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Karp's Interpolation Theorem for Some Classes of Infinitary Languages.

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SUMMARY - We define the notion of ω-consistency property for some classes of infinitary languages. Then we examine such classes with respect to Karp's interpolation theorem.

0. Introduction.

In [7] Karp proved an interpolation theorem for the class of infinitary languages L_{kk}, where k is a strong limit cardinal of cofinality ω.

Karp's result, which generalizes a theorem of Chang [1], says:

« Let σ_1 and σ_2 be sets of relational symbols and let F, G be sentences belonging to the languages $L_{kk}(\sigma_1)$ and $L_{kk}(\sigma_2)$ respectively.

If $\models^{\omega} F \to G$ then there is a $L_{kk}(\sigma_1 \cap \sigma_2)$ sentence H such that $\models F \to H$ and $\models H \to G$, where $\models^{\omega}$ means true with respect to all ω-chains of structures. »

Karp's interpolation theorem cannot be weakened to read $\models F \to G$ in the hypothesis of the theorem (see Malitz [10]): in that case the interpolant might have an infinite number of alternations of quantifiers, as pointed out by Takeuti in [12].

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Karp's result can be improved by writing $\models^{\omega}$ in the thesis of the theorem (Cunningham [2]) and also by considering second order languages (Ferro [3]).

In this paper we use the technique of consistency properties, proving a model existence theorem and a weak interpolation theorem in Karp's style for $L_{\alpha\alpha}$, where α is a regular cardinal greater than ω.

Then we examine some other classes of infinitary languages with respect to Karp's interpolation theorem.

Finally, we give some ideas on a natural extension of ω-satisfiability, furnishing related results.

1. Preliminaries.

Concerning routine definitions and notation see [3].

Here we only want to specify some differences between our terminology and notation and those ones used in [3].

$L_{\alpha\beta}$ stands for the class of infinitary languages in which conjunctions on sets of formulas of cardinality less than α and quantifications on sets of variables of cardinality less than β are permitted.

The language $L_{\alpha\beta}(\sigma)$ consists of the following symbols:

a) individual variables: v_i for $i < \alpha$;

b) a set of cardinality not greater than α of finitary relation symbols and constant symbols;

c) connectives: $\neg$ and $\wedge$;

d) quantifier: $\forall$;

e) equality symbol: $=$;

f) auxiliary symbols: (and).

Atomic formulas are defined in the usual way. The set of $L_{\alpha\beta}(\sigma)$ formulas is the smallest set of strings of symbols which contains atomic formulas and is closed under the negation $\neg F$ of a formula and the rules of formation of formulas described above.

Without loss of generality we assume that a variable free in a formula cannot occur bound in the same formula and that in each formula or set of formulas sets of variables following a quantifier are pairwise disjoint.

An ω-chain of structures $\boldsymbol{A}$ adequate for $L_{\alpha\beta}(\sigma)$ is a pair $\langle A, f\rangle$, where A is a sequence $\langle A_n : n < \omega\rangle$ of universes such that $A_n \subseteq A_{n+1}$ for each $n < \omega$ and f is a function interpreting extralogical symbols occurring in σ.

The function f maps each m-ary relation symbol into a set of m-tuples of elements of $U\{A_n : n < \omega\}$ for $m > 0$ and, for some fixed $n < \omega$, each constant symbol into A_n.

The difference between the interpretation of a formula F in $\boldsymbol{A}$ and the one in the standard structure $\langle U\{A_n : n < \omega\}, f\rangle$ concerns the assignment of values to free variables in F, which is bounded in A, namely each free variable in F must be interpreted, for some fixed n, within A_n.

We say that the ω-chain of structures $\boldsymbol{A} = \langle A, f\rangle$ ω-satisfies an $L_{\alpha\beta}(\sigma)$ formula F (or F is ω-satisfied by $\boldsymbol{A}$) under the bounded assignment g to free variables in F (notation: $\boldsymbol{A}, g \models^{\omega} F$) if, putting $h = f \cup g$, one of the following holds:

a) F is $x_1 = x_2$ and $h(x_1) = h(x_2)$, where x_1 is either a variable or a constant symbol in σ;

b) F is $P(x_1, \ldots, x_n)$ and $\langle h(x_1), \ldots, h(x_n)\rangle \in h(P)$, where P is an n-ary predicate symbol in σ for some $n > 0$ and x_1, is either a variable or a constant symbol in σ;

c) F is $\neg G$ and not $\boldsymbol{A}$, $g \models^{\omega} G$;

d) F is $\bigwedge \boldsymbol{G}$ and for all $G \in \boldsymbol{G}$, $\boldsymbol{A}$, $g \models^{\omega} G$;

e) F is $(\forall \boldsymbol{v})\, G$, where $\boldsymbol{v}$ is a set of variables and for all bounded assignments l to $\boldsymbol{v}$, $\boldsymbol{A}$, $(g \setminus g \restriction \boldsymbol{v}) \cup l \models^{\omega} G$.

A set $\boldsymbol{F}$ of formulas is ω-satisfiable if there is an ω-chain of structures $\boldsymbol{A}$ and a bounded assignment g to free variables occurring in some $F \in \boldsymbol{F}$ such that $\boldsymbol{A}$, $g \models^{\omega} F$ for all $F \in \boldsymbol{F}$.

An $L_{\alpha\beta}(\sigma)$ formula F is ω-valid if $\boldsymbol{A}$, $g \models^{\omega} F$ for each ω-chain of structures $\boldsymbol{A}$ adequate for $L_{\alpha\beta}(\sigma)$ and for each bounded assignment g to free variables in F.

Notation for an ω-valid formula F will be $\models^{\omega} F$.

Further details on the relationship between ω-satisfability and (standard) satisfability can be found in [5].

From now on St $(L_{\alpha\beta}(\sigma))$ stands for the set of $L_{\alpha\beta}(\sigma)$ sentences.

If α is a cardinal number, then α^{+} is the successor cardinal of α.

2. A weak interpolation theorem for $L_{\alpha\alpha}$, where α is a regular cardinal, $\alpha > \omega$.

In this section α stands for a regular cardinal, $\alpha > \omega$.
We assume genaralized continuum hypothesis (GCH).

LEMMA 2.1. If GCH holds and σ is a set of extralogical symbols with cardinality not greater than α then $|\mathrm{St}\,(L_{\alpha\alpha}(\sigma))| = \alpha$.

PROOF. Let For $(L_{\alpha\alpha}(\sigma))$ be the set of all $L_{\alpha\alpha}(\sigma)$ formulas.

For $(L_{\alpha\alpha}(\sigma))$ is a subset of the set of all the strings of length less than α which can be written using an alphabet of cardinality α (the $L_{\alpha\alpha}(\sigma)$ symbols).

Hence $|\mathrm{For}\,(L_{\alpha\alpha}(\sigma))| \leqslant \Sigma\{\alpha: \beta < \alpha\} = \Sigma\{\alpha: \beta < \alpha\} = \alpha$ ($\alpha^\beta = \alpha$ for each $\beta < \alpha$ because β has cofinality strictly smaller than α and GCH holds).

Consider $\boldsymbol{F} = \{(\forall\{v_i: i < \lambda\}) \wedge \{v_i = v_j: i, j < \lambda\}: \lambda < \alpha\}$.

$\boldsymbol{F} \subseteq \mathrm{St}\,(L_{\alpha\alpha}(\sigma))$ for each σ.

From $\mathrm{St}\,(L_{\alpha\alpha}(\sigma)) \subseteq \mathrm{For}\,(L_{\alpha\alpha}(\sigma))$ and $|\boldsymbol{F}| = \alpha$ it follows that

$$|\mathrm{St}\,(L_{\alpha\alpha}(\sigma))| = \alpha. \qquad \square$$

We define now the notion of ω-consistency property for $L_{\alpha\alpha}(\sigma)$.

Let C be the set of constant symbols in σ and let $C' = \cup\{C_n: n < \omega\}$ be a set of constant symbols such that $C \cap C' = \emptyset$ and $|C_{n+1} \setminus C_n| = |C_0| = \alpha$ and $C_n \subseteq C_{n+1}$ for all $n < \omega$.

DEFINITION 2.2. An ω-consistency property for $L_{\alpha\alpha}(\sigma)$ with respect to $\{C_n: n < \omega\}$ is a set of sets of $L_{\alpha\alpha}(\sigma)$ sentences such that for each $s \in S$ there is an n (depending on s) with $s \subseteq \mathrm{St}\,(L_{\alpha\alpha}(\sigma \cup C_n))$ and the following conditions holds:

$C0$) if Z is an atomic sentence then either $Z \notin s$ or $\neg Z \notin s$ and if Z has the form $\neg(x_i = x_i)$ with x_i either a variable or a constant symbol, then $Z \notin s$;

$C1$) $\{\neg\neg F_i: i \in I\} \subseteq s$ then $s \cup \{F_i: i \in I\} \in S$;

$C2$) if $\{\wedge \boldsymbol{F}_i: i \in I\} \subseteq s$ then $s \cup \{F: F \in \boldsymbol{F}_i, i \in I\} \in S$;

$C3$) if $\{\neg(\wedge \boldsymbol{F}_i): i \in I\} \subseteq s$ then there is $g \in \Pi\{\boldsymbol{F}_i: i \in I\}$ such that $s \cup \{\neg g(i): i \in I\} \in S$;

C4) if $\{(\forall \boldsymbol{v}_i) F_i : i \in I\} \subseteq s$ then $s \cup \{F_i(\boldsymbol{v}_i \restriction f_i) : f_i$ is a function from $\boldsymbol{v}_i$ to $C \cup C_n, i \in I\} \in S$ for each $n < \omega$;

C5) if $\{(\neg \forall \boldsymbol{v}_i) F_i : i \in I\} \subseteq s$ and n is the least natural number such that $s \subseteq \mathrm{St}\,(L_{\alpha\alpha}(\sigma \cup C_n))$ then $s \cup \{\neg F_i(\boldsymbol{v}_i \restriction f) : i \in I\} \in S$ for each one-to-one function f from $\cup \{\boldsymbol{v}_i : i \in I\}$ to $C_{n+1} \setminus C_n$.

C6) a) if $\{c_i = d_i : c_i, d_i \in C \cup C', i \in I\} \subseteq s$ then $s \cup \{d_i = c_i, i \in I\} \in S$,

b) if $\{Z_i(c_i), c_i = d_i : Z_i$ atomic or negated atomic formula, $c_i, d_i \in C \cup C', i \in I\} \subseteq s$, then $s \cup \{Z_i(d_i) : i \in I\} \in S$.

THEOREM 2.3 (*model existence theorem*). If s is a set of sentences belonging to an an ω-consistency property for $L_{\alpha\alpha}(\sigma)$ with respect to $\{C_n : n < \omega\}$ then it is ω-satisfied by an ω-chain of structures such that for each $n < \omega$ the cardinality of the n-th universe is not greater than α.

PROOF. Let $s_0 = s$. Suppose $s_n \in S$ and show that $s_{n+1} = s_n^7 \in S$, where $s_n^1, \ldots, s_n^7$ are defined in the following way:

$s_n^1 = s_n \cup \{F : \neg\neg F \in s_n\} \in S$ by C1);

$s_n^2 = s_n^1 \cup \{F : F \in \boldsymbol{F}$ for some $\boldsymbol{F}$ such that $\bigwedge \boldsymbol{F} \in s_n\} \in S$ by C2);

$s_n^3 = s_n^2 \cup \{\neg g(\boldsymbol{F}) : \neg(\bigwedge \boldsymbol{F}) \in s_n\} \in S$, where g is an adequate choice function whose existence is assured by C3);

$s_n^4 = s_n^3 \cup \{F(\boldsymbol{v} \restriction g) : (\forall \boldsymbol{v}) F \in s_n$ and g is a function from $\boldsymbol{v}$ to $C \cup C_{m+1}\} \in S$ by C4), where m is the least natural number such that $s_n \subseteq St\,(L_{\alpha\alpha}(\sigma \cup C_m))$;

$s_n^5 = s_n^4 \cup \{\neg F(\boldsymbol{v} \restriction f) : (\neg \forall \boldsymbol{v}) F \in s_n\} \in S$, where m is the least natural such that $s_n^4 \subseteq \mathrm{St}\,(L_{\alpha\alpha}(\sigma \cup C_m))$ and f is a one-to-one function from $\cup \{\boldsymbol{v} : (\neg \forall \boldsymbol{v}) F \in s_n\}$ to $C_{m+1} \setminus C_m$ as in C5);

$s_n^6 = s_n^5 \cup \{d = c : c = d \in s_n, c, d \in C \cup C'\} \in S$ by C6) a);

$s_n^7 = s_n^6 \cup \{Z(d) : Z(c), c = d \in s_n$ and Z is an atomic or negated atomic formula$\} \in S$ by C6) b).

Note that $s_n \subseteq s_{n+1}$ for each $n < \omega$.

Define $s_\omega = \cup \{s_n : n < \omega\}$. s_ω has the following closure properties (by construction):

a) if Z is an atomic sentence thein either $Z \notin s_\omega$ or $\neg Z \notin s_\omega$ and if x_i is either a variable or a constant symbol then $\neg (x_i = x_i) \notin s_\omega$;

b) if $\neg\neg F \in s_\omega$, then $F \in s_\omega$;

c) if $\bigwedge \boldsymbol{F} \in s_\omega$, then $F \in s_\omega$ for each $F \in \boldsymbol{F}$;

e) if $(\forall \boldsymbol{v})F \in s_\omega$, then $F(\boldsymbol{v} \restriction f) \in s_\omega$ for each $n < \omega$ and for each function f from $\boldsymbol{v}$ to $C \cup C_n$;

f) if $(\neg\forall \boldsymbol{v})F \in s_\omega$, then there are $m < \omega$ and a one-to-one function from $\boldsymbol{v}$ to $C_{m+1} \setminus C_m$ such that $\neg F(\boldsymbol{v} \restriction f) \in s_\omega$;

g) if $c = d \in s_\omega$, $c, d \in C \cup C'$, then $d = c \in s_\omega$;

h) if Z is an atomic or negated atomic formula and $Z(c)$, $c = d \in s_\omega$, $c, d \in C \cup C'$, then $Z(d) \in s_\omega$.

s_ω and $\{C_n : n < \omega\}$ can be used to define an ω-chain of structures.

We define a relation $\tilde{}$ on $C \cup C'$ in this way: $c\tilde{\ }d$ if $c = d \in s_\omega$ or c and d are the same constant symbol.

Recalling the definition of s_n^6 and s_n^7 we can easily verify that $\tilde{}$ is an equivalence relation on $C \cup C'$.

Let $c^\sim$ be the equivalence class of c and let $A_n = \{c^\sim : c \in C \cup C_n\}$.

Take $A = \langle A_n : n < \omega \rangle$ and define for each $n < \omega$ the following (bounded) assignments of values: $f_n : C \cup C_n \to A_n$, $c \to c^\sim$.

For each $0 < m < \omega$ and for each m-ary predicate symbol P in ω let $f(P)$ be

$$\{\langle \tilde{c}_1, \dots, \tilde{c}_m \rangle : P(c_1, \dots, c_m) \in s_\omega,\ c_1, \dots, c_m \in C \cup C'\}\,.$$

f is a well-defined function by definition of s_n^7.

Recalling *a*)-*h*) and using induction on the rank of formulas, we can prove that if F is any sentence in $s_\omega \cap \mathrm{St}\,(L_{\alpha\alpha}(\sigma \cup C_n))$ for some $n < \omega$, then the ω-chain of structures $\langle A, f \cup f_n \rangle$ ω-satisfies F.

Details of the proof are omitted (a similar proof can be found in [5]).

Since $s = s_0 \subseteq \mathrm{St}\,(L_{\alpha\alpha}(\sigma \cup C_n))$ for some $n < \omega$ and $s_0 \subseteq s_\omega$, it follows that $\langle A, f \cup f_n \rangle$ ω-satisfies F. $\square$

Let $L_n = L_{\alpha^{++}\alpha^{+}}(\sigma \cup C_n)$ for each $n < \omega$.

DEFINITION 2.4. Let $s \subseteq \mathrm{St}\,(L_{\alpha\alpha}(\sigma \cup C_n))$ for some $n < \omega$. A sentence H belonging to $\mathrm{St}\,(L_n)$ is a weak interpolant for s with respect to the partition (s_1, s_2) of s if:

a) each extralogical symbol occurring in H occurs in both of s_1 and s_2

and

b) $s_1 \cup \{\neg H\}$ and $s_2 \cup \{H\}$ are not satisfiable.

The next result is fundamental.

THEOREM 2.5. Let S be the set of all the sets s of $L_{\alpha\alpha}(\sigma \cup C')$ sentences such that:

1) there is $n < \omega$ such that $s \subseteq \mathrm{St}\,(L_{\alpha\alpha}(\sigma \cup C_n))$

and

2) s has a partition (s_1, s_2) without weak interpolant.

Then S is an ω-consistency property for $L_{\alpha\alpha}(\sigma)$.

PROOF. We must show that $C0)$-$C6)$ hold for each $s \in S$.

Here we shall consider only the proofs of $C3)$, $C6)$ *b*) and sketches of the proofs of $C4)$ and $C5)$, the other conditions being easy to prove.

$C3)$ Proof of $C3)$ shows why we obtain a weak interpolation theorem.

We must show that if $\{\neg\,(\bigwedge \boldsymbol{F}_i)\colon i \in I\} \subseteq s \in S$ then there is a function $f \in \Pi\,\{\boldsymbol{F}_i\colon i \in I\}$ such that $s' = s \cup \{\neg\, f(i)\colon i \in I\} \in S$.

Let (s_1, s_2) be a partition of s without weak interpolant and let $I_1 = \{i \in I\colon \neg\,(\bigwedge \boldsymbol{F}_1) \in s_1\}$, $I_2 = I \setminus I_1$.

First we show the existence of a $g \in G = \Pi\{\boldsymbol{F}_i\colon i \in I_1\}$ such that the partition $(s_1 \cup \{\neg\, g(i)\colon i \in I_1\}, s_2)$ of $s_g = s \cup \{\neg\, g(i)\colon i \in I_1\}$ does not have a weak interpolant.

If such g does not exist then, for each $g \in G$, we can find a weak interpolant H_g for the partition $(s_1 \cup \{\neg\, g(i)\colon i \in I_1\}, s_2)$ of s_g.

Then $s_1 \cup \Big\{\neg\,\big(\bigwedge\{\neg\,(\bigwedge\{\neg\, g(i)\colon\ i \in I_1\})\colon\ g \in G\}\big)\Big\} \cup \{\bigwedge\{\neg\, H_g\colon\ g \in G\}\}$ and $s_2 \cup \{\neg\,(\bigwedge\{\neg\, H_g\colon\ g \in G\})\}$ are not satisfiable.

Let H be $\neg\,(\bigwedge\{\neg\, H_g\colon g \in G\})$. We have that $s_1 \cup \{\neg\, H\}$ and $s_2 \cup \{H\}$ are not satisfiable and each extralogical symbol occuring in H occurs in both of s_1 and s_2, contradicting the assumption on (s_1, s_2).

Note that $|G|$ could be α^+; for this reason we can only assure the existence of a weak interpolant, namely an interpolant which is in a language richer than $L_{\alpha\alpha}(\sigma)$ with respect to the expressive power.

Note also that we have tacitly assumed distributivity axioms for L_n to obtain the contradiction.

In the same way we can find $h \in \Pi\,\{\boldsymbol{F}_i : i \in I_2\}$ such that the partition $(s_1 \cup \{\neg\, g(i) : i \in I_1\}, s_2 \cup \{\neg\, h(i) : i \in I_2\})$ of s' does not have a weak interpolant.

Thus $f = g \cup h$ is the required function and $C3$) is proved.

$C4$) We show that if $\{(\forall \boldsymbol{v}_i)\boldsymbol{F}_i : i \in I\} \subseteq s$ then $s \cup \{\boldsymbol{F}_i(\boldsymbol{v}_i \restriction g_i) : i \in I$ and g_i is a function from $\boldsymbol{v}_i$ to $C \cup C_n\} \in S$, for each $n < \omega$.

Let $I_1 = \{i \in I : (\forall \boldsymbol{v}_i)\,\boldsymbol{F}_i \in s_1\}$, $I_2 = I \setminus I_1$.

Let n be a fixed natural and let, $s'_j = s_j \cup \{\boldsymbol{F}_i(\boldsymbol{v}_i \restriction g_i) : i \in I_j$ and g_i is a function from $\boldsymbol{v}_i$ to $C_n\}$, for $j = 1, 2$.

To show that (s'_1, s'_2) is a partition of s' without weak interpolant we prove first that (s'_1, s_2) is a partition of $s'_1 \cup s_2$ without weak interpolant and then that the same holds for the partition (s'_1, s'_2).

Let $\boldsymbol{H}$ be a weak interpolant for (s'_1, s_2) and let $\boldsymbol{D}$ be the set of constant symbols in $\boldsymbol{H}$ not occurring on s_1.

Let h be a bijective function from $\boldsymbol{D}$ to a set $\boldsymbol{w}$ of new variables.

Thus $s_1 \cup \{(\neg\forall \boldsymbol{w})\boldsymbol{H}(\boldsymbol{D} \restriction h)\}$ and $s_2 \cup \{(\forall \boldsymbol{w})\boldsymbol{H}(\boldsymbol{D} \restriction h)$ would not be satisfiable and each extralogical symbol occurring in $(\forall \boldsymbol{w})\boldsymbol{H}(\boldsymbol{D} \restriction h)$ occurs in both of s_1 and s_2; a contradiction.

In the same way we can prove that $s'_1 \cup s'_2 \in S$.

Note thet it could be necessary to replace a set of cardinality α of constant symbols occurring in $\boldsymbol{H}$ by a set of new variables and then to quantify on that set of variables: for this reason the interpolant could have quantifications on sets of α variables.

Similar remarks apply to $C6$) b).

$C5$) We show that if $\{(\neg\forall \boldsymbol{v}_i)\,\boldsymbol{F}_i : i \in I\} \subseteq s \in S$ and n is the least natural number such that $s \subseteq \mathrm{St}\,(L_{\alpha\alpha}(\sigma \cup C_n))$ then $s \cup \{\neg\, \boldsymbol{F}_i(\boldsymbol{v}_i \restriction f)\} \in S$ for each one-to-one function f from $\cup\,\{\boldsymbol{v}_i : i \in I\}$ to $C_{n+1} \setminus C_n$.

Let f be a one-to-one function of type described above and let (s_1, s_2) be a partition of s without weak interpolant.

Define $s' = s \cup \{\neg\, \boldsymbol{F}_i(\boldsymbol{v}_i \restriction f) : i \in I\}$; $I_1 = \{i \in I : (\neg\forall \boldsymbol{v}_i)\,\boldsymbol{F}_i \in s_1\}$ and $I_2 = I \setminus I_1$.

Let $s'_j = s_j \cup \{\neg\, \boldsymbol{F}_i(\boldsymbol{v}_i \restriction f) : i \in I_j\}$, for $j = 1, 2$.

We want to show that (s'_1, s'_2) is a partition of s' without weak interpolant.

For, if not, there would be a weak interpolant H for (s_1', s_2') and each extralogical symbol occurring in H would occur in both of s_1 and s_2, since passing from s to s' we introduce by a one-to-one function constant symbols not occurring in s and we assume that in each set of formulas sets of variables following a quantifier are pairwise disjoint (see preliminaries).

It follows that $s_1 \cup \{\neg H\}$ and $s_2 \cup \{H\}$ would be not satisfiable (if they were satisfiable then $s_1' \cup \{\neg H\}$ and $s_2 \cup \{H\}$ would be satisfiable), contradicting the assumption on (s_1, s_2).

C6) b) We show that if $\{Z_i(c_i), c_i = d_i: Z_i$ atomic or negated atomic formula, $c_i, d_i \in C \cup C', i \in I\} \subseteq s \in S$, then $s' = s \cup \{Z_i(d_i): i \in I\} \in S$.

Let (s_1, s_2) be a partition of s without weak interpolant.

Define

$$I_1 = \{i \in I: Z_i(c_i),\ c_i = d_i \in s_1\} ;$$

$$I_2 = \{i \in I: Z_i(c_i),\ c_i = d_i \in s_2\} ;$$

$$I_3 = \{i \in I: Z_i(c_i) \in s_1,\ c_i = d_i \in s_2\} ;$$

$$I_4 = \{i \in I: Z_i(c_i) \in s_2,\ c_i = d_i \in s_1\} .$$

Let $t_j = s_j \cup \{Z_i(d_i): i \in I_j\}$, for $j = 1, 2$. It's easy to show that $t_1 \cup t_2 \in S$. Let $s_1' = t_1 \cup \{Z_i(d_i): i \in I_3\}$; $s_2' = t_2 \cup \{Z_i(d_i): i \in I_4\}$.

We show that (s_1', s_2') is a partition of s' without weak interpolant.

Suppose the existence of a weak interpolant H for (s_1', s_2') and let H' be $(\wedge\{c_i = d_i: i \in I_4\}) \wedge \left(\neg \left((\wedge\{c_i = d_i: i \in I_3\}) \wedge (\neg H)\right)\right)$.

$t_1 \cup \{\neg H'\}$ and $t_2 \cup \{H'\}$ are not satisfiable (if they were satisfiable, then $t_1 \cup \{\neg H\}$ and $t_2 \cup \{H\}$ would be satisfiable: a contradiction).

Let $D_1 (D_2)$ be the set of constant symbols in H' which occur in $t_2 (t_1)$, but not in $t_1 (t_2)$ and let h_1, h_2 be two bi$_k$ective maps from D_1, D_2 to disjoint sets of new variables $\boldsymbol{w}_1, \boldsymbol{w}_2$ respectively.

Then $t_1 \cup \{(\neg\forall \boldsymbol{w}_1) H'(D_1 \restriction h_1)\}$ and $t_2 \cup \{(\forall \boldsymbol{w}_2) H'(D_1 \restriction h_1)\}$ are not satisfiable and the same holds for $t_1 \cup \{(\forall \boldsymbol{w}_2)(\neg\forall \boldsymbol{w}_1) H'(D_1 \restriction h_1)(D_2 \restriction h_2)\}$ and $t_2 \cup \{(\neg\forall \boldsymbol{w}_2) (\neg\forall \boldsymbol{w}_1) H'(D_1 \restriction h_1)(D_2 \restriction h_2)\}$.

Moreover each extralogical symbol occurring in

$$(\neg\forall \boldsymbol{w}_2) (\neg\forall \boldsymbol{w}_1) H'(D_1 \restriction h_1)(D_2 \restriction h_2)$$

occurs in both t_1 and t_2, but $t_1 \cup t_2 \in S$, so we get a contradiction. □

From theorem 2.5 it follows:

COROLLARY 2.6 (*weak interpolation theorem*). If $F \to G$ is an ω-valid $L_{\alpha\alpha}(\sigma)$ sentence then there is a $L_{\alpha^{++}\alpha^{+}}(\sigma)$ sentence H whose extralogical symbols occur in both F and G and $\models F \to H$ and $\models H \to G$.

PROOF. From the hypothesis $\models F \to G$ it follows that $\{F, \neg G\}$ is not ω-satisfiable, hence, by theorem 2.3, $\{F, \neg G\} S$, where S is the ω-consistency property defined in theorem 2.5.

Therefore there is a $L_{\alpha^{++}\alpha^{+}}(\sigma)$ sentence H which weakly interpolates the partition $(\{F\}, \{\neg G\})$ of $\{F, \neg G\}$, namely each extralogical symbol occurring in H occurs in both F and G and $\{F, \neg H\}$, $\{\neg G, H\}$ are not satifiable, but the last two conditions are equivalent to saying $\models F \to H$ and $\models H \to G$. □

3. Interpolation theorems for other classes of infinitary languages and counterexamples.

Recall that a strongly inaccessible cardinal is a regular strong limit cardinal greater than ω.

In this section we examine, among others, the class $L_{\beta\beta}$, with β a strongly inaccessible cardinal.

Let $L_{\beta\beta}(\sigma)$ be a language belonging to that class (recall that σ is a set of constant symbols and finitary predicate symbols of cardinality not greater than β).

Let C be the set of constant symbols in σ and let $C' = \cup \{C_n : n < \omega\}$ be a set of constant symbols disjoint from C such that $C_n \subseteq C_{n+1}$ and $|C_{n+1} \setminus C_n| = \beta = |C_0|$ for each $n < \omega$.

DEFINITION 3.1. S is an ω-consistency property for $L_{\beta\beta}(\sigma)$ with respect to $\{C_n : n < \omega\}$ if S is a set of sets of $L_{\beta\beta}(\sigma)$ sentences such that for each $s \in S$, $|s| < \beta$, $s \subseteq \mathrm{St}\,(L_{\beta\beta}(\sigma \cup C_n))$ for some $n < \omega$ and the following hold:

$C0$), $C1$), $C2$), $C3$), $C5$), $C6$) as in definition 1.2;

$C4$) *If* $\{\forall \boldsymbol{v}_i F_i : i \in I\} \subseteq S$ then, for each set of constant symbols D such that $|D| < \beta$ and $D \subseteq C \cup C_n$ for some $n < \omega$, $s \cup \{F_i(\boldsymbol{v}_i \restriction f_i) : i \in I$ and f_i is a function from $\boldsymbol{v}_i$ to $D\} \in S$.

Note that the assumption that β be stronbly inaccessible assures that each element of S has cardinality strictly smaller than β (problems are with clauses $C2$) and $C4$)).

By using the previous definition, one can prove, via a model existence theorem a strong interpolation theorem for $L_{\beta\beta}(\sigma)$. Here strong means that it is not necessary to jump to a more expressive language in order to find the interpolant.

Proofs are omitted. We only point out that the assumptions on β enable us to show that every member of an ω-consistency property for $L_{\beta\beta}(\sigma)$ is ω-satisfied by an ω-chain of structure whose n-th universe has cardinality strictly smaller than β.

Anyway, this strong interpolation theorem can be immediately derived from Karp's result quoted in the introduction: suppose β a strongly inaccessible cardinal, $F \to G$ a sentence of $L_{\beta\beta}(\sigma)$, for some σ. Then there are some strong limit cardinal λ of cofinality ω and some $\sigma' \subseteq \sigma$ such that $F \to G$ is a sentence of $L_{\lambda\lambda}(\sigma')$.

If $\models^{\omega} F \to G$, then, by Karp's result, there is an $L_{\lambda\lambda}(\sigma')$ sentence H in the common vocabulary such that $\models (F \to H) \wedge (H \to G)$.

Notice also that Cunningham's improvement of Karp's interpolation theorem paermits to derive a stronger version of the result, where $\models^{\omega}$ appears also in the thesis of the theorem.

We still assume GCH and $\alpha > \omega$ to be any uncountable regular cardinal. Then the following holds:

Corollary 3.2. Let $F \to G$ be an ω-valid sentence of $L_{\alpha^+\alpha}(\sigma)$, where σ is a set of constant and finitary relation symbols such that $|\sigma| < \alpha$. Then there exists a sentence H in the common vocabulary σ' of F and G and belonging to $L_{\alpha^{++}\alpha^+}(\sigma')$ such that $\models (F \to H) \wedge (H \to G)$.

Proof. Similar to the one of corollary 2.6. □

Let's state now two results (see [9]) that will be useful for constructing some counterexamples to interpolation theorems for $L_{\alpha^+\beta}(\sigma)$, $\omega < \beta < \alpha$.

Proposition 3.3. Let $A = \langle A \rangle$ be a standard structure and let k and λ be cardinal numbers such that $k > \lambda$. If $|\langle A \rangle| > \lambda$, then A is an elementary substructure with respect to the language $L_{k\lambda}(\emptyset) = L_{k\lambda}$ of $\langle A \cup B \rangle$, for any B.

Proposition 3.4. Let G be a sentence of $L_{k\lambda}(\emptyset)$ (i.e. only equality symbol occurs in G). Then G is true either in all standard structures of cardinality $\geqslant \lambda$ or in none of them.

We can prove now:

THEOREM 3.5. Let k and λ be cardinal numbers such that $k > \lambda^+$. If the set of constant symbols in σ has cardinality $\geqslant \lambda^+$, then there exists an ω-valid sentence in $L_{k\omega}(\sigma)$ without interpolant in any $L_{\mu\lambda}(\sigma)$.

PROOF. Let $\{c_i: i < \lambda\} \subseteq \sigma$ be a set of constant symbols. Let F be the sentence

$$(\forall v_0 \bigvee \{v_0 = c_i;\ i < \lambda\}) \to \bigvee \{c_m = c_n: \lambda < m < n < \lambda^+\} .$$

F is a valid sentence of $L_{k\omega}(\sigma)$, therefore F is ω-valid because $\models$ and $\models^\omega$ do coincide for languages where only finite quantifications are allowed.

Assume now that F has an interpolant H in $L_{\mu\lambda}$, for some μ. Then H has standard models of cardinality λ (because the antecedent of F does have some), but not all standard models of cardinality λ^+ are among its models (because not all of them are models of the consequent of F).

Then, by proposition 3.4, H does not have any model of cardinality λ^+, contradicting proposition 3.3. □

An immediate corollary of theorem 3.5 is the following:

COROLLARY 3.6. Let α and β be cardinal numbers such that $\alpha > \beta^+$.

If the set of constant symbols in σ has cardinality $> \beta^+$ there exists an ω-valid sentence $F \to G$ in $L_{\alpha^+\omega}(\sigma)$, and hence in $L_{\alpha^+\beta}(\sigma)$, with no interpolant in $L_{\mu\beta^+}(\sigma)$, for any σ. □

Corollary 3.6 can also be proved by assuming that σ contains only two binary relation symbols, as proved by Karp in [8].

REMARK: In [10] Malitz shows that, under the standard notion of satisfiability, Craig's interpolation theorem fails for all the languages $L_{k\lambda}(\sigma)$, if $k \geqslant \lambda > \omega$ or $k > \omega_1$.

He proves that there exist F and G sentences in $L_{\omega_1\omega_1}(\sigma_1)$ and $L_{\omega_2\omega_2}(\sigma_2)$ respectively such that $\models F \to G$ and there is no interpolant in $L_{k\lambda}(\sigma_1 \cap \sigma_2)$ for any k and λ.

Malitz's counterexample essentially appeals to the existence of at most one isomorphism between two well-ordered sets.

That is false in case of ω-satisfiability: the notion of isomorphism between two ω-chains of structures (see [2]) permits the existence of more than one isomorphism: that's the reason why Malitz's counterexample cannot be applied in case of ω-satisfiability.

Large classes of counterexamples to Craig's interpolation theorem in infinitary languages can also be found in [11].

Concerning Beth's definability theorem in infinitary languages, see [6].

4. λ-satisfiability.

A natural extension of ω-satisfiability is λ-satisfiability, where λ is an infinite cardinal distinct from ω.

The notions of λ-chain of structures and λ-satisfiability can be immediately derived from those in the introduction. There exist valid, but not λ-valid sentences: an example is the negation of the following sentence:

$$\bigwedge\{\exists v_0 P_i(v_0): i<\lambda\}\vee(\forall\{v_i: i<\lambda\}\bigvee\{\neg P_i(v_i): i<\lambda\}) .$$

Clearly this sentence is not satisfiable, but it is λ-satisfiable by $\langle A_i: i<\lambda\rangle$, with $A_i=i$ (intended as an ordinal), and $f(P_i)=\{i\}$.

If we work in the class of languages $L_{kk}(\sigma)$, where k is a strong limit cardinal of cofinality λ and σ is a set of finitary relation symbols with $|\sigma|<k$, then λ-satisfiability and ω-satisfiability are two distinct notions.

Let G be the sentence

$$\forall v_0\,\exists v_1 P(v_0, v_1)\wedge\forall\{v_i: i<\omega\}\bigvee\{\neg P(v_i, v_{i+1}): i<\omega\} .$$

G is ω-satisfiable, but not λ-satisfiable. If G were λ-satisfiable there would be a λ-chain of structures $\boldsymbol{A}$ such that $|A_i|<\omega$ for all $i<\lambda$.

For all $0<i<\omega$, let $\lambda_i=|\{A_j: j<\omega \text{ and } |A_j|=i\}|$.

It turns out that $\lambda_i<\lambda$ for all i's and $\Sigma\{\lambda_i: i<\omega\}=\lambda$, contradicting the regularity of λ.

On the other hand, let H be the following:

$$\forall\{v_i: i<\lambda\}\bigvee\{v_i=v_j: i<j<\lambda\}\wedge \left(\bigwedge\{\exists\{v_i: i<\delta\}\bigwedge\{v_i=v_j: i<j<\delta\}: \delta<\lambda\}\right).$$

The λ-chain of structures $\langle A_i : i < \lambda \rangle$ with $A_i = i$ (intended as an ordinal) λ-satisfies H. Since λ is a regular cardinal, there exists no ω-chain of structures that satisfies H.

The notion of λ-consistency property for $L_{kk}(\sigma)$ can be given in such a way that every member of the property is λ-satisfiable by a λ-chain of structures with every member of the chain having cardinality strictly smaller than k.

The tecnique is similar to the one described in [5]. The clauses defining the consistency property are essentially the same as in [5], but, in order to guarantee a model existence theorem, one must add the following:

$C7$) If μ is a limit ordinal less than λ and $\{s_i : i < \mu\}$ is a sequence of elements of S such that, for all $i < \mu$, s_{i+1} is obtained from s_i by means of finitely many applications of some clauses among $C1$-$C6$ and for any limit i, s_i is $\cup \{s_j : j < i\}$, then $\cup \{s_i : i < \mu\} \in S$.

5. Conclusion.

It is an open question whether Karp's interpolation theorem can also be proved in a strong form for $L_{\alpha\alpha}(\sigma)$, for a regular cardinal $\alpha > \omega$.

Our feeling is that the technique of consistency properties does not succeed in this case.

Another open problem is how to improve theorem 2.6 in such a way as to write $\models^{\omega}$ also in the thesis of the theorem, following the improvement of Karp's result given by Cunningham and Ferro in two different ways.

At first glance, this problem seems less difficult to attack than the former one.

Undoubtely, we do not yet a clear picture of all infinitary languages with respect to Karp's interpolation theorem, as we have for example with respect to Craig's interpolation theorem (dealing with standard satisfability), by virtue of the results in [10] and [11].

Concerning the notion of λ-satisfiability and related results, the model existence theorem seems to suggest the possibility of applying such a notion in the framework of interpolation theorems.

Anyway, the additional closure property $C7$) makes the notion hard to use.

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