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On a Hamilton-Jacobi equation

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On a Hamilton-Jacobi Equation.

GIOVANNI BASSANELLI (*)

SUMMARY - We prove a theorem of existence and uniqueness for the problem:

$$\frac{\partial}{\partial u}\varphi\frac{\partial}{\partial v}\varphi+\sum a_{jk}(u,v,z)\frac{\partial}{\partial z_j}\varphi\frac{\partial}{\partial z_k}\varphi=0$$

with data: $\varphi(u, v, z, \eta) = z\cdot\eta$ for $uv = 0$, on a strip $[0, U]\times\overline{\mathbf{R}}^{+}\times\mathbf{R}^{n}\times\dot{\mathbf{R}}^{n}$ where $A = (a_{jk})$ is a C^{∞}, $n\times n$, negative definite real matrix.

1. Introduction.

Let $A = A(u, v, z)(u, v\geqslant 0,\ z\in\mathbf{R}^{n},\ n\geqslant 1)$ be an $n\times n$ matrix. In this note we shall study the following problem:

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial u}\varphi\,\dfrac{\partial}{\partial u}\varphi+\left\langle A\,\dfrac{\partial}{\partial z}\varphi\,,\ \dfrac{\partial}{\partial z}\varphi\right\rangle=0\,, \\ \varphi(u, v, z, \eta)=z\cdot\eta\,, \qquad \text{if } uv=0\,, \end{cases}$$

with $\eta\in\dot{\mathbf{R}}^{n}$. First we want to show, briefly, the reason for our interest in (1.1).

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The formulation of quantum field theories in light-cone coordinates is useful in some subjects of physics (see [4], [6], [9]) and such a reformulation involves a study of the most important equations (Klein-Gordon, Dirac, etc.) in this frame. For example, let us consider the following Cauchy problem:

$$(1.2)\qquad \begin{cases} (\partial^2_{uv} - \Delta_z) f = 0 & \text{in } [0, U]\times \boldsymbol{R}\times \boldsymbol{R}^n,\\ f(0, v, z) = g(v, z), \end{cases}$$

(see [5]), this is a natural problem for the wave equation in light-cone coordinates. For a more general operator

$$P = \partial^2_{uv} + \sum a_{jk}\partial^2_{z_j z_k} + \sum b_j \partial_{z_j} + c\,,$$

with smooth coefficients, which are constant outside a compact subset of $\boldsymbol{R}_u\times\boldsymbol{R}_v\times\boldsymbol{R}^n_z$, and $A = (a_{jk})$ negative definite (there is no restriction assuming that A is real and symmetric), the corresponding Cauchy problem is

$$(1.3)\qquad \begin{cases} Pf = 0 & \text{in } [0, U]\times \boldsymbol{R}\times \boldsymbol{R}^n,\\ f(0, v, z) = g(z, u)\,. \end{cases}$$

This is a characteristic Cauchy problem and if, as $v \to -\infty$, g satisfies some growth conditions, then it is « well-posed » (see [1]). A formal computation performed on (1.2) suggests that, in order to construct a parametrix for (1.3), we must look for an operator of the form

$$(Eg)(u, v, z) = \\ = g(v, z) - \int_{-\infty}^{v}\int_{\eta} \exp[i\varphi^{\pm}(u, v, v', z, \eta)]\, a^{\pm}(u, v, v', z, \eta)\, g(v', \eta)\, d\eta\, dv'.$$

By a well known argument (see e.g. [2], [7]) φ^+ and φ^- have to be real and homogeneous functions of degree 1 in η, and they must satisfy equation (1.1) with the boundary condition

$$\varphi^{\pm}(u, v, v', z, \eta) = z\cdot\eta\,,\qquad \text{if } u(v - v') = 0\,.$$

But actually v' is only a parameter, which, at present, we can overlook; thus we have exactly problem (1.1).

Our main result is the following *Theorem* (see Theorem 6.1): there exists $U > 0$ such that (1.1) has two solutions $\varphi^{\pm}(u, v, z, \eta)$, real and homogeneous of degree 1 in η, defined on the strip $[0, U]\times\overline{\boldsymbol{R}}^+\times\boldsymbol{R}^n\times\times\dot{\boldsymbol{R}}^n$, which are smooth for $uv > 0$, and with a suitable regularity at the boundary; every solution φ, satisfing the above conditions, coincides with φ^+ or φ^- on each connected component of $[0, U]\times\overline{\boldsymbol{R}}^+\times\times\boldsymbol{R}^n\times\dot{\boldsymbol{R}}^n$.

In order to prove this theorem we shall study the Hamilton-Jacobi problem (2.1) which, as we point out in § 3, is very different from the classical case, and it seems us to be of some interest.

2. Firstly we show that problem (1.1) can be reduced to a Hamilton-Jacobi equation, then we shall study the associated bicharacteristic curves.

Let $A = A(u, v, z) \in C^\infty(\overline{\boldsymbol{R}}^+\times\overline{\boldsymbol{R}}^+\times\boldsymbol{R}^n)$ be a $n\times n$ real symmetric matrix, negative definite and constant outside a compact subset of $\overline{\boldsymbol{R}}^+\times\overline{\boldsymbol{R}}^+\times\boldsymbol{R}^n$. By means of

$$x_1 = u + v\,, \qquad x_2 = u - v\,, \qquad x_{2+j} = z_j \qquad (j = 1, \dots, n)\,,$$

and writing $x' = (x_3, \dots, x_{2+n})$, $x = (x_1, x_2, x')$, $\psi(x, \eta) = \varphi(u, v, z, \eta)$, $B(x) = A(u, v, z)$, problem (1.1) becomes

$$\begin{cases} \left(\dfrac{\partial}{\partial x_1}\psi\right)^2 - \left(\dfrac{\partial}{\partial x_2}\psi\right)^2 + \left\langle B(x)\dfrac{\partial}{\partial x'}\psi, \dfrac{\partial}{\partial x'}\psi\right\rangle = 0\,, \\ \psi(|x_2|, x_2, x', \eta) = x'\cdot\eta\,. \end{cases}$$

This suggests that we fix $\eta\in\dot{\boldsymbol{R}}^n$, from now on, and consider the following equations

$$\frac{\partial}{\partial x_1}\psi \pm \left[\left(\frac{\partial}{\partial x_1}\psi\right)^2 - \left\langle B(x)\frac{\partial}{\partial x'}\psi, \frac{\partial}{\partial x'}\psi\right\rangle\right]^{\frac{1}{2}} = 0 \tag{2.1}$$

whit data at the boundary:

$$\psi(|x_2|, x_2, x', \eta) = x'\cdot\eta\,. \tag{2.2}$$

(For simplicity we shall study only the case with sign « — » and, in the last paragraph, we shall look at the dependence on η). Let us remark that $B(x)$ is defined in $\bar{\Omega}$ where

$$\Omega = \{x \in \boldsymbol{R}^{2+n};\, x_1 > |x_2|\}$$

and the hypotheses on A give: there exist positive constants M_j ($j = 0, 1, 2$) and a matrix B_0 such that

$$\text{if } \|x\| \geqslant M_0\,, \qquad \text{then } B(x) = B_0\,, \tag{2.3}$$

and, for every $(x, \eta) \in \bar{\Omega} \times \boldsymbol{R}^n$,

$$-M_1\|\eta\|^2 \geqslant \langle B(x)\eta, \eta\rangle \geqslant -M_2\|\eta\|^2\,. \tag{2.4}$$

The symbol $p\colon \dot{T}^*(\bar{\Omega}) = \bar{\Omega} \times \dot{\boldsymbol{R}}^{2+n} \to \boldsymbol{R}$ of equation (2.1) is

$$p(x, \xi) = \xi_1 - (\xi_2^2 - \langle B(x)\xi', \xi'\rangle)^{\frac{1}{2}}\,.$$

Since $(\partial/\partial\xi_1)p = 1$, we can choose x_1 as parameter of the null bicharacteristic curves, which are all the curves in $\dot{T}^*(\bar{\Omega})$ of the form

$$x = x(t)\,, \qquad \xi = \xi(t) \qquad \text{with } x_1(t) = t \tag{2.5}$$

such that

$$\dot{x}_2 = -\xi_1^{-1}\xi_2\,,\ \dot{x}' = \xi_1^{-1}B(x)\xi'\,,\ \dot{\xi}_j = -\frac{1}{2}\xi_1^{-1}\left\langle\frac{\partial}{\partial x_j}B(x)\xi', \xi'\right\rangle \qquad (j = 1, \ldots, 2+n)\,. \tag{2.6}$$

2.1. Proposition. Let $(x^0, \xi^0) \in p^{-1}(0)$. Then there exists $\bar{t} \geqslant 0$ such that the bicharacteristic curve, of the form (2.5), passing through (x^0, ξ^0) is defined on $[\bar{t}, +\infty)$ and $x(\bar{t}) \in \partial\Omega$. Moreover there are two cases:

(i) If $\xi^{0\prime} = 0$, then $\bar{t} = 0$ and

$$(x(t), \xi(t)) = ((t, \pm t, x^{0\prime}), (\xi_1, \mp\xi_1, 0)) \in \partial\dot{T}^*(\partial\Omega)\,,$$

for every $t \geqslant 0$.

(ii) If $\xi^{0\prime} \neq 0$, then, for every $t > \bar{t}$, $x(t) \in \Omega$ and $\xi'(t)$ never vanishes.

In both the cases

$$|x_2(t) - x_2^0| \leqslant |t - x_1^0| \tag{2.7}$$

and, for a suitable constant $K > 0$ (independent from (x^0, ξ^0)),

$$\|\xi^{0\prime}\| \exp(-K|t - x_1^0|) \leqslant \|\xi'\| \leqslant \|\xi^{0\prime}\| \exp(K|t - x_1^0|) . \tag{2.8}$$

PROOF. First we verify (i). From (2.4) and (2.6) we get

$$\frac{1}{2}\frac{d}{dt}\|\xi'\|^2 \leqslant \left| -\frac{1}{2}\sum_{j=3}^{n+2} \xi_1^{-1} \left\langle \frac{\partial}{\partial x_j} B(x)\xi', \xi' \right\rangle \xi_j \right| \leqslant \frac{1}{2} M_1^{-\frac{1}{2}} \left(\sup_{x.j} \left\| \frac{\partial}{\partial x_j} B(x) \right\| \right) \|\xi'\|^2$$

from which (2.8) follows. Thanks to (2.6) and $p(x(t), \xi(t)) = 0$, it is straightforward to prove (2.7) and that $(x(t), \xi(t))$ is bounded on every bounded interval. So we can extend the curve to an interval of the form $[\bar{t}, +\infty)$, with $x(\bar{t}) \in \partial\Omega$. Q.E.D.

3. In this section we state all the machinary which we need in order to prove our main theorem. Write

$$\partial\Omega = \partial_0\Omega \cup \partial_+\Omega \cup \partial_-\Omega$$

where

$$\partial_0\Omega = \{(0, 0)\} \times \boldsymbol{R}^n , \qquad \partial_\pm\Omega = \{x \in \Omega;\ 0 < x_1 = \pm x_2\} .$$

Consider the $(n + 1)$-dimensional submanifold $L_0(\eta) = \{(x^0, \xi^0) \in p^{-1}(0);$ $x^0 \in \partial_0\Omega$ and $\xi^{0\prime} = \eta\}$ and the union $L(\eta)$ of the all null-bicharacteristic curves issued from $L_0(\eta)$; we point out that $\pi(L(\eta))$ (here π is the natural projection of $\dot{T}^*(\bar{\Omega})$ onto Ω) is a family of curves issued from the wedge $\partial_0\Omega$, such that there are continuously many curves starting from each point of $\partial_0\Omega$ and as we shall see, on each of these curves the solution $\psi(x)$ must be constant. (So problem (2.1-2) is very different from the classical one for a Hamilton-Jacobi equation: in fact, in the latter, you usually give the solution ψ on a hypersurface L_0 and, for every $P \in L_0$, $\psi(P)$ propagates constant along a unique curve issued from P (see e.g. [8]). Unfortunately, by Proposition 2.1, $\pi(L(\eta)) \cap \partial_\pm\Omega = \emptyset$, so in order to study $\pi^{-1}(x) \cap L(\eta)$, for x near $\partial_\pm\Omega$, it is necessary to introduce the following machinary:

3.1 DEFINITION.

(3.1) $\tilde{L}_0(\eta) =$

$$= \{(x^0, \tilde{\xi}^0);\ x_0 = (0, 0, y),\ \tilde{\xi}^0 = (1, \cos\theta, h(y)\sin\theta\eta),\ y \in \mathbf{R}^n,\ 0 \leqslant \theta \leqslant \pi\}$$

where $h(y) = (-\langle B(x^0)\eta, \eta\rangle)^{-\frac{1}{2}}$

Remark that

$$L_0(\eta) = \{(x^0, \xi^0); (x_0, \tilde{\xi}^0) \in \tilde{L}_0(\eta),\ 0 < \theta < \pi,\ \xi^0 = (h(y)\sin\theta)^{-1}\tilde{\xi}^0\}\ .$$

We shall denote by $(x(t, \theta, y),\ \xi(t, \theta, y))$ and $(\tilde{x}(t, \theta, y),\ \tilde{\xi}(t, \theta, y))$ respectively the coordinates of the bicharacteristic of the form (2.5) issued from $(x^0, \xi^0) \in L_0(\eta)$ and $(x^0, \tilde{\xi}^0) \in \tilde{L}_0(\eta)$. From (2.6), by a homogeneity argument it follows

$$\begin{cases} \tilde{x}(t, \theta, y) = x(t, \theta, y)\ , \\ \tilde{\xi}(t, \theta, y) = h(y)\sin\theta\xi(t, \theta, y)\ , \end{cases} \qquad 0 \leqslant t\ ,\ 0 < \theta < \pi\ ,\ y \in \mathbf{R}^n\ .$$

Now we choose positive constants ε, ω in a suitable way, as specified in Theorem 3.2 below, and let Σ be the union of the following subsets of $\dot{\mathbf{R}}^+ \times (0, \pi) \times \mathbf{R}^n$:

$$(0, \varepsilon) \times (0, \pi) \times \mathbf{R}^n\ , \qquad (0, M_0 + 1] \times (0, \omega) \times \mathbf{R}^n$$

and

$$\{(t, \theta);\ M_0 + 1 \leqslant t \text{ and } 0 < \theta < \omega(t - M_0)^{-\frac{1}{2}}\} \times \mathbf{R}^n\ ;$$

(here M_0 is given from (2.3)); moreover let us define the following C^∞ functions: for every $(t, \theta, y) \in \overline{\mathbf{R}}^+ \times [0, \pi] \times \mathbf{R}^n$,

$$F(t, \theta, y) := x(t, \theta, y) \qquad \text{and} \qquad u(t, \theta, y) := \tfrac{1}{2}[x_1(t, \theta, y) + x_2(t, \theta, y)]\ .$$

So we can state a technical theorem which would be proved in next section:

3.2. THEOREM. There exist positive constants C, ε, δ, ω such that $\omega < \pi$, $\varepsilon < M_0 + 1$ and

(i) dF is non-singular on the closure of Σ in $\dot{\mathbf{R}}^+ \times (0, \pi) \times \mathbf{R}^n$;

(ii) for every (t, θ, y) such that $0 \leqslant t$, $\theta \in [0, \pi]$, $\|y\| \geqslant C$,

$$F(t, \theta, y) = (t, -t, h_0 t \sin\theta\, B_0 \eta + y)$$

(where $h_0 = (-\langle B_0 \eta, \eta \rangle)^{-\frac{1}{2}}$);

(iii) on $\overline{\boldsymbol{R}}^+ \times [0, \omega] \times \boldsymbol{R}^n$, $u(t, \theta, y) \geqslant \delta t \theta^2$.

4. In order to prove Theorem 3.2 we need some preliminary results.

If $\|y\| \gg 1$, then we can explicity compute $F(t, \theta, y)$:

4.1. Proposition. There exists a positive constant C such that

$$x_2(t, \theta, y) = -t \cos\theta\,, \qquad x'(t, \theta, y) = h_0 t \sin\theta\, B_0 \eta + y$$

for every $t \geqslant 0$, $\theta \in [0, \pi]$, $\|y\| \geqslant C$.

Proof. It is easy to see from (2.3), (2.4) and (2.6) that there exists a suitable positive constant C such that $\|x(t, \theta, y)\| \geqslant M_0$, for $t \geqslant 0$, $\theta \in [0, \pi]$, $\|y\| \geqslant C$; therefore (2.6) can be directly integrated. Q.E.D.

Since $x_1(t, \theta, y) = t$, we get

4.2. Corollary. $dF(t, \theta, y)$ is non-singular for $t > 0$, $\theta \in (0, \pi)$, $\|y\| \geqslant C$.

By definition

$$x_j, \xi_j \in C^\infty(\overline{\boldsymbol{R}}^+ \times [0, \pi] \times \boldsymbol{R}^n) \qquad \text{and} \qquad \xi_j \in C^\infty(\dot{\boldsymbol{R}}^+ \times (0, \pi) \times \boldsymbol{R}^n)\,,$$

$(j = 1, \ldots, 2 + n)$; a straightforward computation gives the following.

4.3. Lemma. There exist smooth functions r, R, s_1, s_2, S such that

$$x_2(t, \theta, y) = -t - \left[\frac{1}{2} h^2(y) \int_0^t \langle B(t', -t', y)\eta, \eta \rangle\, dt' \right] \theta^2 + r(t, \theta, y)\, \theta^3\,,$$

$$x'(t, \theta, y) = y + \left[h(y) \int_0^t B(t', -t', y)\eta\, dt' \right] \theta + R(t, \theta, y) \theta^2\,,$$

$$\xi_j(t, \theta, y) = 1 + s_j(t, \theta, y)\, \theta^2\,, \qquad (j = 1, 2)\,,$$

$$\xi'(t, \theta, y) = h(y) \eta \theta + S(t, \theta, y)\, \theta^2\,,$$

and

$$\text{if } t=0, \text{ then } r=0,\ R=0,\ S=(\operatorname{sen}\theta-\theta)\theta^{-2}h(y)\eta,$$

$$\text{if } \|y\|\geqslant C, \text{ then } r,\ R \text{ do not depend on } y.$$

Moreover $\xi'(t,\theta,y)$ can be smoothly extended to $\overline{\boldsymbol{R}}^{+}\times[0,\pi]\times\boldsymbol{R}^{n}$ be means of $\xi'(t,0,y)=\xi'(t,\pi,y)=\eta$.

4.4. Proposition. Let $t_0>0$. Then there exists a positive constant ω_1 such that $dF(t,\theta,y)$ is non singular on $(0,t_0]\times[0,\omega_1]\times\boldsymbol{R}^{n}$.

Proof. Let

$$A(t,\theta,y)=\begin{pmatrix}\theta^{-1}t^{-1}\dfrac{\partial}{\partial\theta}x_2 & \theta^{-1}t^{-1}\dfrac{\partial}{\partial y}x_2\\[2ex] \dfrac{\partial}{\partial\theta}x' & \dfrac{\partial}{\partial y}x'\end{pmatrix},$$

$$B(t,\theta,y)=\begin{pmatrix}t^{-1}\dfrac{\partial^2}{\partial\theta^2}x_2(t,0,y) & 0\\[2ex] \dfrac{\partial}{\partial\theta}x'(t,\theta,y) & I_n\end{pmatrix}.$$

From Lemma 4.3 and (2.4) it follows that $t^{-1}(\partial^2/\partial\theta^2)x_2(t,0,y)\geqslant M_1/M_2$; therefore, for $(t,\theta,y)\in[0,t_0]\times[0,\pi]\times\boldsymbol{R}^{n}$, $B(t,\theta,y)$ is in a compact set of non singular matrices. Note that $\det\left((\partial x_2\,\partial x')/(\partial\theta\,\partial y)\right)=\theta t\det A$. Now it is enough to show that for a suitable $M>0$, $\|A-B\|\leqslant M\theta$ on $[0,t_0]\times[0,\pi]\times\boldsymbol{R}^{n}$.

Since $x_2=0$ for $t=0$, from Lemma 4.3 we get

$$\theta^{-1}\frac{\partial}{\partial\theta}x_2(t,\theta,y)-\frac{\partial^2}{\partial\theta^2}x_2(t,0,y)=$$

$$=\int_0^t\left[\theta^{-1}\frac{\partial}{\partial\theta}\dot{x}_2(t',\theta,y)-\frac{\partial^2}{\partial\theta^2}\dot{x}_\theta(t',0,y)\right]dt'=\int_0^t\left(\theta\frac{\partial}{\partial\theta}\dot{r}+3\dot{r}\right)dt'\,\theta\,.$$

But r does not depend on y for $\|y\|\gg$, so

$$\left|\theta^{-1}\frac{\partial}{\partial\theta}x_2(t,\theta,y)-\frac{\partial^2}{\partial\theta^2}x_2(t,0,y)\right|\leqslant Mt\theta\,.$$

In a similar way one can show

$$\left|\theta^{-1}\frac{\partial}{\partial y}x_2(t,\theta,y)\right|\leqslant Mt\theta \qquad \text{and} \qquad \left\|\frac{\partial}{\partial y}x'(t,\theta,y)-I_n\right\|\leqslant M\theta\,. \qquad \text{Q.E.D.}$$

4.5. REMARK. Since an analogous of Lemma 4.3 holds for θ near π, we get: for every $t_0>0$ there exists ω_1 such that: $0<\omega_1<\pi$ and $dF(t,\theta,y)$ is non-singular on

$$(0,t_0]\times\big((0,\omega_1]\cup[\pi-\omega_1,\pi)\big)\times\boldsymbol{R}^n\,.$$

4.6 PROPOSITION. There exists a positive constant ω_2 such that $dF(t,\theta,y)$ is not singular for $t\geqslant M_0+1$, $0<\theta\leqslant\omega_2(t-M_0)^{-\frac{1}{2}}$, $y\in\boldsymbol{R}^n$.

In order to prove this Proposition we need some preparation. For $t\geqslant M_0$, all bicharacteristics are straight-line, so

$$\frac{\partial}{\partial\theta}x(t,\theta,y)=\frac{\partial}{\partial\theta}x(M_0,\theta,y)+\frac{\partial}{\partial\theta}\dot{x}(M_0,\theta,y)(t-M_0)$$

and an analogous formula holds for $(\partial/\partial y)x$. More precisely, thank to Lemma 4.3:

$$\begin{aligned}
\frac{\partial}{\partial\theta}x_2(t,\theta,y)\quad &=c_{00}\theta+d_{00}\theta(t-M_0)\,,\\
\frac{\partial}{\partial y_k}x_2(t,\theta,y)\quad &=c_{0k}\theta^2+d_{0k}\theta^2(t-M_0)\,,\\
\frac{\partial}{\partial\theta}x_{2+j}(t,\theta,y)&=c_{j0}+d_{j0}(t-M_0)\,,\\
\frac{\partial}{\partial y_k}x_{2+j}(t,\theta,y)&=\delta_{jk}+c_{jk}\theta+d_{jk}\theta(t-M_0)\,,\qquad (j,k=1,\dots,n)\,.
\end{aligned}$$

here c_{jk} and d_{jk} are C^∞ functions of $\theta\in[0,\pi]$, $y\in\boldsymbol{R}^n$, $(j,k=1,\dots,n)$. Denote by J_k, C_k, D_k respectively the k-th column of matrices

$$J=\begin{pmatrix}0&0\\0&I_n\end{pmatrix},\ (c_{jk})_{jk}\,,\ (d_{jk})_{jk}\,;$$

let

$$E_{jk} = \begin{cases} J_k & \text{if } j = 0 \\ C_k & \text{if } j = 1 \\ D_k & \text{if } j = 2 \end{cases} \qquad \begin{array}{l} \text{and } E_{j_0 \dots j_n} = [E_{j_0 0}, \dots, E_{j_n n}] \\ \text{for } j_0, \dots, j_n \in \{0, 1, 2\} . \end{array}$$

Finally let

$$\gamma = \gamma_{j_1 \dots j_n} = \# \{k \geqslant 1;\ j_k = 1\} , \qquad \delta = \delta_{j_1 \dots j_n} = \# \{k \geqslant 1;\ j_k = 2\} ,$$

and

$$\alpha^* = \begin{cases} \alpha & \text{if } \alpha \geqslant 0 , \\ 0 & \text{if } \alpha < 0 . \end{cases}$$

PROOF OF PROPOSITION 4.6. Since $[\partial/\partial y_k \log h(y)]\, C_0(0, y) = C_k(0, y)$, $k = 1, \dots, n$ and the same formula holds for D_0, D_k, it follows that

$$\varphi_{1,j_1,\dots,j_n}(\theta, y) = \theta^{-\gamma-(\delta-1)^*} \det E_{1,j_1,\dots,j_n}$$

and

$$\varphi_{2,j_1,\dots,j_n}(\theta, y) = \theta^{-(\gamma-1)^*-\delta} \det E_{1,j_1,\dots,j_n}$$

are C^∞ functions. Therefore

$$\begin{aligned} &\det\left(\frac{\partial x_2}{\partial \theta}\frac{\partial x'}{\partial y}\right)(t, \theta, y) = \\ &= \sum_{j_1 \dots j_n} [\theta^{2\gamma+\delta+(\delta-1)^*+1}(t - M_0)^\delta \varphi_{1,j_1,\dots,j_n}(\theta, y) + \\ &+ \theta^{\gamma+(\gamma-1)^*+2\delta+1}(t - M_0)^{\delta+1}\varphi_{2,j_1,\dots,j_n}(\theta, y)] = \\ &= \theta[\varphi_{1,0,\dots,0}(\theta, y) + \sum_{\delta=0,\gamma>0} \theta^{2\gamma}\varphi_{1,j_1,\dots,j_n}(\theta, y)] + \\ &+ \theta(t - M_0)\Big[\varphi_{2,0,\dots,0}(\theta, y) + \sum_{\gamma+\delta>0} \theta^{\gamma+(\gamma-1)^*+2\delta}(t - M_0)^\delta \varphi_{2,j_1,\dots,j_n}(\theta, y) + \\ &\qquad + \sum_{\delta>0} \theta^{2\gamma+2\delta-1}(t - M_0)^{\delta-1}\varphi_{1,j_1,\dots,j_n}(\theta, y)\Big] . \end{aligned}$$

Now it is enough to make the following remarks:

$$\varphi_{1,0,\dots,0}(0, y) = c_{00}(0, y) = -h^2(y)\langle B_0\eta, \eta, \eta\rangle \geqslant \frac{M_1}{M_2},$$

$$\varphi_{2,0,\dots,0}(0, y) = d_{00}(0, y) = -h^2(y)\int_0^{M_0}\langle B(t, -t, y)\eta, \eta\rangle\, dt \geqslant \frac{M_0 M_1}{M_2},$$

and $\varphi_{j_0,j_1,\dots,j_n}(\theta, y)$ are bounded on $[0, \pi]\times \boldsymbol{R}^n$. Q.E.D.

4.7. Proposition. Let ω be such that $0 < \omega < \pi/2$. Then there exists $\varepsilon > 0$ such that $dF(t, \theta, y)$ is nonsingular on $(0, \varepsilon]\times[\omega, \pi-\omega]\times \boldsymbol{R}^n$.

Proof. Initial data (3.1) give

$$\frac{\partial}{\partial\theta}x_2(0, \theta, y) = 0 \quad \text{and} \quad \frac{\partial}{\partial y}x_2(0, \theta, y) = 0\ ;$$

then from (2.6) and (3.1) it follows that

$$\lim_{t\to 0+} t^{-1}\det\left(\frac{\partial x_2}{\partial\theta}\frac{\partial x'}{\partial y}\right)(t, \theta, y) =$$

$$= \det\begin{pmatrix} \dfrac{\partial}{\partial\theta}\dot{x}_2 & \dfrac{\partial}{\partial y}\dot{x}_2 \\ \dfrac{\partial}{\partial\theta}x' & \dfrac{\partial}{\partial y}x' \end{pmatrix}(0, \theta, y) = \operatorname{sen}\theta\,. \qquad \text{Q.E.D.}$$

The above results prove Theorem 3.2 (i) and (ii). For the last statement we show the following

4.8. Proposition. There exist positive constants ω_3, δ such that $u(t, \theta, y) \geqslant \delta t\theta^2$ on $\overline{\boldsymbol{R}}^+\times[0, \omega_3]\times \boldsymbol{R}^n$.

Proof. Since

$$-\frac{1}{2}h^2(y)\int_0^t\langle B(t', -t', y)\eta, \eta\rangle\, dt' \geqslant \frac{tM_1}{2M_2} \qquad \text{and } r(0, \theta, y) = 0$$

we get

$$u(t,\theta,y)\geqslant\frac{1}{2}\left[\frac{tM_1}{2M_2}\theta^2+\int_0^t\dot{r}(t',\theta,y)\,dt'\,\theta^3\right].$$

It is enough to remark that, for $t\geqslant M_0$, $\dot{x}_2$ (and so $\dot{u}$ too) not depend on t, therefore $\dot{u}$ is bounded on $\bar{\boldsymbol{R}}^+\times[0,\pi]\times\boldsymbol{R}^n$. Q.E.D.

5. In this section we shall show that, for all fixed $\eta\in\dot{\boldsymbol{R}}^n$, there exists a (unique) solution $\psi(x)$ of (2.1-2), which is definite in a suitable domain.

5.1. Lemma:

(i) $F|_\Sigma:\Sigma\to F(\Sigma)$ is locally invertible and $F(\Sigma)$ is an open subset of Ω;

(ii) $F|_{\bar\Sigma}:\bar\Sigma\to\bar\Omega$ is a closed map;

(iii) For every $x\in F(\Sigma)$, $F^{-1}(x)\cap\Sigma$ is finite.

Proof. (i) follows from Theorem 3.2 (i) and Proposition 2.1.

(ii) Let $P_\nu=(t_\nu,\theta_\nu,y_\nu)$, $\nu\in N$, be a sequence in $\bar\Sigma$ with $\lim_{\nu\to+\infty}F(P_\nu)=\bar{x}$; then $t_\nu=x_1(P_\nu)\to\bar{x}_1$; therefore, if $\{P_\nu\}$ is not bounded, $\{y_\nu\}$ is not bounded too and one can apply Theorem 3.2 (ii) in order to see that $\{P_\nu\}$ has a limit point; hence, in every way, $\{P_\nu\}$ has a limit point.

(iii) Since F is injective for $\|y\|\geqslant C$, it follows that $F^{-1}(x)\cap\Sigma$ is bounded. Theorem 3.2 (i) says that $F^{-1}(x)\cap\Sigma$ has no accumulation points in $\bar\Sigma\cap(\dot{\boldsymbol{R}}^+\times(0,\pi)\times\boldsymbol{R}^n)$; thanks to Proposition 2.1 it does not have any in $\bar\Sigma$ either. Q.E.D.

5.2. Theorem. The map $F|_\Sigma:\Sigma\to F(\Sigma)$ is a diffeomorphism.

Proof. By lemma 5.1, $F|_\Sigma$ is a covering map between arcwise connected spaces, so the cardinality of the fiber does not depend on the base-point.

Let $\Sigma_\varepsilon=(0,\varepsilon)\times(0,\pi)\times\boldsymbol{R}^n$, $\Omega^\varepsilon=\{x\in\Omega;\,x_1<\varepsilon\}$. From lemma 5.1 (ii) it follows that $F(\bar\Sigma_\varepsilon)=\overline{F(\Sigma_\varepsilon)}$, and from Proposition 2.1. $F(\partial\Sigma_\varepsilon)\cap\Omega^\varepsilon=\emptyset$; therefore $\overline{F(\Sigma_\varepsilon)}\cap\Omega^\varepsilon=F(\Sigma_\varepsilon)\cap\Omega^\varepsilon$; hence $F(\Sigma_\varepsilon)$ is an open and closed subset of Ω^ε, i.e. $F(\Sigma_\varepsilon)=\Omega^\varepsilon$.

This proves that $F|_{\Sigma_\varepsilon}: \Sigma_\varepsilon \to \Omega^\varepsilon$ is a covering map between simply connected spaces, i.e. it is a homeomorphism. Q.E.D.

5.3. PROPOSITION. There exists $U > 0$ such that $\Omega_U \subseteq F(\Sigma)$.

PROOF. Choose $U < \min\{\varepsilon\delta\omega^2, \delta\omega^2\}$. Arguing as above it is enough to show $F(\partial\Sigma) \cap \Omega_U = \emptyset$.

From Theorem 3.2 (iii) it follows immediately that $u\big((\partial\Sigma - \partial\Sigma_\varepsilon) \cap \cap (\dot{\boldsymbol{R}}^+ \times (0, \pi) \times \boldsymbol{R}^n)\big) > U$. We shall see

$$u(\varepsilon, \theta, y) > U \tag{5.1}$$

for $\theta \geqslant \omega$, $y \in \boldsymbol{R}^n$. Both the hypersurface $\theta = \omega$ and $u(\varepsilon, \theta, y) = \varepsilon\delta\omega^2$ disconnect the strip $(0, \pi) \times \boldsymbol{R}^n$; then from $u(\varepsilon, 0, y) = 0$ and $u(\varepsilon, \omega, y) \geqslant \geqslant \varepsilon\delta\omega^2$ it follows (5.1). Q.E.D.

Let now $t = t(x)$, $\theta = \theta(x)$, $y = y(x)$ be the components of the inverse map $\Omega_U \to F^{-1}(\Omega_U)$; they are smooth, moreover

5.4. LEMMA. (i) $t(x) = x$;

(ii) $\theta(x)$ can be continuously extended to $\bar{\Omega}_U - \partial_0\Omega$ by means of $\theta(\partial_\pm\Omega \cap \bar{\Omega}_U) = \arccos(\pm 1)$;

(iii) $y(x)$ can be continuously extended to $\bar{\Omega}_U$ by means of $y(|x_2|, x_2, x') = x'$.

PROOF. Check the continuity of extensions (ii) and (iii) on a sequence $\{x_\nu\}$ in Ω_U converging to the boundary. Use Theorem 5.1 (ii) and Proposition 2.1.

Finally we can establish that

5.5. THEOREM. For every $\eta \in \dot{\boldsymbol{R}}^n$ there exists a unique real solution $\psi \in C^\infty(\Omega_U)$ of (2.1-2) such that

(i) $\psi \in C(\bar{\Omega}_U)$,

(ii) $\dfrac{\partial}{\partial x_1}\psi \pm \dfrac{\partial}{\partial x_2}\psi \in C\big((\bar{\Omega}_U \cap \partial_\pm\Omega) \cup \Omega_U\big)$ and $\dfrac{\partial}{\partial x_j}\psi \in C(\bar{\Omega}_U)$,

$(j = 3, \ldots, 2 + n)$.

PROOF. *Existence.* Since the symplectic form $\sum_{j=1}^{2+n} d\xi_j \wedge dx_j$ vanishes identically on $L_0(\eta)$ we can apply Theorem 6.4.3 in [3] in order to see that the 1-form

$$\Omega_U \in x \to \big(x, \xi(t(x), \theta(x), y(x))\big) \in L(\eta) \subseteq p^{-1}(0) \subseteq \dot{T}^*(\bar{\Omega})$$

is closed. Thus there is a function $\psi \in C^\infty(\Omega_U)$, unique up to a constant, such that

$$d\psi(x) = \xi\big(t(x), \theta(x), y(x)\big) ; \tag{5.2}$$

then ψ satisfies (2.1). Check that $(\partial/\partial t)[\psi(x(t, \theta, y))] = 0$, therefore

$$\frac{\partial}{\partial\theta}[\psi(x(t, \theta, y))] = \lim_{t\to 0+} \frac{\partial}{\partial\theta}[\psi(x(t, \theta, y))] = \sum_{j=1}^{2+n} \xi_j(0, \theta, y)\,\partial_\theta x_j(0, \theta, y) = 0$$

and, in the same way,

$$\frac{\partial}{\partial y_k}[\psi(x(t, \theta, y))] = \eta_k \qquad (k = 1, \ldots, n) .$$

Hence, up to a constant, $\psi(x(t, \theta, y)) = y\cdot\eta$. Next, for a suitable choice of the constant,

$$\psi(x) = y(x)\cdot\eta , \quad \text{for every } x \in \Omega_U . \tag{5.3}$$

From Lemma 4.3.

$$\frac{\partial}{\partial x_1}\psi(x) - \frac{\partial}{\partial x_2}\psi(x) =$$

$$= \big[s_1\big(t(x), \theta(x), y(x)\big) - s_2\big(t(x), \theta(x), y(x)\big)\big] \frac{\theta^2(x)}{h\big(y(x)\big)\sin\theta(x)} ,$$

$$\frac{\partial}{\partial x'}\psi(x) = \frac{\big[\eta + S\big(t(x), \theta(x), y(x)\big)\,\theta(x)\,h^{-1}\big(y(x)\big)\big]\theta(x)}{\sin\theta(x)} .$$

If $\bar{x} \in \bar{\Omega}_U \cap \partial_-\Omega$ (resp. $\bar{\Omega}_U \cap (\partial_0\Omega \cup \partial_-\Omega)$) apply Lemma 5.4 to see that

$$\lim_{x\to\bar{x}} \frac{\partial}{\partial x_1}\psi(x) - \frac{\partial}{\partial x_2}\psi(x) = 0 \qquad \left(\text{resp. } \lim_{x\to\bar{x}} \frac{\partial}{\partial x'}\psi(x) = \eta\right) .$$

Uniqueness. Let ψ satisfy all the hypotheses. Certainly there exists $x^0 \in \Omega_U$ with $d\psi(x^0) \neq 0$. Let $(x(t), \xi(t))$, with $\bar{t} \leqslant t < +\infty$ and $x(\bar{t}) \in \partial\Omega$, be the null bicharacteristic passing through $(x^0, d\psi(x^0))$. As is well known, $\xi(t) = d\psi(x(t))$.

Suppose for a moment that $x(\bar{t}) \in \partial_\pm \Omega$. Now $x(\bar{t}) = (\bar{t}, \pm\bar{t}, x'(\bar{t}))$ with $\bar{t} > 0$, therefore, from (2.2)

$$\xi'(\bar{t}) = \frac{\partial}{\partial x'}\psi(x(\bar{t})) = \frac{\partial}{\partial x'}(x'\cdot\eta)\big|_{x'=x'(\bar{t})} = \eta$$

and

$$\xi_1(\bar{t}) \pm \xi_2(\bar{t}) = \left(\frac{\partial}{\partial x_1}\psi(x(\bar{t})) \pm \frac{\partial}{\partial x_2}\psi(x(\bar{t}))\right) =$$

$$= \frac{d}{dt}[\psi(t, \pm t, x'(\bar{t}))]\big|_{t=\bar{t}} = \frac{d}{dt}x'(\bar{t})\cdot\eta\big|_{t=\bar{t}} = 0 ;$$

but this is in contraddiction with $p(x(\bar{t}), \xi(\bar{t})) = 0$.

We conclude that $\bar{t} = 0$, then $(x(0), \xi(0)) \in L_0(\eta)$. This proves $(x^0, d\psi(x^0)) \in L(\eta)$. From (2.8) it follows that

$$\left\|\frac{\partial}{\partial x'}\psi(x^0)\right\| \geqslant \|\eta\|\exp(-kx_1^0),$$

so $d\psi$ never vanishes and the above argument can be repeated for every point of Ω_U. Hence we obtain (5.2) for every $x \in \Omega_U$; but, as already recalled, this determines ψ up to a constant. The thesis follows since $\psi = x\cdot\eta$ on $\partial\Omega$. Q.E.D.

6. Finally we have to consider the dependence on $\eta \in \dot{\boldsymbol{R}}^n$, then we write $\psi_{(\eta)}$ instead of ψ, which is defined on $\bar{\Omega}_U$ with $U = U(\eta)$.

It is easy to see, from (2.6), that $F(t, \theta, y)$ and $y(t, \theta, y)$ are homogeneous of degree 0 in η, therefore, by (5.3), $\psi_{(\eta)}$ is of degree 1. Thus, since $\{\eta \in \dot{\boldsymbol{R}}^n; \|\eta\| = 1\}$ is compact, there exists $U > 0$ which works for every $\eta \neq 0$. So we have proved the following:

6.1. THEOREM. (i) There exists $U > 0$ such that (1.1) has two solutions

$$\varphi^\pm(u, v, z, \eta) \in C^\infty((0, U] \times \dot{\boldsymbol{R}}^+ \times \boldsymbol{R}^n \times \dot{\boldsymbol{R}}^n) \cap C([0, U] \times \bar{\boldsymbol{R}}^+ \times \boldsymbol{R}^n \times \dot{\boldsymbol{R}}^n),$$

real and homogeneous of degree 1 in η. Moreover

$$\frac{\partial}{\partial u}\varphi^{\pm}\in C\big((0,\,U]\times\overline{\boldsymbol{R}}^{+}\times\boldsymbol{R}^{n}\times\dot{\boldsymbol{R}}^{n}\big)\,,\qquad \frac{\partial}{\partial v}\varphi^{\pm}\in C\big([0,\,U]\times\dot{\boldsymbol{R}}^{+}\times\boldsymbol{R}^{n}\times\dot{\boldsymbol{R}}^{n}\big)\,,$$

$$\frac{\partial}{\partial z_j}\varphi^{\pm}\in C\big([0,\,U]\times\overline{\boldsymbol{R}}^{+}\times\boldsymbol{R}^{n}\times\dot{\boldsymbol{R}}^{n}\big)\,,\qquad (j=1,\dots,n)\,.$$

(ii) If φ satisfies the above conditions, then φ coincides with φ^{+} or φ^{-} in each connected component of $[0,\,U]\times\overline{\boldsymbol{R}}^{+}\times\boldsymbol{R}^{n}\times\dot{\boldsymbol{R}}^{n}$.

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