathfrak{D}'(F_*)_{x_0}, \alpha_*\big)\Big)$$

is infinite dimensional.

REFERENCES

[1] A. Andreotti - C. D. Hill - S. Łojasiewicz - B. MacKichan, *Complexes of differential operators*, Invent. Math., **35** (1976), pp. 43-86.

[2] A. Andreotti - M. Nacinovich, *Complexes of partial differential operators*, Ann. Scuola Norm. Sup. Pisa, serie 4, **3** (1976), pp. 653-621.

[3] A. Andreotti - M. Nacinovich, *Noncharacteristic hypersurfaces for complexes of differential operators*, Annali di Mat. Pura e Appl., (IV), **125** (1980), pp. 13-83.

[4] G. Bratti, *Su di un teorema di Hartogs*, Rend. Sem. Mat. Univ. Padova, **79** (1988), pp. 59-70.

[5] L. Hörmander, *The analysis of linear partial differential operators I, II*, Springer-Verlag, Berlin, 1983.

[6] M. Nacinovich, *On boundary Hilbert differential complexes*, Annales Polonici Mathematici, **46** (1985), pp. 213-235.

[7] M. NACINOVICH, *Cauchy problem for overdetermined systems*, to appear in Annali di Mat. Pura e Appl.

[8] M. NACINOVICH - G. VALLI, *Tangential Cauchy-Riemann complexes on distributions*, Annali di Mat. Pura e Appl., (IV), **146** (1987), pp. 123-160.

[9] J. C. TOUGERON, *Ideaux de fonctions differentiables*, Springer-Verlag, Berlin, 1972.

[10] A. KANEKO, *On continuation of regular solutions of partial differential equations with constant coefficients*, J. Math. Soc. Japan, **26** (1974), pp. 92-123.

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