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Finite Groups in which Subnormalizers are Subgroups.

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Let H be a subgroup of a group G. We put

$$S_G(H) = \{g \in G;\ H \operatorname{sn} \langle H, g\rangle\}$$

and call it the « subnormalizer » of H in G (see [7; p. 238]). In general, $S_G(H)$ is not a subgroup (see [7]). The aim of this paper is to study the class of groups (which we call sn-groups) in which the subnormalizer of every subgroup is a subgroup. As observed in [7; p. 238], $S_G(H)$ is a subgroup of G if and only if H is subnormal in $\langle U, V\rangle$, whenever $H \operatorname{sn} U \leqslant G$ and $H \operatorname{sn} V \leqslant G$. Furthermore, if G is finite and $S_G(H) \leqslant G$ then, by a subnormality criterion of H. Wielandt [10], H is subnormal in $S_G(H)$; thus $S_G(H)$ is the maximal subgroup of G in which H is embedded as a subnormal subgroup.

From now on, « group » will mean « finite group ».

In the first section of this paper we show that the property of being an sn-group has a local character. Namely, we define for every prime p, the class of sn(p)-groups of those groups in which the subnormalizer of every p-subgroup is a subgroup, and prove that G is an sn-group if and only if G is an sn(p)-group for every prime p dividing G. This in turn leads to the following characterization of sn-groups:

a group G is an sn-group if and only if the intersection of any two Sylow subgroups of G is pronormal in G.

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(We recall that a subgroup H of G is said to be pronormal if, for every $g \in G$, H is conjugate to H^g in $\langle H, H^g \rangle$.)

At this stage we observe that the class of sn-groups has already been studied. In fact, in [8], T. Peng defined the class of $\boldsymbol{E}$-groups as the class of those groups G in which $E_G(x) := \{g \in G; [g, {}_n x] = 1$ for some $n \in \mathbb{N}\}$ is a subgroup for every $x \in G$; and, for any prime p, the class $\boldsymbol{E}_p$ of those groups in which $E_G(x)$ is a subgroup for every p-element x of G. It turns out that the class of $\boldsymbol{E}$-groups is the same as the class of sn-groups, and that $\boldsymbol{E}_p$-groups are just the sn(p)-groups, for every prime p. As a consequence, we have that G is an $\boldsymbol{E}$-group if and only if it is an $\boldsymbol{E}_p$-group for every prime p, giving a positive answer to a question raised by Peng.

On the basis of these characterizations, we then give a more detailed description of sn-groups, both in the soluble and in the general case, extending some of the results obtained by Peng. In particular, we have that the only non abelian simple sn-groups are those of type $PSL(2, 2^n)$ and $Sz(2^{2m+1})$; also an sn-group has generalized Fitting length at most four, and all chief factors abelian or simple. Finally, we give necessary and sufficient conditions for a group G to be an sn-group in terms of the automorphisms groups induced by G on its chief factors; as a sample we quote the following: a group G of odd order is an sn-group if and only if the group of automorphisms induced by G on each chief factor is a $\boldsymbol{T}$-group.

After this paper was written, I was informed that H. Heineken, in his study on $\boldsymbol{E}$-groups, had obtained indipendently part of the results which appear in sections 2 and 3 of this paper. In his forthcoming paper [1], other informations on (not necessarily finite) $\boldsymbol{E}$-groups are to to be found.

Notation is mostly standard. We shall make use of P. Hall's closure operations. A $\boldsymbol{T}$-group is a group in which every sunbormal subgroup is normal. For any group G, $F(G)$ denotes the Fitting subgroup of G, and, if U/V is a chief factor of G, $A_G(U/V)$ denotes the group of automorphisms of U/V induced by conjugation by G (thus $A_G(U/V) \simeq G/C_G(U/V)$).

1. A characterization of sn-groups.

Our first Lemma gives some elementary properties of the set $S_G(H)$. The proof, which is straightforward, is omitted.

1.1 LEMMA. *Let H be a subgroup of the group G:*

(i) *If T sn H, then $S_G(T) \supseteq S_G(H)$.*

(ii) *If $N \trianglelefteq G$, then $S_{G/N}(HN/N) \supseteq S_G(H)N/N$; if further $N \leqslant H$, then $S_G(H)/N = S_{G/N}(H/N)$*

(obviously here $S_G(H)/N := \{gN \in G/N;\ g \in S_G(H)\}$).

The next result, although very elementary, is fundamental.

1.2 LEMMA. *Let H be a subnormal subgroup of G, and P a Sylow p-subgroup of H, for some prime p; then*
$\in S_G(H)\})$.

$$G = H\langle S_G(P)\rangle .$$

PROOF. By induction on the defect n of H in G. If H is normal in G apply the usual Frattini argument. Let $n > 1$; then H has defect $n-1$ in its normal closure H^G. By inductive hypothesis:

$$(+) \qquad H^G = H\langle S_{H^G}(P)\rangle .$$

Take now $P_0 \in \mathrm{Syl}_p(H^G)$ such that $P \leqslant P_0$; then P sn $N_G(P_0)$ and so $N_G(P_0) \subseteq S_G(P)$. By the Frattini argument we have also

$$G = H^G N_G(P_0) = H^G\langle S_G(P)\rangle$$

which, together with (+), gives:

$$G = H\langle S_{H^G}(P)\rangle\langle S_G(P)\rangle = H\langle S_G(P)\rangle . \quad \blacksquare$$

We recall, from the introduction, that a group G is an sn-group if $S_G(H)$ is a subgroup of G for every $H \leqslant G$, and G is an sn(p)-group, p a prime, if $S_G(H) \leqslant G$ for every p-subgroup H of G.

It then follows from 1.1 (ii) that the class of sn-groups is $\boldsymbol{Q}$-closed (that is every homomorphic image of an sn-group is an sn-group). The class of sn-groups is also $\boldsymbol{S}$-closed (that is every subgroup of an sn-group is an sn-group), in fact, in any group G, if $H \leqslant K \leqslant G$, then $S_K(H) = S_G(H) \cap K$. Thus, for every prime p, the class of sn(p)-groups is $\boldsymbol{S}$-closed.

1.3 LEMMA. *For every prime p, the class of* sn(p)*-groups is* $\boldsymbol{Q}$*-closed.*

PROOF. Let G be an sn(p)-group, and $N \trianglelefteq G$. Let H/N be a p-subgroup of G/N. Since the class of sn(p)-groups is s-closed, we may assume $G = \langle S_G(H) \rangle$, that is $G/N = \langle S_{G/N}(H/N) \rangle$. Let P be a Sylow p-subgroup of H, then $NP = H$. If $H \operatorname{sn} V \leqslant G$, by 1.2 we have $V = H\langle S_V(P) \rangle$ and so, since G is an sn(p)-group,

$$V = HS_V(P) = NS_V(P) .$$

Thus:

$$NS_G(P) \geqslant \langle NS_V(P);\, H \operatorname{sn} V \leqslant G \rangle = \langle V;\, H \operatorname{sn} V \leqslant G \rangle = \langle S_G(H) \rangle = G .$$

Hence $G/N = NS_G(P)/N \cong S_G(P)/(S_G(P) \cap N)$. Since $P(S_G(P) \cap N)$ is subnormal in $S_G(P)$, we conclude that H/N is subnormal in G/N, and so $S_{G/N}(H/N) = \langle S_{G/N}(H/N) \rangle$, which means that G/N is an sn(p)-group. ■

1.4 PROPOSITION. *Let G be a group, p a prime number. The following are equivalent.*

(i) *G is an* sn(p)*-group;*

(ii) *for any intersection R of Sylow p-subgroups of G,* $S_G(R) = N_G(R)$;

(iii) *for every $P \in \operatorname{Syl}_p(G)$ and $g \in G$, $P \cap P^g \trianglelefteq N_G(P)$;*

(iv) *for every $P \in \operatorname{Syl}_p(G)$, $g_1, \ldots, g_n \in G$, $\langle P, g_1, \ldots, g_n \rangle$ normalizes* $P \cap P^{g_1} \cap \ldots \cap P^{g_n}$.

PROOF. (i) $\Rightarrow$ (ii). Let G be an sn(p)-group and $R = P_1 \cap \ldots \cap P_r$ with $P_i \in \operatorname{Syl}_p(G)$, $i = 1, \ldots, r$. Clearly $N_G(R) \subseteq S_G(R)$. Set $S_G(R) = S$; since R is a p-subgroup of G, $S \leqslant G$ and $R \operatorname{sn} S$. Thus R^S is a normal p-subgroup of S and it is contained in every Sylow p-subgroup of S. Now, for every $i = 1, \ldots, r$, R is subnormal in P_i and so $P_i \subseteq S$. Hence $P_i \in \operatorname{Syl}_p(S)$, yielding

$$R^S \leqslant P_1 \cap \ldots \cap P_r = R .$$

Thus $R^S = R$, that is $S \subseteq N_G(R)$ and, consequently, $S = N_G(R)$.

(ii) $\Rightarrow$ (iii). Let $P \in \mathrm{Syl}_p(G)$ and $g \in G$; where G satisfies condition (ii). Then $N_G(P) \subseteq S_G(P \cap P^g) = N_G(P \cap P^g)$ and so $P \cap P^g$ is normal in $N_G(P)$.

(iii) $\Rightarrow$ (iv). Let G satisfy condition (iii) and let P be a Sylow p-subgroup of G; $g_1, \ldots, g_n \in G$. Set $R = P^{g_0} \cap P^{g_1} \cap \ldots \cap P^{g_n}$, where $g_0 = 1$. Then for every $i = 0, 1, \ldots, n$, we may write

$$R = \bigcap_{j=0}^{n} (P^{g_i} \cap P^{g_j}) .$$

Thus, by (iii), $R \trianglelefteq N_G(P^{g_i})$ for every $i = 0, 1, \ldots, n$ and so

$$R \trianglelefteq \langle N_G(P), N_G(P^{g_1}), \ldots, N_G(P^{g_n})\rangle = T .$$

Now, since for every $g \in G$, $g \in \langle N_G(P), N_G(P)^g\rangle$, we have

$$T = \langle N_G(P), g_1, \ldots, g_n\rangle ,$$

whence, in particular, $\langle P, g_1, \ldots, g_n\rangle$ normalizes R.

(iv) $\Rightarrow$ (i). Let G satisfy condition (iv) and let H be a p-subgroup of G. Denote by $\mathcal{S}$ the set of those Sylow p-subgroups of G, which contain H, and put $R = \bigcap_{P \in \mathcal{S}} P$. We show that $S_G(H) = N_G(R)$ and so that $S_G(H)$ is a subgroup of G.

Since R is a p-subgroup of G, containing H, we have at once $H \operatorname{sn} R \trianglelefteq N_G(R)$, whence $N_G(R) \subseteq S_G(H)$.

Conversely, let $V \leqslant G$ such that $H \operatorname{sn} V$. Then H is contained in every Sylow p-subgroup of V. If $P \in \mathcal{S}$ and $v \in V$, we have: $H \leqslant P \cap V \leqslant P^g \cap V$, where $g \in G$ is such that $P^g \cap V$ is a Sylow p-subgroup of V. Thus:

$$H \leqslant (P^g \cap V)^v = P^{gv} \cap V .$$

In particular

$$H \leqslant P \cap P^g \cap P^{gv} .$$

But, since G satisfies (iv), $P \cap P^g \cap P^{gv}$ is normalized by $\langle g, gv\rangle = \langle g, v\rangle$. In particular, it is normalized by v and so $H \leqslant P^v$, whence $P^v \in \mathcal{S}$. This shows that V permutes by conjugation the elements of $\mathcal{T}$ and so $V \leqslant N_G(R)$. We have therefore:

$$S_G(H) = \bigcup \{V \leqslant G;\, H \operatorname{sn} V\} \subseteq N_G(R)\,.$$

Thus $S_G(H) = N_G(R)$, as we wanted. ∎

Now we can state our main characterization of sn(p)-groups.

1.5 THEOREM. *Let G be a group. Then G is an* sn(p)*-group if and only if for every $P, Q \in \mathrm{Syl}_p(G)$, $P \cap Q$ is pronormal in G.*

PROOF. Let G be an sn(p)-group. Let P, Q be Sylow p-subgroups of G and put $R = P \cap Q$. Let also $h \in G$ and write $L = \langle R, R^h \rangle$. Choose $T \in \mathrm{Syl}_p(G)$ such that $R \leqslant T \cap L \in \mathrm{Syl}_p(L)$. Since R^h is a p-subgroup of L, there exists $x \in L$ such that $R^h \leqslant (T \cap L)^x$. In particular $R^h \leqslant T^x$ and so $R \leqslant T^{xh^{-1}}$. Thus

$$R = T \cap T^{xh^{-1}} \cap P \cap Q\,.$$

By 1.4 (i) $\Rightarrow$ (iv), xh^{-1} normalizes R; then $R^x = R^h$, proving that R is pronormal in G.

Conversely, assume that $P \cap Q$ is pronormal in G, for every $P, Q \in \mathrm{Syl}_p(G)$. Then $P \cap Q$ is both pronormal and subnormal in $N_G(P)$. This forces $P \cap Q$ to be normal in $N_G(P)$. By 1.4 (iii) $\Rightarrow$ (i), we have that G is an sn(p)-group. ∎

1.6 LEMMA. *Assume that G is an* sn(p)*-group for every prime p dividing $|G|$, and let H be a perfect subnormal subgroup of G. Then $H \trianglelefteq G$.*

PROOF. Assume, firstly, that H is simple and non abelian. Then, by a well known result of H. Wielandt (see [7, p. 54]), $H^G = H \times K$, where K is the direct product of the conjugates of H, distinct from H. Let p be a prime divisor of $|H|$ and $P \in \mathrm{Syl}_p(H)$. Since G is an sn(p)-group, Lemma 1.2 yields

$$G = HS_G(P)\,.$$

Set $S = S_G(P)$ and $Q = P^S$. Since P is subnormal in S, Q is a p-group and it is contained in every Sylow p-subgroup of S. Let $T = Q \cap H^G$; then T is a p-group and $T \geqslant P$. Since P is a maximal p-subgroup of H, we have:

$$T = P \times (T \cap K) = P \times (Q \cap K) .$$

On the other hand, K centralizes P and so $K \leqslant S$, whence $Q \cap K \trianglelefteq K$. But $O_p(K) = 1$, because K is either trivial or the direct product of non abelian simple groups. Thus $Q \cap K = 1$ and so $Q \cap H^G = P$, yielding $P \trianglelefteq S$. This gives $S = N_G(P)$ and, consequently, $G = HN_G(P)$. Now, if $g \in N_G(P)$, $1 \neq P \leqslant H \cap H^g$. Simplicity of H now forces $H = H^g$ Hence $N_G(P) \leqslant N_G(H)$ and thus $H \trianglelefteq G$.

We now turn to the case in which H is just a perfect subnormal subgroup of G, and proceed by induction on $|G|$. By 1.3 we may therefore assume $H_G = 1$. Let R be the soluble radical of H^G. Assume $R \neq 1$; then, as R is normal in G, HR/R is subnormal in G/R and $HR/R \simeq H/H \cap R$ is perfect. By inductive hypothesis, $HR \trianglelefteq G$ and so $HR = H^G$. Let $g \in G$ such that $H \trianglelefteq \langle H, H^g \rangle$. Then $HH^g/H \simeq$ $\simeq H^g/(H \cap H^g)$ is perfect; but also $HH^g/H = (HH^g \cap R)H/H \simeq$ $\simeq (HH^g \cap R)/(H \cap R)$ is soluble. Thus $HH^g = H$, that is $H = H^g$. Since H is subnormal in G, this implies that H is normal in G.

It remains the case in which $R = 1$. Then, if A is a minimal subnormal subgroup of G contained in H, A is simple non abelian and so, by the case discussed above, A is normal in G. Now, by inductive hypothesis: $H/A \trianglelefteq G/A$, yielding $H \trianglelefteq G$. ■

1.7 LEMMA. *Let G be a group, $G = TS$ with $S \leqslant G$, T sn G. If P sn S then $\langle T, P \rangle$ is subnormal in G.*

PROOF. By induction on the defect n of T in G. If $n = 1$, T is normal and $G/T = TS/T \simeq S/T \cap S$; in this isomorphism, $\langle T, P \rangle/T =$ $= TP/T$ corresponds to $P(S \cap T)/S \cap T$ which is subnormal in $S/S \cap T$; thus $\langle T, P \rangle$ is subnormal in G.

Let now $n > 1$. Assume firstly that P normalizes T. Let $K = T[G, {}_{n-1}T]$, then $T \trianglelefteq K$, K has defect $n - 1$ in G and $G = KS$. Hence, by inductive hypothesis $A = \langle K, P \rangle = KP$ is subnormal in G. Now $A \cap S = KP \cap S = (K \cap S)P$ and P sn $(K \cap S)P$. Further:

$$(K \cap S)PT = (KP \cap S)T = KP \cap ST = KP = A .$$

Since $T \trianglelefteq A$ we have, by the case $n = 1$, $\langle T, P \rangle = TP$ sn A and so TP sn G.

In the general case, set $\overline{T} = T^P = \langle T^h; h \in P\rangle$. Then $\overline{T}$, being generated by subnormal subgroups, is subnormal in G, $G = \overline{T}S$ and $\overline{T}$ is normalized by P. By the case discussed above $\langle T, P\rangle = \overline{T}P$ is subnormal in G. ■

1.8 PROPOSITION. *Suppose that G is not an* sn-*group. If $H \leqslant G$ is minimal such that $S_G(H) \nleqslant G$, then either H is perfect or it is cyclic of order a power of a prime.*

PROOF. Let G, H be as in our hypothesis, and suppose that H is not a perfect group. Then there exists a maximal normal subgroup T of H, such that $|H:T| = p$ for some prime p. Now $S_G(T) \supseteq S_G(H)$ and so, by our choice of H:

$$\langle S_G(H)\rangle \leqslant S_G(T) \leqslant G .$$

Set $L = \langle S_G(H)\rangle$; then $T \operatorname{sn} L$. Let $P \in \mathrm{Syl}_p(H)$, thus $TP = H$.

Suppose that $P \neq H$; then $S_G(P) \leqslant G$. Let $V \leqslant G$ such that $H \operatorname{sn} V$; then $S_V^*(P) = S_G(P) \cap V \leqslant V$ and, by Lemma 1.2, $V = HS_V(P)$. Thus H permutes with $K = \langle S_V(P); H \operatorname{sn} V \leqslant G\rangle$. Now:

$$HK \geqslant \langle HS_V(P); H \operatorname{sn} V\rangle = \langle V; H \operatorname{sn} V\rangle = L \geqslant HK ,$$

whence $HK = L$ and so $L = TK$. Moreover, $K \leqslant S_G(P)$ and so $P \operatorname{sn} K$. Since also T is subnormal in L, Lemma 1.7 yields $H = \langle T, P\rangle \operatorname{sn} L$; thus $S_G(H) = \langle S_G(H)\rangle \leqslant G$, contradicting our choice of H.

Therefore, we must have $H = P$. If H admits two distinct maximal subgroups R and Q, then both are subnormal in H and so $L = \langle S_G(H)\rangle \leqslant S_G(R) \cap S_G(Q)$. In particular R, Q are subnormal in L and thus $H = \langle R, Q\rangle$ is subnormal in L, again contradicting the choice of H. Hence H has a unique maximal subgroup and it is therefore cyclic. ■

We now recall the definitions given by Peng in [8]. Let G be a group, $x \in G$; put $E_G(x) = \{g \in G; [g, {}_nx] = 1, n \in \mathrm{N}\}$. Then $\boldsymbol{E}$ is the class of groups G in which $E_G(x)$ is a subgroup, for every $x \in G$. For any prime p, $\boldsymbol{E}_p$ is the class of groups in which $E_G(x) \leqslant G$ for every p-element x of G.

Peng raises the question as to whether $G \in \boldsymbol{E}$ if (and only if) $G \in \boldsymbol{E}_p$ for every prime p; he gives a positive answer for soluble groups of 2-length at most 1 [8, Corollary 3, p. 328].

1.9 LEMMA. *Let G be a group, $x \in G$. If G is an* $\mathrm{sn}(p)$*-group for every prime p dividing $|x|$, then:*

$$E_G(x) = S_G(x) \leqslant G .$$

PROOF. Let $S = S_G(\langle x \rangle)$. If $g \in S$ then $\langle x \rangle \,\mathrm{sn}\, \langle x, g \rangle$ and so $[g, {}_n x] = 1$ for some $n \in \mathbb{N}$; thus $g \in E_G(x)$ and $S \subseteq E_G(x)$. Write $x = x_1 \dots x_r$, where $\langle x_i \rangle$ $(i = 1, \dots, r)$ are the primary components of $\langle x \rangle$, and set $S_i = S_G(\langle x_i \rangle)$. Then, for any i, S_i is a subgroup of G, by our hypothesis. Let $T = \bigcap_{i=1}^{r} S_i$. Now, for any $i = 1, \dots, r$, $\langle x_i \rangle$ is subnormal in T; thus $\langle x \rangle = \langle x_1, \dots, x_r \rangle \,\mathrm{sn}\, T$, whence $T \subseteq S$. Conversely, if $g \in S$, $\langle x_i \rangle \trianglelefteq \langle x \rangle \,\mathrm{sn}\, \langle x, g \rangle$ and so $g \in S_i$ for every $i = 1, \dots, r$; thus $g \in T$, yielding $S \subseteq T$ and, consequently, $S = T \leqslant G$.

Let now $g \in G$ such that $x^g \in S$; we show that $g \in S$. In fact $x^g = x_1^g \dots x_r^g \in S$ and, the x_i's being suitable powers of x, $x_i^g \in S$ for every i; in particular $x_i^g \in S_i$ for every i. Let $p_1, \dots, p_r$ be primes such that, for any $i = 1, \dots, r$, x_i is a p_i-element. Further, for any i, choose a Sylow p_i-subgroup P_i of S_i such that $x_i^g \in P_i$. Then, since $\langle x_i \rangle \,\mathrm{sn}\, S_i$, $x_i \in P_i$ and so $\langle x_i \rangle \leqslant P_i \cap P_i^{g^{-1}}$. By 1.4, $P_i \cap P_i^{g^{-1}} \trianglelefteq$ $\trianglelefteq \langle P_i, g \rangle$; in particular $\langle x_i \rangle$ is subnormal in $\langle P_i, g \rangle$ and so $g \in S_i$. This is true for every $i = 1, \dots, r$ whence $g \in S$.

Let now $y \in E_G(x)$; then, for some $n \in \mathbb{N}$, $[y, {}_n x] = 1$. Let m be the minimal natural number such that $[y, {}_m x] \in S$ (we put $[y, {}_0 x] = y$). Suppose $m > 0$, then:

$S \ni [y, {}_m x] = [y, {}_{m-1} x, x] = (x^{-1})^{[y, {}_{m-1} x]} x$, and this implies $x^{[y, {}_{m-1} x]} \in S$. Thus $[y, {}_{m-1} x] \in S$, contradicting the choice of m. Hence $m = 0$ and, therefore, $y \in S$. This yields $E_G(x) \subseteq S$ completing the proof that $E_G(x) = S$. ■

An immediate consequence is the following.

1.10 COROLLARY. *A group G is an* $\mathrm{sn}(p)$*-group for some prime p if and only if $G \in \boldsymbol{E}_p$.*

PROOF. If G is an $\mathrm{sn}(p)$-group, then Lemma 1.9 implies at once that $E_G(x)$ is a subgroup for every p-element x of G, and so $G \in \boldsymbol{E}_p$.

Conversely, let $G \in \boldsymbol{E}_p$ and suppose that G is not an $\mathrm{sn}(p)$-group. Thus, let H be a minimal p-subgroup of G such that $S_G(H)$ is not a subgroup. Then, as in the proof of 1.8, H has a unique maximal subgroup, so $H = \langle x \rangle$ for some p-element x of G. Now $E_G(x) \leqslant G$ and x is a left Engel element in $E_G(x)$; thus $\langle x \rangle$ sn $E_G(x)$ and $E_G(x) \subseteq$ $\subseteq S_G(\langle x \rangle)$. Since, clearly, $S_G(\langle x \rangle) \subseteq E_G(x)$, we get $S_G(H) = E_G(x) \leqslant G$, a contradiction. Hence G is an sn(p)-group. ■

1.11 THEOREM. *Let G be a group. The following are equivalent.*

(i) *G is an* sn(p)*-group for every p dividing $|G|$;*

(ii) *G is an* sn*-group;*

(iii) *G is an **E**-group;*

(iv) *G is an $\mathbf{E}_p$-group for every p dividing $|G|$.*

PROOF. (i) ⇒ (ii). Let G be an sn(p)-group for every p dividing $|G|$ and suppose, by contradiction, that G is not an sn-group. Let $H \leqslant G$ be minimal such that $S_G(H)$ is not a subgroup. Then, Proposition 1.8 implies that H is perfect. Let H sn$V \leqslant G$. Since the class of sn(p)-groups is **S**-closed, V is an sn(p)-group for every prime dividing its order, hence, by Lemma 1.6, H is normal in V. Thus:

$$\langle S_G(H) \rangle = \langle V \leqslant G ; H \,\mathrm{sn}\, V \rangle = N_G(H) \subseteq S_G(H)\,,$$

and so $S_G(H) = N_G(H)$ is a subgroup, a contradiction. Thus G is an sn-group.

(ii) ⇒ (iii). This follows from Lemma 1.9.

(iii) ⇒ (iv). This is obvious.

(iv) ⇒ (i). This follows from Corollary 1.10. ■

1.12 COROLLARY. *A group G is an* sn*-group if and only if every intersection of two Sylow subgroups of G is pronormal in G.*

PROOF. Immediate from 1.5 and 1.11. ■

1.13 COROLLARY. *Every chief factor of an* sn*-group is simple or abelian.*

PROOF. Follows from 1.11 and Lemma 1.6. ■

1.14 COROLLARY. (*a*) *For any prime p, the class of* sn(p)*-groups is a formation.*

(*b*) *The class of* sn*-groups is a formation.*

PROOF. (*a*) Since the class of sn(p)-groups is both $\boldsymbol{S}$ and $\boldsymbol{Q}$-closed, it is sufficient to show that the direct product of two sn(p)-groups is again an sn(p)-group, and this is clearly true in view of the identification of sn(p)-groups with $\boldsymbol{E}_p$-groups stated in 1.10.

(*b*) Follows in the same manner from 1.11 and the fact that the class of sn-groups is both $\boldsymbol{S}$ and $\boldsymbol{Q}$-closed. ∎

For further reference we state here some elementary consequences of the results obtained in this section.

1.15 LEMMA. *Let p be a prime: each of the following conditions imply that the group G is an* sn(p)*-group.*

(*a*) *The Sylow p-subgroups of G are disjoint from their conjugates.*

(*b*) *The Sylow p-subgroups of G are cyclic.*

(*c*) *$G/O_p(G)$ is an* sn(p)*-group.*

(*d*) *$G/Z(G)$ is an* sn(p)*-group.*

(observe that the last two conditions are also necessary for G to be an sn(p)-group).

PROOF. (*a*) and (*b*) follow immediately from Theorem 1.5.

(*c*) This also follows from 1.5; in fact if R is an intersection of Sylow p-subgroups of G, then $R \geqslant O_p(G)$.

(*d*) Let $Z = Z(G)$, H a p-subgroup of G and $U, V \leqslant G$ such that $H \operatorname{sn} U$ and $H \operatorname{sn} V$. Then $ZH \operatorname{sn} \langle U, V\rangle Z$, because G/Z is an sn(p)-group; thus there exists $n \in \mathbb{N}$ such that $[\langle U, V\rangle, {}_nH] \leqslant ZH$. Then $[\langle U, V\rangle, {}_{n+1}H] \leqslant H$, whence $H \operatorname{sn} \langle U, V\rangle$. This implies that $S_G(H)$ is a subgroup of G. ∎

2. Simple sn-groups.

The main result to be proved in this section is the following.

2.1 THEOREM. *A nonabelian simple group is an* sn*-group if and only if it is one of the following groups*:

$$PSL(2,2^n), \quad n \geqslant 2\,; \qquad Sz(2^{2m+1}), \quad m \geqslant 1\,.$$

One way of proving this Theorem is to check the list of all simple groups. Instead, we have chosen to use a Theorem of Goldschmidt on strongly closed subgroups, which we will quote in due course. Before, we proceed to eliminate some groups.

2.2 LEMMA. *Let G be a group with no subgroups of index two and dihedral Sylow 2-subgroups of order at least* 8. *Then G is not an* sn (2)-*group.*

PROOF. Let Q be a Sylow 2-subgroup of G; x, y two involutions such that $Q = \langle x, y\rangle$. Then (see [5; 7.7.3]) there exists $g \in G$ such that $x^g = y$ and so $y \in Q \cap Q^g = R$. If G were an sn (2)-group, then, by 1.4: $R \trianglelefteq \langle Q, g\rangle$. In particular, $x = y^{g^{-1}} \in R^{g^{-1}} = R$. Thus $R = \langle x, y\rangle = Q$ and $g \in N_G(Q)$. This implies that x and y are conjugate in Q, which is not the case. Hence G is not an sn (2)-group. ■

2.3 LEMMA. *$PSL(2, q)$ is an* sn (2)-*group if and only if $q = 3, 5, 2^n$.*

PROOF. $PSL(2, 3) \simeq A_4$ is an sn-group.

If $q = 2^n$ then the Sylow 2-subgroups of $PSL(2, q)$ are disjoint from their conjugates; thus, by 1.15, $PSL(2, q)$ is an sn (2)-group. Also, $PSL(2, 5) \simeq PSL(2, 4)$ is an sn (2)-group.

Conversely, let $G = PSL(2, \mathrm{q})$, with $q = p^n > 3$, $p \neq 2$. We distinguish two cases.

(*a*) Let $q \not\equiv 3.\ 5 \pmod 8$. Then (see [5; 15.1.1]), the Sylow 2-subgroups of G are dihedral of order at least 8. Since G is simple, Lemma 2.2 implies that G is not an sn (2)-group.

(*b*) Let $q \equiv 3, 5 \pmod 8$. In this case, the Sylow 2-subgroups of G are elementary abelian of order 4 and they coincide with their centralizer in G (see [5; 15.1.1]). Let $Q \in \mathrm{Syl}_2(G)$ and suppose that G is an sn ((2)-group. Assume that there exists $1 \neq x \in Q$ such that $C_G(x) > Q$ and let $g \in C_G(x) \setminus Q$. Then $x \in Q \cap Q^g = R$. If $R = Q$, then $g \in N_G(Q)$. But $N_G(Q) \simeq A_4$ and so $g \in Q$, a contradiction. Hence $R = \langle x\rangle$; by 1.4 this implies $\langle x\rangle \trianglelefteq N_G(Q)$, which is not possible. Thus, for avery $1 \neq x \in Q$, $C_G(x) = Q$. By a Theorem of Suzuki (see [5; 9.3.2]), G is a Zassenhaus group of degree $|Q| + 1 = 5$ and so $G \simeq PSL(2, 5)$. This completes the proof. ■

DEFINITION. Let Q be a subgroup of $P \in \mathrm{Syl}_p(G)$, G a group. Then Q is said to be *strongly closed* in P (with respect to G) if, for every $x \in Q$ and $g \in G$, $x^g \in P$ implies $x^g \in Q$.

The Theorem of Goldschmidt that we are going to use is the following (see [6; Theorem 4. 128]).

THEOREM (Goldschmidt). *Let G be a simple group; if a Sylow 2-subgroup S of G contains a non trivial elementary abelian subgroup which is strongly closed in S with respect to G, then G is one of the following groups.*

(*a*) $PSL(2, 2^n)$, $PSU(3, 2^{2n})$, $n > 1$; $Sz(2^{2m+1})$, $m \geqslant 1$.

(*b*) $PSL(2, q)$, $q \equiv 3, 5 \pmod 8$.

(*c*) *The first Janko group J_1 or a Ree group ${}^2G_2(3^n)$, n odd, $n > 1$.*

2.4 PROPOSITION. *Let G be a simple non abelian group; then G is an* sn(2)*-group if and only if G is one of the following groups.*

$$PSL(2, 2^n)\,, \qquad PSU(3, 2^{2n})\,, \qquad Sz(2^{2m+1})\,; \qquad n > 1,\ m \geqslant 1\,.$$

PROOF. Let G be a simple non abelian sn(2)-group, and let S be a Sylow 2-subgroup of G. Take $R \leqslant S$ to be a non trivial intersection of Sylow 2-subgroups of G of minimal possible order (thus $R = S$ if the Sylow 2-subgroups of G are pairwise disjoint). Let $A = \Omega_1(Z(R))$; then A is a nontrivial elementary abelian characteristic subgroup of R. We show that A is strongly closed in S. Let $1 \neq x \in A$, $g \in G$ and suppose $x^g \in S$. Then $x \in A \cap S^{g^{-1}} \leqslant R \cap S^{g^{-1}}$. Now, by our choice of R, we get $R \cap S^{g^{-1}} = R$ and so, by Proposition 1.4, g normalizes R, whence g normalizes A. Thus $x^g \in A$, showing that A is strongly closed in S with respect to G.

Therefore, G is one of the groups listed in Goldschmidt's Theorem. Now, groups in (*a*) are indeed sn(2)-groups, because in each of them the Sylow 2-subgroups are disjoint from their conjugates. Groups in (*b*) are not sn(2)-groups by Lemma 2.2, except when $q = 5$, but then $G \cong PSL(2, 4)$.

The Janko group J_1 is not an sn(2)-group because, for instance, it has a subgroup isomorphic to $PSL(2, 11)$. Finally, groups of Ree type ${}^2G_2(3^n)$ are not sn(2)-groups: in fact ${}^2G_2(3^n)$ contains a subgroup isomorphic to $PSL(2, 3^n)$, $n > 1$. ∎

PROOF OF THEOREM 2.1. First, groups of type $PSL(2, 2^n)$ and $Sz(2^{2m+1})$ are sn-groups. In fact, in both cases, the Sylow 2-subgroups are disjoint from conjugates and the Sylow p-subgroups, p odd, are cyclic. By 1.15 these groups are sn-groups.

Conversely, if G is a nonabelian simple sn-group, it is, in particular, an sn(2)-group, hence one of those listed in Proposition 2.4. Thus, to complete the proof of the Theorem, we have to show that groups of type $PSU(3, q^2)$, $q = 2^n$, $n > 1$, are not sn-groups.

Now, by 1.15 (d) and 1.11, $PSU(3, q^2)$ is an sn-group if and only if $SU(3, q^2)$ is an sn-group; we deal with this latter group, and thus put $G = SU(3, q^2)$ ($q = 2^n$, $n > 1$). Let also $K = GF(q^2)$ be the field with q^2 elements.

We have $|G| = q^3(q^3 + 1)(q^2 - 1)$. Let p be an (odd) prime dividing $q + 1$ and let p^r be the highest power of p which divides $q + 1$. We observe that, as $n > 1$, we may always choose p in such a way that $p^r > 3$. The order of a Sylow p-subgroup of G is now p^{2r} if $p \neq 3$, and p^{2r+1} if $p = 3$. Since p^r divides $q^2 - 1$, the field K contains a primitive p^r-th root of unity, which we denote by u. Let also $v = u^{(p^r-1)/2}$ and observe that, since $p^r > 3$, $u \neq v$. In $SL(3, q^2)$ we take the matrices:

$$a = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v \end{pmatrix}, \qquad b = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix}, \qquad \pi = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $|a| = |b| = p^r$, $|\pi| = 3$, $[a, b] = 1$ and $P = \langle a, b \rangle$ is an abelian group of order p^{2r} which is normalized by π. Moreover $a, b, \pi \in SU(3, q^2)$ (obviously, we are assuming that a base of the vector space over K has been chosen in such a way the matrix of the Hermitian product is the identity).

Thus, if $p \neq 3$, P is a Sylow p-subgroup of G; if $p = 3$, $Q = \langle P, \pi \rangle$ is a Sylow p-subgroup of G. Now take $z \in K$ a root of the polynomial $x^2 + x + 1$ over $GF(2)$; then $z \neq 0, 1$ and, since $q \geqslant 4$, $z^q = z$. Thus the matrix

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & z+1 \\ 0 & z+1 & z \end{pmatrix}$$

is a unitary matrix, so $h \in G$. Moreover, $|h| = 2$ and $a^h = a$.

Let $p \neq 3$, then one easily checks that

$$P^h \cap P = \langle a \rangle .$$

If G were an sn-group, $\langle a\rangle$ should be normalized by $N_G(P)$; in particular, π should normalize $\langle a\rangle$, which is not the case. Hence G is not an sn-group.

If $p = 3$, then $Q^h \cap Q = Z\langle a\rangle$, where Z is the centre of G, which in this case has order 3. Again $Z\langle a\rangle$ is not normalized by $\pi \in N_G(Q)$, and so G is not an sn-group.

The proof is now complete. ■

We observed in Corollary 1.13 that a nonabelian chief factor of an sn-group is simple. We end this section by describing the automorphisms group induced by an sn-group on its nonabelian (simple) chief factors. By Theorem 2.1 such factors are isomorphic to groups of type $PSL(2, 2^n)$ or $Sz(2^{2m+1})$. It is well known that the group of automorphisms of any of these groups is the semidirect product of the group of inner automorphisms by a cyclic group of those automorphisms induced by the automorphisms of the underlying field.

2.5 Proposition. *Let U/V be a nonabelian (simple) chief factor of an* sn-*group G. Then the group of outer automorphisms induced by G on U/V is cyclic of order coprime to the order of U/V, or one of the following two exceptions occurs:*

(*a*) *$U/V \cong PSL(2, 2^3)$ and $A_G(U/V)$ is isomorphic to the semidirect product of U/V by a (field) automorphism of order* 3.

(*b*) *$U/V \cong Sz(2^5)$ and $A_G(U/V)$ is isomorphic to the semidirect product of U/V by a (field) automorphism of order* 5.

For the proof we need the following simple observation.

2.6 Lemma. *Let p be an odd prime;*

(*a*) *if p divides $|PSL(2, 2^p)|$, then $p = 3$;*

(*b*) *if p divides $|Sz(2^p)|$, then $p = 5$.*

Proof. (*a*) If p is an odd prime dividing $|PSL(2, 2^p)| = 2^p(2^{2p} - 1)$, then p divides $2^{2p} - 1$. Now: $3^p = (2^2 - 1)^p \equiv 2^{2p} - 1 \pmod p$. Thus $p|3^p$ and so $p = 3$.

(*b*) If p is an odd prime dividing $|Sz(2^p)| = 2^{2p}(2^{2p} + 1)(2^p - 1)$, then $p|(2^{2p} + 1)(2^p - 1)$ ans so, a fortiori, $p|2^{4p} - 1$. Now: $15^p = (2^4 - 1)^p \equiv 2^{4p} - 1 \pmod p$. Thus $p|15$. Since 3 does not divide the order of $Sz(2^p)$, we have $p = 5$. ■

PROOF OF PROPOSITION 2.5. By the Q-closure of the class of sn-groups, we may assume $V = 1$; also, since $C_G(U) \cap U = 1$, we may assume $C_G(U) = 1$ and so view G as a subgroup of $\mathrm{Aut}(U)$. Thus U is identified with $\mathrm{Inn}(U)$, and $U \cong PSL(2, 2^n)$ or $U \cong Sz(2^{2m+1})$. Then G is a semidirect product $U \rtimes \langle x \rangle$, where x is an automorphism of U induced by an automorphism of the underlying field K (thus $K = GF(2^n)$ or $K = GF(2^{2m+1})$). Without loss of generality, we may also assume that $|x| = p$, p a prime number. Now, if $U \cong PSL(2, 2^n)$, $p|n$ (and, if $U \cong Sz(2^{2m+1})$, $p|2m+1$). We write $n = ap$ (respectively, $2m + 1 = ap$). Then we may take x as the automorphism induced on U by the field automorphism mapping every $u \in K$ to u^{2^a}. Let $C = C_U(x)$; then $C \cong PSL(2, 2^a)$ or, respectively, $C \cong Sz(2^a)$ (here we consider also $Sz(2)$, which is soluble of order 20). Suppose that p divides the order of U.

Let $P \in \mathrm{Syl}_p(G)$ such that $x \in P$, and $D = U \cap P$; then $D \trianglelefteq P$ and $P = DT$ where $T = \langle x \rangle$. Let $h \in C$ and suppose that $D^h \neq D$; thus, since in U the Sylow p-subgroups are disjoint from their conjugates (this is indeed true for every prime dividing $|U|$), $D^h \cap D = 1$. Hence

$$P \cap P^h \cap U = (P \cap U)^h \cap P \cap U = D^h \cap D = 1 .$$

Now, $T \leqslant P \cap P^h$, because $x^h = x$, so $T = P \cap P^h$.

If G is an sn-group, then $T \trianglelefteq N_G(P)$. In particular, $[D, T] \leqslant D \cap T = 1$ and thus $D \leqslant C$, which is not the case, because $D \in \mathrm{Syl}_p(U)$ while p divides $|U:C|$. Thus, in order to have an sn-group, C must normalize D. But in U the normalizers of Sylow subgroups are soluble, so this forces $C \cong PSL(2, 2)$ or, respectively, $C \cong Sz(2)$. Hence $a = 1$ and $n = p$ (or $2m + 1 = p$).

If p is odd, by Lemma 2.6, we have therefore $U \cong PSL(2, 2^3)$ and $p = 3$, or $U \cong Sz(2^5)$ and $p = 5$.

If $p = 2$, then $U \cong PSL(2, 4) \cong A_5$, and $G \cong S_5$ is not an sn-group (in this case, in the notation used above, $C \cong S_3$ does not normalize any subgroup of U).

Conversely, let G be a split extension of a group U of type $PSL(2, 2^n)$ or $Sz(2^{2m+1})$ by a group of automorphisms induced by field automorphisms, such that $(|U|, |G:U|) = 1$. Then G is an sn-group because its Sylow p-subgroups are disjoint from conjugates if $p \mid |U|$, and cyclic if $p \mid |G:U|$.

Finally, let G be one of the groups in (a) or (b) of our statement;

then one checks that the Sylow 3-subgroups of the group in (*a*) and the Sylow 5-subgroups of the group in (*b*) are, in fact, disjoint from their coniugates, and so the two groups are sn-groups. ■

3. Automorphism groups induced on chief factors.

In this section we study the structure of sn-groups by looking at the automorphism groups induced by conjugation on each chief factor. By 1.13, the chief factors of an sn-group are simple or abelian. The case of a simple nonabelian chief factor has already been treated in Proposition 2.5; thus, from now on, we deal with abelian chief factors. We remid that, if U/V is a chief factor of a group G, we put $A_G(U/V) = G/C_G(U/V)$.

3.1 LEMMA. *Let U/V be an abelian chief factor of an* sn(p)*-group G, and let $A = A_G(U/V)$. Then $O_p(A)$ acts as a group of fixed point free (f.p.f.) automorphisms on U/V.*

PROOF. In view of the Q-closure of the class of sn(p)-groups, we may assume $V=1$. Hence U is a minimal normal subgroup of G and it is an elementary abelian q-group, for some prime q. If $q = p$, it is well known that $O_p(A) = 1$. Let $p \neq q$ and assume, by contradiction, that there exists $1 \neq \bar{x} \in O_p(A)$ such that $K = C_U(\bar{x}) \neq 1$. If $C = C_G(U)$, let $\bar{x} = Cx$ with $x \in G \setminus C$; write $L = \langle C, x \rangle$ and take a Sylow p-subgroup p of L. Then $[K, P] = 1$. But $S_G(P)$ is a subgroup of G and $P \operatorname{sn} S_G(P)$. Thus: $S_G(P) \cap U = C_U(P) = K$; whence, in particular, $K \trianglelefteq S_G(P)$. Now, $L/C \leqslant O_p(G/C)$, so L is subnormal in G and, by 1.2, $G = LS_G(P)$, which implies that K is normal in G. Minimality of U gives $K = 1$ or $K = U$, both contradicting our choice of $\bar{x}$. ■

We denote by $l(G)$ the Fitting length of the group G.

3.2 COROLLARY. *Let G be an* sn*-group. Then:*

(*a*) *for every abelian chief factor U/V of G, $F(A_G(U/V))$ acts as a group of f.p.f. automorphisms on U/V;*

(*b*) *if G is soluble, then $l(G) \leqslant 4$ and if, further, S_4 is not involved in G, $l(G) \leqslant 3$.*

PROOF. (*a*) This follows at once from Lemma 3.1.

(*b*) It is a consequence of (*a*) that, for every chief factor U/V of a soluble sn-group G, $F(A_G(U/V))$ is either cyclic or the direct product of a cyclic group of odd order and a generalized quaternion group. In particular, the chief factors of $A_G(U/V)$ are cyclic or of order 4, and so the automorphism group induced by $A_G(U/V)$ on each of its chief factors is abelian or it is isomorphic to S_3 (and it is always abelian if S_4 is not involved in G, see [4; Lemma 6]). Thus $A_G(U/V)'' \leqslant F(A_G(U/V))$ and $A_G(U/V)' \leqslant F(A_G(U/V))$ if S_4 is not involved in $A_G(U/V)$. Hence $l(A_G(U/V)) \leqslant 3$ and $l(A_G(U/V)) \leqslant 2$ if S_4 is not involved in $A_G(U/V)$.

Since $F(G) = \bigcap C_G(U/V)$, U/V the chief factors of G, we get $l(G) \leqslant 4$ and $l(G) \leqslant 3$ if S_4 is not involved in G. ■

REMARK. 4 is the best possible bound for the Fitting length of a soluble sn-group. Indeed, we shall see that every soluble Frobenius group is an sn-group; and there exist soluble Frobenius groups of Fitting length 4.

DEFINITION. (*a*) Let G be a group. Following Robinson [9], we say that G satisfies condition $\mathfrak{C}_p$, p a prime, if every subgroup of a Sylow p-subgroup P of G is normal in $N_G(P)$. We quote from [9] the following results.

1) (J. Rose). *A group G satisfies $\mathfrak{C}_p$ if and only if every p-subgroup of G is pronormal in G.*

2) (D. Robinson). *A group G is a soluble **T**-group if and only if it satisfies $\mathfrak{C}_p$ for every prime (dividing $|G|$).*

(*b*) We say that the group G satisfies condition $\mathfrak{C}_2^*$ if

(i) every Sylow 2-subgroup P of G is either abelian or $P = Q \times A$, where Q is a generalized quaternion group and A is elementary abelian; and

(ii) $\Omega_1(P) \leqslant Z(N_G(P))$ (we observe that, if P is abelian, then (ii) implies $P \leqslant Z(N_G(P))$).

It follows from Proposition 1.4 that a group satisfying $\mathfrak{C}_p$, for a prime p, is an sn(p)-group. Indeed we can say a little more.

3.3 LEMMA. *Let p, q be prime numbers and let M be a normal q-subgroup of the group G; $C = C_G(M)$. If G/M is an* sn(p)*-group and G/C satisfies condition $\mathfrak{C}_p$, then G is an* sn(p)*-group.*

PROOF. If $p = q$, then $M \leqslant O_p(G)$ and so, by the Q-closure of the class of sn(p)-groups, $G/O_p(G)$ is an sn(p)-group. By 1.15, G is an sn(p)-group.

Let $p \neq q$; we prove that $P \cap P^g \trianglelefteq N_G(P)$ for every $P \in \mathrm{Syl}_p(G)$ and $g \in G$; by Proposition 1.4, this implies that G is an sn(p)-group.

Put $N = N_G(P)$ and $R = P \cap P^g$; we have that PM/M and $P^g M/M$ are Sylow p-subgroups of G/M and $NM/M = N_{G/M}(PM/M)$. Thus, since G/M is an sn(p)-group:

$$(1) \qquad L = PM \cap P^g M \quad \text{is normalized by } N .$$

Now, $(L \cap P)M = L \cap PM = L$ and so $L \cap P$ and, analogously, $L \cap P^g$ is a Sylow p-subgroup of L; whence there exists $u \in M$ such that $L \cap P^g = (L \cap P)^u = L \cap P^u$. Thus:

$$(2) \qquad R = P \cap P^g = L \cap P \cap P^g = L \cap P \cap P^u .$$

Set $R_0 = P \cap P^u$. Now, G/C satisfies condition $\mathfrak{C}_p$; thus, since $PC/C \in \mathrm{Syl}_p(G/C)$ and

$$R_0 C/C \leqslant PC/C \leqslant NC/C \leqslant N_{G/C}(PC/C) ,$$

we have: $R_0 C/C \trianglelefteq NC/C$. Hence $R_0 C$ is normalized by N, and, consequently, $P \cap R_0 C = R_0(P \cap C) \trianglelefteq N$. But, since $[C, u] = 1$, $P^u \cap C = (P \cap C)^u = P \cap C$. Thus $P \cap C \leqslant P \cap P^u = R_0$, yielding:

$$R_0 = R_0(P \cap C) \trianglelefteq N .$$

This, together with (1) and (2), shows that N normalizes $R = P \cap P^g$, concluding the proof. ■

We observe that condition $\mathfrak{C}_2^*$ alone is not enough to ensure that a group satisfying it is an sn(2)-group. Let $H = SL(2, 3)$ and M be an odd order elementary abelian group on which H acts irreducibly and in such a way $C_H(M) = Z(H)$. Take $G = MH$ the semidirect product; then G satisfies $\mathfrak{C}_2^*$ but G is not an sn(2)-group (the reason for that will be clear soon).

In the next Lemma we isolate an argument which will be frequently used in the sequel.

3.4 Lemma *Let M be an abelian normal subgroup of the sn-group G, and write $\bar{G} = G/C_G(M)$. Then the following condition* $(*)$ *is satisfied*:

$(*)$ *For every prime p, with $(p, |M|) = 1$, and every $\bar{H} \leqslant \bar{P} \in \mathrm{Syl}_p(\bar{G})$, $C_M(\bar{H})$ is invariant for $N_{\bar{G}}(\bar{P})$* (see Peng [8; Lemma 4]).

Proof. Let $\bar{N} = N_{\bar{G}}(\bar{P})$ and let H, N be the inverse images of $\bar{H}, \bar{N}$ respectively, in the canonical homomorphism $G \to \bar{G}$. Then H is subnormal in N. Since N is an *sn*-group, by Lemma 1.2 we have: $N = HS_N(Q)$, where $Q \in \mathrm{Syl}_p(H)$. Now, since $(p, |M|) = 1$: $S_N(Q) \cap M = C_M(Q) = C_M(H)$, whence $C_M(H)$ is normal in $S_N(Q)$, yielding: $C_M(Q) \trianglelefteq HS_N(Q) = N$, as we wanted. ■

Let G be a group; we denote by $F^*(G)$ the generalized Fitting subgroup of G (see [3; §13]). Then [3; 13.14]: $F^*(G) = E(G)F(G)$, where $F(G)$ is the Fitting subgroup of G, and $E(G)$ is a perfect characteristic subgroup of G such that $E(G)/Z(E(G))$ is the direct product of simple non abelian groups. Further, $[E(G), F(G)] = 1$ and $E(G) \cap F(G) = Z(E(G))$.

We now consider the groups $A_G(U/V)$, where U/V is an abelian chief factor of an *sn*-group G. In view of Lemma 3.4, and in order to simplify notations, we state here the following common hypothesis for the next results:

(I) *A is an sn-group acting faithfully and irreducibly on a $\mathbb{F}_q A$-module M (q a prime), in such a way condition $(*)$ is satisfied, for every subgroup K of A acting on M viewed as a $\mathbb{F}_q K$-module.*

Now, Lemma 3.4 ensures that hypothesis (I) is satisfied when $M = U/V$ is an abelian chief factor of an sn-group G, and $A = A_G(U/V)$. Also Lemma 3.1 follows from hypothesis (I); in fact if $H \leqslant O_p(A)$, then H is contained in every Sylow p-subgroup of A and so hypothesis (I) yields $C_M(H)$ invariant by $(N_A(P))^A = A$ (where $P \in \mathrm{Syl}_p(A)$); thus $C_M(H) = 1$. We shall refer to this fact as to Lemma 3.1.

The next Lemma may be compared to Corollary 3.2.

3.5 Lemma. *Assume hypothesis* (I). *Then*:

(*a*) *For every prime p, $p \neq q$, the Sylow p-subgroups of $F^*(A)$ are cyclic or generalized quaternion.*

(*b*) *Let $\bar{A} = A/F^*(A)$; then $|\bar{A}'| \leqslant 3$ and, if A is not soluble, $\bar{A}' = 1$.*

PROOF. (a) If $E(A) = 1$, then $F^*(A) = F(A)$ acts, by Lemma 3.1, as a group of f.p.f. automorphisms on M; thus the result follows.

Hence assume $E(A) \neq 1$ and let $Z = Z(E(A))$. Since A is an sn-group, $E(A)/Z$ is the direct product of groups of type $PSL(2, r)$ or $Sz(r)$, where r is a power of 2. Since the Schur multiplicator of such groups is elementary abelian of order 1, 2 or 4 (this last case occurring only for $Sz(8)$), we have that Z is an elementary abelian 2-group; as $Z \leqslant F(A)$, this yields $|Z| \leqslant 2$.

Let p be odd, and assume, by contradiction, that the Sylow p-subgroups of $F^*(A)$ are not cyclic; let $P \in \mathrm{Syl}_p(F^*(A))$. Then there exists a non trivial component S of $E(A)$ (see [3; X.13.17]) such that $D = P \cap S \neq 1$. D is cyclic, because such are the Sylow p-subgroups (p odd) of the groups of type $PSL(2, 2^n)$ and $Sz(2^{m+1})$. Further, since P is not cyclic, there exists an element x of order p in P, such that $S \cap \langle x \rangle = [S, x] = 1$ (here we use the fact that Z is a 2-group and, in particular, $D \cap F(G) = 1$). Let y be an element of order p in D and put $B = \langle x, y \rangle$. Then B is an elementary abelian p-group acting faithfully on M. Let $M_0 = C_M(B)$ and $\bar{M} = M/M_0$; thus $\bar{M}$ is non trivial and, since $p \neq q$: $\bar{M} = \langle C_{\bar{M}}(a); 1 \neq a \in B \rangle$ (see [3; X.1.9]). Let $N = N_S(D)$; now, N normalizes B (since it fixes $\langle y \rangle$ and centralizes $\langle x \rangle$) and so it acts on $\bar{M}$. Hypothesis (I) implies that, for every $1 \neq a \in B$, $C_{\bar{M}}(a)$ is N-invariant and thus $\langle a \rangle^N$ acts trivially on $C_{\bar{M}}(a)$. Now, $\langle a \rangle^N = \langle a \rangle$ or $\langle a \rangle^N = B$; if the second case occurs, $C_{\bar{M}}(a) = 1$. Hence, if $1 \neq a \in B$ and $C_{\bar{M}}(a) \neq 1$, then $\langle a \rangle \trianglelefteq N$. In this case, suppose $\langle a \rangle \neq \langle x \rangle$ and $\langle a \rangle \neq \langle y \rangle$; since N centralizes $B/\langle y \rangle$, we have that N centralizes $\langle a \rangle$ and so N centralizes $\langle a, x \rangle = B$, which is not possible because N does not centralize the cyclic group D. Thus, if $1 \neq a \in B$ and $C_{\bar{M}}(a) \neq 1$, then $a \in \langle u \rangle \cup \langle y \rangle$; hence:

$$\bar{M} = \langle C_{\bar{M}}(x), C_{\bar{M}}(y) \rangle .$$

Therefore, if $M_1 = C_M(x)$, M_1 is N-invariant and y centralizes M/M_1. Since $[S, x] = 1$, we may apply the same argument for every conjugate of D in S. M_1 is S-invariant and $\langle y \rangle^S$ centralizes M/M_1. This implies $[M, \langle y \rangle^S] \neq M$. But $\langle y \rangle^S = S$ is normal in A, being a perfect subnormal subgroup. This contradicts the fact that M is a faithful irreducible module for A. Thus, if p is odd, the Sylow p-subgroups of $F^*(A)$ are cyclic.

Let now $p = 2$, then $q \neq 2$. Let $P \in \mathrm{Syl}_2(F^*(A))$; we show that P acts as a group of f.p.f. automorphisms on M. Let $Q \in \mathrm{Syl}_2(A)$ such that $P \leqslant Q$ and let $L = N_A(Q)$. Take $x \in P$ of order 2 such that $|C_M(x)|$ is maximal. Write $M_0 = C_M(x)$ and $K = \langle x \rangle^L$; then $K \leqslant P = Q \cap \cap F^*(A)$ and, since M_0 is L-invariant, $K \leqslant C_L(M_0)$. If $\langle x \rangle^L \neq \langle x \rangle$, then $\langle x \rangle^L$ is not cyclic and there exists $1 \neq y \in \langle x \rangle^L$ such that $M_1/M_0 = C_{M/M_0}(y) \neq 1$. Now, $[M_1, y, y] \leqslant [M_0, K] = 1$ and, since $q \neq 2$, $[M_1, y] = 1$. Our choice of x gives $M_1 = M_0$, a contradiction. Hence K is cyclic, that is $\langle x \rangle$ is normal in L.

Let now $g \in A$ such that $x^g \in Q$; then $x^g \in P = Q \cap F^*(A)$ and $|C_M(x^g)| = |C_M(x)|$. Thus, again, $\langle x^g \rangle \trianglelefteq L$. By a classical Theorem of Burnside, this implies $\langle x \rangle = \langle x^g \rangle$ and, since $|x| = 2$, $x = x^g$. Therefore, x is an isolated involution of Q. By Glauberman Z^*-Theorem (see [6; Th. 4.95]):

$$x \in Z^*(A), \quad \text{where } Z^*(A)/O_{2'}(A) = Z(A/O_{2'}(A)) .$$

Hence, for every $h \in A$:

$$[h, x] \in O_{2'}(A) \cap F^*(A) = O_{2'}(F(A));$$

since $F(A)$ is the hypercentre of $F^*(A)$, we conclude that x belongs to $F(A)$ and so, by Lemma 3.1, $C_M(x) = 1$. This shows that the Sylow 2-subgroups of $F^*(A)$ are cyclic or generalized quaternion, concluding the proof of point (*a*).

(*b*) Since A is an *sn*-group, by Proposition 2.5, we have that A' induces a group of inner automorphisms on every non abelian chief factor of A. Now, if A is not soluble, $E(A) \neq 1$, and so, by point (*a*), $F(A)$ is cyclic (in fact, this follows from 3.1 if $q = 2$, and the fact that (*a*) implies that any Sylow 2-subgroup of $F^*(A)$ is a Sylow 2-subgroup of $E(A)$ if $E(A) \neq 1$ and $q \neq 2$; we recall that $E(A) \cap F(A) = Z(E(A))$). Thus A' induces a group of inner automorphisms on every chief factor of A, and so $A' \leqslant F^*(A)$.

If A is soluble, then $F^*(A) = F(A)$ and there is at most one non cyclic composition factor U/V of A, between 1 and $F(A)$ in every chief series of A through $F(A)$ (this follows from Lemma 3.1). In this case $|U/V| = 4$ and, if $C = C_A(U/V)$, A/C is isomorphic to a subgroup of $\mathrm{Aut}(U/V) \simeq S_3$. Now, $A' \cap C$ centralizes every chief factors of A and so $A' \cap C \leqslant F(A)$, proving our assertion: $|\bar{A}'| \leqslant 3$. ∎

We denote by $l^*(G)$ the generalized Fitting length of the group G.

3.6 Corollary. *Let G be an* sn-*group. Then $l^*(G) \leqslant 4$. More precisely, if $F = F(G)$ and $H/F = F^*(G/F)$, then G/H is metabelian.*

PROOF. Let $F^* = F^*(G)$. By definition, F^* is the set of those elements of G which act, by conjugation, as an inner automorphism on every chief factor of G. By 2.5 and 3.5 (together with Lemma 3.4), the chief factors of G/F^* are simple or of order 4. Hence, the same is true for G/F, because all chief factors of G lying between F and F^* are non abelian and thus simple. So, if $H/F = F^*(G/F)$, G/H is metabelian.

3.7 LEMMA. *Assume hypothesis* (I) *and let p be a prime $p \neq q$. If p is odd, then A satisfies $\mathcal{C}_p$; if $p = 2$, then A satisfies $\mathcal{C}_2^*$.*

PROOF. (*A*) p odd. Let $P \in \mathrm{Syl}_p(A)$ and $N = N_A(P)$. Firstly, we observe that P is abelian. In fact, as $p \neq q$, M is completely reducible as a $\mathbb{F}_q P$-module. Now, hypothesis (I) implies, via Lemma 3.1, that $P/C_P(U)$ is cyclic, for every P-component U of M. Since P is faithful on M, it follows that P is abelian.

Now, in order to prove that A satisfies $\mathcal{C}_p$, it is enough to show that N fixes by conjugation every cyclic subgroup of $B = \Omega_1(P)$.

Let $D = P \cap F^*(A)$; then, by 3.5, D is cyclic.

Let $1 \neq x \in B$; thus $|x| = p$. If $x \in D$ then $\langle x \rangle$ car $D \trianglelefteq N$, and so $\langle x \rangle \trianglelefteq N$. Hence assume $x \notin D$, so $\langle x \rangle \cap D = 1$. Let $D \neq 1$.

If $D \leqslant S$ for some component S of $E(A)$, then, since P is abelian, x centralizes D and it follows from 2.5 that x induces on S an inner automorphism. Without loss of generality, we may assume that x centralizes S; now, the argument used in the proof of 3.5(*a*) leads to a contradiction.

Thus $D \cap E(A) = 1$ and so $D \leqslant F(A)$. Suppose that $p \neq 3$ or A is non soluble. By Lemma 3.5(*b*), we have in this case $[P, N] \leqslant P \cap F^*(A) = D$ and so $P/D \leqslant Z(N/D)$. In particular, $\langle D, x \rangle \trianglelefteq N$. Take $z \in D$ of order p, and set $L = \langle x, z \rangle$. Then L is elementary abelian of order p^2, whence

$$M = \langle C_M(a); 1 \neq a \in L \rangle .$$

Since z acts f.p.f. on M, there exist $u, v \in L \setminus \langle z \rangle$ such that $\langle u \rangle \neq \langle v \rangle$ and $C_M(u) \neq 1 \neq C_M(v)$. Now, hypothesis (I) implies $C_M(u) = C_M(\langle u \rangle^N)$, and the same for v. But $L \trianglelefteq N$, because $L = \Omega_1(\langle x, D \rangle) \trianglelefteq N$. Hence $\langle u \rangle^N \leqslant L$ and $\langle v \rangle^N \leqslant L$. Since $z \in L$ acts f.p.f. on M this yields $\langle u \rangle \trianglelefteq N$ and $\langle v \rangle \trianglelefteq N$. Thus, N normalizes the non trivial pairwise disjoint subgroups $\langle z \rangle$, $\langle u \rangle$, $\langle v \rangle$ of L. Since $|L| = p^2$, it follows that N acts as a group of powers on L; in particular $\langle x \rangle$ is normalized by N, as we wanted. Observe that if $D = 1$, then, as in this case $P \leqslant Z(N)$, condition $\mathcal{C}_p$ follows trivially.

Now assume that A is soluble and $p = 3$. Then, since $x \notin F(A) = F^*(A)$, x does not centralize some $O_r(A)$, $r \neq p$. Suppose that x does not centralize $K = O_r(A)$ for some $r \neq 2$, and let $Q = \Omega_1(K)$; since Q is cyclic, $\langle Q, x\rangle$ is a Frobenius group acting faithfully on M (observe that we certainly have $r \neq q$), and so $C_M(x) \neq 1$ (see [4; 3.4.4]). Now, $C_M(x) = C_M(T)$, where $T = \langle x\rangle^N \leqslant B$. If $T \neq \langle x\rangle$, then $C_T(Q) \neq 1$, because $T/C_T(Q)$ is cyclic. If $M_0 = C_M(C_T(Q))$, $M/M_0 \neq 1$ and, by the same argument used before $C_{M/M_0}(x) \neq 1$. But $C_{M/M_0}(x) = C_M(x)M_0/M_0$, contradicting the fact that $C_M(x) = C_M(T) \leqslant C_M(C_T(Q)) = M_0$. Thus $T = \langle x\rangle$ and so $\langle x\rangle$ is normalized by N.

Finally, suppose that the only r-component, $r \neq 3$, of $F(A)$ not centralized by x is $R = O_2(A)$. This implies that R is a quaternion group of order 8. Also, we have that $\langle F(A), x\rangle$ is normal in A (this is because x centralizes $O_{2'}(F(A))$, so $\langle F(A), x\rangle/F(A) = (A/F(A))'$, by 3.5). Now, arguing as in the case $p \neq 3$, we conclude that $\langle x\rangle$ is also normalised by N. This completes the proof for p odd.

(B) $p = 2$. Again, let $P \in \mathrm{Syl}_2(A)$ and set $N = N_A^{\mathfrak{F}}(P)$, $D = P \cap F^*(A)$.

(1) $\Omega_1(P) \leqslant Z(N)$. Write $B = \Omega_1(P)$.

Let U be an $\mathbb{F}_q N$-chief factor of M. Then hypothesis (I) and lemma 3.1 imply that $B/C_B(U)$ is cyclic or generalized quaternion. Since B is generated by elements of order 2, we get $|B/C_B(U)| \leqslant 2$ and so $[B, N] \leqslant C_B(U)$. This holds for every N-chief factor of M. Since $q \neq 2$ and B acts faithfully on M, we have $[B, N] = 1$.

If P is abelian (1) implies that condition $\mathcal{C}_2$ is satisfied. Hence assume, for the fest of the proof, that P is not abelian.

(2) $P/D \leqslant Z(N/D)$. In fact, Lemma 3.5(b) implies, in particular, that the derived subgroup of $NF^*(A)/F^*(A)$ has order 1 or 3. This entails: $[P, N] \leqslant P \cap F^*(A) = D$, thus giving (2).

If $D = 1$, we are done. Thus assume, from now on, $D \neq 1$.

(3) Let $g \in P$, $|g| = 4$; then $g^2 \in D$.

Suppose, by contradiction $g^2 \notin D$. Then $\langle g\rangle \cap D = 1$ and, if $\langle z\rangle = \Omega_1(D)$, $\langle g\rangle \cap \langle z\rangle = 1$. Let $a = g^2$; $\langle z, a\rangle$ is an elementary abelian group of order 4, and so $M = \langle C_M(x); 1 \neq x \in \langle z, a\rangle\rangle$. Because z

does not fix any element of M, it follows that $C_M(a) \neq 1$. Since $a \in Z(P)$, there exists a non trivial irreducible $\mathbb{F}_q P$-submodule V of M, such that a acts trivially on V. Let $C^* = C_P(V)$ and $R = C^* \cap \langle z, g\rangle$. Then $R = \langle g\rangle$ or $R = \langle zg\rangle$, because $a \in R$, $\langle z, g\rangle/R$ is cyclic by Lemma 3.1, and $z \notin R$. Further, $C^* \trianglelefteq P$, $C^* \cap D = 1$ and so, by (2): $[C^*, P] \leqslant P' \cap C^* \leqslant D \cap C^* = 1$, whence $C^* \leqslant Z(P)$. Thus, $g \in Z(P)$ or $zg \in Z(P)$; since $z \in Z(P)$, we get $g \in Z(P)$.

Let now U be an irreducible $\mathbb{F}_q P$-submodule of M such that $a \notin C_p(U)$ (this certainly exists, because M is faithful and completely reducible as an $\mathbb{F}_q P$-module). Then also we have $C_P(U) \cap D = 1$ and, by 3.1, $P/C_P(U)$ is cyclic or generalized quaternion. Now, $gC_P(U)$ is a central element of order 4 in $P/C_P(U)$ and so $P/C_P(U)$ is cyclic. But then, $P' \leqslant D \cap C_P(U) = 1$, which is not the case. This contradiction shows that $a = g^2 \in D$.

(4) Conclusion. Let K be a subgroup of P maximal in order to contain D and such that $\Omega_1(K) = \Omega_1(D) = \langle z\rangle$. Then, by (2), $K \trianglelefteq N$. Let W be a non trivial irreducible $\mathbb{F}_q P$-submodule of M and let $C = C_P(W)$. Then $C \cap D = 1$ and so $C \cap K = 1$; by (3), C is elementary abelian. Moreover, since P is not abelian and $P' \leqslant D$, P/C is generalized quaternion. The proof is completed by showing that $P = KC$. Suppose, by contradiction, that $KC \neq P$. Since P/C is generated by elements of order 4, there exists $y \in P$ such that $|yC| = 4$ and $y \notin KC$. Now, $y^4 \in C$ and, since $\langle z, C\rangle/C = \Omega_1(P/C)$, $y^2 \in KC$. Hence $y^4 \in K \cap C = 1$ and so, by (3), $y^2 = z \in D$. Consider now $L = \langle K, y\rangle$. Since $K \trianglelefteq P$ and $y \notin K$, $|L:K| = 2$ and, by our choice of K, $\Omega_1(L) > \Omega_1(K) = \langle z\rangle$. Thus $L = K\Omega_1(L) \leqslant K\Omega_1(P)$ and, consequently, $y \in K\Omega_1(P) = K\langle z, C\rangle = KC$, contradicting the choice of y. Thus $P = KC$ and the Lemma is proved. ∎

Before stating the next Theorem, we observe the following trivial property of groups satisfying $\mathfrak{C}_2^*$.

3.8 Lemma. *Let G be a group satisfying condition $\mathfrak{C}_2^*$. Let $P \in \mathrm{Syl}_2(G)$ and $N = N_G(P)$. If $H \leqslant P$, then $H \trianglelefteq N$, or $P = Q \times A$, where Q is generalized quaternion, and $H \geqslant Z(Q)$.*

Proof. If P is abelian, then $P \leqslant Z(N)$ and the result is trivial. Hence assume $P = Q \times A$, with Q generalized quaternion and A elementary abelian; also suppose $Z(Q) \nleqslant H \leqslant P$. Then $H \cap Q = 1$ and so $H = H/(H \cap Q) \cong HQ/Q$ is elementary abelian. Thus $H \leqslant \Omega_1(P) \leqslant Z(N)$; in particular $H \trianglelefteq N$. ∎

3.9 THEOREM. *Let G be a group. Then G is an* sn-*group if and only if for every chief factor U/V of G the following conditions hold*:

(*a*) *if U/V is non abelian, then it is simple and $A_G(U/V)$ is as described in Proposition* 2.5;

(*b*) *if U/V is an elementary abelian q-group, then $A_G(U/V)$ satisfies $\mathfrak{C}_p$ for every odd prime p, $p \neq q$, and it satisfies $\mathfrak{C}_2^*$ if $q \neq 2$.*

PROOF. ($\Rightarrow$) This follows at once from Proposition 2.5 and Lemma 3.5 (via Lemma 3.4).

($\Leftarrow$) Suppose that for every chief factor of the group G conditions (*a*) and (*b*) are satisfied. We proceed by induction on G. Let M be a minimal normal subgroup of G. Then, by inductive hypothesis, G/M is an sn-group. Let $C = C_G(M)$. If M is simple non abelian, then $C \cap M = 1$. By 2.5 and condition (*a*), we have that G/C is an sn-group. Since the class of sn-groups is a formation, we conclude that G is an sn-group.

Otherwise M is an elementary abelian q-group, for some prime q. Now, by Lemma 3.3 and condition (*b*), G is an sn(p)-group for every odd prime p, $p \neq q$. Also, by 1.15, since $M \leqslant O_q(G)$, G is an sn(q)-group. In order to apply Theorem 1.11 and conclude that G is an sn-group, we have to show that, if $q \neq 2$, G is an sn(2)-group. Let P be a Sylow 2-subgroup of G, and $\overline{P} = PC/C$. If $\overline{P}$ is abelian, then G/C actually satisfies condition $\mathfrak{C}_2$ and we may apply Lemma 3.3. Thus, assume $\overline{P} = \overline{P} = \overline{Q} \times \overline{A}$, with $\overline{Q}$ generalized quaternion and $\overline{A}$ elementary abelian. Let $z \in P$, such that, if $\bar{z} = Cz$, $\langle \bar{z} \rangle = Z(\overline{Q})$. We show that z acts as the inversion on M. Let $\overline{G} = G/C$, $\overline{F} = F(G)$. Since $\overline{G}$ satisfies $\mathfrak{C}_p$ for every odd prime p, $p \neq q$, it follows that every subgroup of $O_{2'}(\overline{F})$ is normal in $\overline{G}$; in fact any such subgroup is both subnormal (being contained in $\overline{F}$) and pronormal (by Rose's characterization of groups satisfying $\mathfrak{C}_p$, see [8; p. 936]). Thus $\overline{G}$ centralizes $O_{2'}(\overline{F})$; in particular $\bar{z} \in \overline{P}'$ centralizes $O_{2'}(\overline{F})$. Moreover, it is easy to check that $\bar{z}$ is an isolated involution of $\overline{P}$. By Glauberman's Z^*-Theorem, $\bar{z} \in Z^*(\overline{G}) = \overline{K}$. Now, $\overline{K}$ is 2-nilpotent and $\bar{z}$ centralizes $O_{2'}(F(\overline{K}))$. It follows that $\bar{z} \in F(\overline{K}) \leqslant \overline{F}$; this in turn implies $\bar{z} \in Z(\overline{G})$. Thus, since M is a minimal normal subgroup of G, z acts as the inversion on M.

Now, we have to show that, for every $g \in G$, $P \cap P^g \trianglelefteq N$, where $N = N_G(P)$. Arguing as in the proof of Lemma 3.3, it is enough to show that this is true when $g \in M$. Let $R = P \cap P^g$. Suppose that

RC/C is not normalized by $NC/C \leqslant N_{\bar{G}}(\bar{P})$. Then, by Lemma 3.8, $RC/C \ni \bar{z}$; hence there exists $y \in R$ acting as the inversion on M. In particular, $g^y = g^{-1}$. Since $R = P \cap P^g$, this implies $g = 1$, and so $R = P \trianglelefteq N$. Otherwise RC is normalized by N. Then, arguing as in the proof of Lemma 3.3, N normalizes $RC \cap P = R(C \cap P) = R$. This completes the proof of the Theorem. ■

We now exploit Theorem 3.9 (and the preceding lemmas) to give some more explicit descriptions of the groups $A_G(U/V)$, for an *sn*-group G.

3.10 Theorem. *Let G be a group of odd order. Then G is an* sn*-group if and only if $A_G(U/V)$ is a* **T**-*group for every chief factor U/V of G.*

Proof. Let G be an odd order sn-group, U/V a chief factor of G and set $\bar{G} = A_G(U/V)$. Let U/V be a q-group, q a prime. Then, by Theorem 3.9, and the fact that $2 \nmid |G|$, $\bar{G}$ satisfies condition $\mathcal{C}_p$ for every $p \neq q$. Moreover, by Corollary 3.2, $\bar{F} = F(\bar{G})$ is cyclic. Thus $\bar{G}' \leqslant \bar{F}$; hence, if $\bar{Q}$ is a Sylow q-subgroup of $\bar{G}$, $[\bar{Q}, N_{\bar{G}}(\bar{Q})] \leqslant \bar{Q} \cap \bar{F} = 1$. Then $\bar{G}$ satisfies also condition $\mathcal{C}_q$. If now follows from Robinson [9; Theorem 1] that $\bar{G}$ is a (soluble) **T**-group.

Conversely, a soluble **T**-group satisfies condition $\mathcal{C}_p$ for every prime p ([9; Theorem 1*]). Thus, by Theorem 3.9, a soluble group in which $A_G(U/V)$ is a **T**-group, for every chief factor U/V, is an *sn*-group; in particular this is true for groups of odd order. ■

Remarks (*a*) Arguing as in the first part of the proof of 3.10, it is easy to show that, in a soluble *sn*-group G, $A_G(U/V)$ satisfies condition $\mathcal{C}_2^*$ also when U/V is a 2-group (in this case it indeed satisfies $\mathcal{C}_2$).

(*b*) Every soluble Frobenius group is an *sn*-group. This is not true for Frobenius groups in general. In fact the non split extension of $SL(2, 5)$ by a group of order 2 is a Frobenius complement, but it is not an sn-group. Accordingly to Zassenhaus's results on Frobenius groups, every such group has a subgroup of index 2 which is an *sn*-group.

Finally, we describe $A_G(U/V)$ in (non soluble) *sn*-groups. To avoid heavy notations we come back to hypothesis (I).

3.11 Theorem. *Assume that hypothesis* (I) *holds for the group A and the $\mathbb{F}_q A$-module M. Then*:

(a) If $q \neq 2$, then A is soluble or $F^*(A) \simeq SL(2,5) \times K$, where K is cyclic and $(|K|, |SL(2,5)|) = 1$. Also $A \simeq SL(2,5) \times H$, where H is a $\boldsymbol{T}$-group, $K = F(H)$, the Sylow 2-subgroups of H are elementary abelian and the only possible prime common divisors of $(|H|, |SL(2,5)|)$ are q and 2.

(b) If $q = 2$, then A is soluble or $F^*(A) = R \times K$ where K is cyclic of odd order and R is either simple or $R = S \times T$ with $S \simeq PSL(2, 2^n)$, $T \simeq Sz(2^m)$ and n, m are odd and coprime (this ensures $(|S|, |T|) = 2^i$). Moreover A is a semidirect product $R \rtimes H$, where H is a $\boldsymbol{T}$-group, acting on R in the way described in 2.5, $K = F(H)$ and $(|R|, |H|) = 2^j$.

PROOF. (*a*) $q \neq 2$. If A is not soluble, then, by 3.5 (*b*), $E(A) \neq 1$, and, as we observed in the first part of the proof of 3.5, $Z = Z(E(A))$ has order at most 2. Furthermore, by 3.5(*a*), the Sylow 2-subgroups of $F^*(A)$ are generalized quaternion (if they were cyclic, $F^*(A)$, and so A, would be soluble). It follows that the Sylow 2-subgroups of $E(A)$ are generalized quaternion. Now, $E(A)/Z$ is the direct product of groups of type $PSL(2, 2^n)$ and $Sz(2^{2m+1})$. Thus, the only possibility is $E(A)/Z \simeq PSL(2,4) \simeq PSL(2,5)$ and $E(A) \simeq SL(2,5)$.

If $K = O_{2'}(F(A))$, then, by 3.5(*a*), K is cyclic and $(|K|, |E(A)|) = 1$; also: $F^*(A) = E(A) \times K$. Now, let $C = C_A(E(A))$. By 2.5, $A/C \simeq PSL(2,5)$, so $A = E(A)C$ and $E(A) \cap C = Z$. Clearly $F(C) = F(A)$, which is cyclic; whence C is 2-nilpotent. Moreover, by 3.9, A satisfies $\mathfrak{C}_2^*$. This implies that the Sylow 2-subgroups of C are elementary abelian. It follows that $Z = O_2(F(C))$ has a normal complement H in C. Now, clearly, $F(H) = K$ and $A \simeq E(A) \times H$. Furthermore, since, by 3.9, A satisfies $\mathfrak{C}_p$ for every prime $p \neq 2, q$, we have that, for all such primes, A is either p-nilpotent or p-perfect (Robinson [8; Theorem 3]). Since $H' \leqslant K$ and $(|E(A)|, |K|) = 1$, we get $(|E(A)|, |H|) = 2^i q^j$, $i, j \in \mathbb{N}$. Finally, keeping in mind that H satisfies $\mathfrak{C}_p$ (see Robinson [9; Corollary p. 936]) for every prime p dividing $F(H)$ and that $F(H) \geqslant H'$, it is easy to see that H is a $\boldsymbol{T}$-group.

(*b*) $q = 2$. Suppose that A is not soluble; then $E(A) \neq 1$, and, since in this case $O_2(A) = 1$, $Z(E(A)) = 1$. Thus $F^*(A) = R \times K$, where $K = F(A)$ is cyclic of odd order, and $R = R_1 \times R_2 \times \ldots \times R_s$ is the direct product of simple sn-groups. But, by 3.5(*a*), the Sylow p-subgroup of $F^*(A)$ are cyclic, for every odd prime p. Hence

$(|R|, |K|) = 1$ and, for every $i, j \in \{1, 2, ..., s\}$, $i \neq j$, the only prime dividing $(|R_i|, |R_j|)$ is 2. Now, for every $n \in \mathbb{N}$, 3 divides $|PSL(2, 2^n)|$ and 5 divides $|Sz(2^{2n+1})|$. Thus either R is simple or $s = 2$, $R = S \times T$, with $S \simeq PSL(2, 2^n)$, $T \simeq Sz(2^m)$, m odd, $m \geqslant 3$, and $(|S|, |T|) = 2^i$, $i \in \mathbb{N}$; it is easily seen that this last condition is satisfied if and on y if n, m are coprime odd numbers.

Let now $C = C_A(R)$; then $C \cap R = 1$ and it follows from Proposition 2.5 that A/C is the semidirect product of RC/C by an abelian group H/C. Hence $A \simeq R \rtimes H$, the semidirect product of R by H, where the action of H on R is as described in 2.5.

Now, $K \leqslant C \leqslant H$ and, by 3.5(*b*), $H' \leqslant F^*(A) \cap H = K$, whence, $K = F(H)$. Finally, by 3.9, A satisfies $\mathcal{C}_p$ for every odd prime p and so does H. Thus A and H are either p-nilpotent or p-perfect, for every odd prime p. This yields at once $(|R|, |H|) = 2^j$ (recall that $(|R|, |K|) = 2^i$) and, since $H' \leqslant F(H)$ and $F(H)$ is cyclic of odd order, implies that H is a T-group. ■

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