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Unitary Independence in the Study of Finitely Generated and of Finite Rank Torsion-Free Modules over a Valuation Domain.

PAOLO ZANARDO (*)

Introduction.

Let R be a valuation domain, S a fixed maximal immediate extension of R; if $u_1, \ldots, u_n$ are units of S, and I is an ideal of R, then $u_1, \ldots, u_n$ are said to be *unitarily independent* (briefly: *u-independent*) *over* I if the following property is satisfied:

(*) if $c_0 + \sum_{i=1}^{n} c_i u_i \in IS$, with $c_0, c_1, \ldots, c_n \in R$, then $c_0, c_1, \ldots, c_n \in P$, where P is the maximal ideal of R.

The notion of unitary independence was first introduced, in a slightly different way, in [10], in order to construct indecomposable finitely generated R-modules (see Prop. 4, Theorem 6 and Prop. 7 of [10]). Unitary independence was investigated by Facchini, Salce and the author in [2, 5], and played a fundamental role for the classification of certain classes of indecomposable finitely generated R-modules (see [11]).

L. Salce and the author made evident, in [7, 8], a resemblance between the theory of finitely generated R-modules and the theory

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of torsion-free R-modules of finite rank (see, in particular, Theorem 6.2 of [8]). This similarity suggests that the results on finitely generated modules can be carried, with suitable modifications, over finite rank torsion-free modules, and vice versa.

The purpose of this paper is to employ u-independence for the investigation of finite rank torsion-free modules, obtaining results analogous to the ones that hold for finitely generated modules. We shall consider finite rank torsion-free modules which are ***homogeneous and of co-rank*** 1 (see the preliminary Section 1).

In Section 2 we shall see that to any such module M, with rank $n+1$, we can associate an n-tuple $(u_1, \dots, u_n)$ of units of S, and an ideal I, in such a way that M is indecomposable if $u_1, \dots, u_n$ are u-independent over I (Theorem 2). Conversely, starting from an ideal I and an n-tuple $(u_1, \dots, u_n)$ of units of S u-independent over I, we construct in Prop. 3 an indecomposable homogeneous module of rank $n+1$ and co-rank 1. We note that Prop. 3 generalizes results by Arnold [1], Prop. 4.3, and by Viljoen [9]. The starting point of all these results is the classical construction of a rank-two torsion-free indecomposable module over a discrete valuation ring, given by Kaplansky in [4], p. 46.

The results in Section 3 show the central role of u-independence for the investigation of both finitely generated and finite rank torsion-free modules. In fact, if the n-tuple $(u_1, \dots, u_n)$ and the ideal I are associated to the finite rank torsion-free module M, and $u_1, \dots, u_n$ are u-dependent over I, then, not only M is decomposable, but also *the relations of u-dependence among the u_i's produce, in a canonical way, a decomposition of M into indecomposable summands* (Theorem 4). Finally, in Theorem 5 we prove an analogous result for finitely generated modules, giving a remarkable improvement of Prop. 7 of [10].

§ 1. For general facts about valuation domains and their modules we refer to the book by Fuchs and Salce [3].

Let us fix some terminology. In the sequel, we shall always denote by R a valuation domain, by P the maximal ideal of R, by Q the field of fractions of R, by S a fixed maximal immediate extension of R, and by $U(R)$, $U(S)$ the sets of units of R, S, respectively.

If $u \in U(S)\setminus R$, the ***breadth ideal*** $B(u)$ of u is defined as follows (see [5]):

$$B(u) = \{a \in R\colon u \notin R + aS\} .$$

We recall that an ideal I of R is the breadth ideal of a suitable unit u of S not in R, $I = B(u)$, if and only if R/I is not complete in the R/I-topology (see Prop. 1.4 of [5]).

Let us note that, if $u_1, \ldots, u_n \in U(S)$ are u-independent over an ideal I, then no u_i belongs to R; if, moreover, $j \leqslant n$ is such that $B(u_j) \leqslant I$, then, necessarily, $B(u_j) = I$.

In Ch. X of [3], it is introduced the notion of *basic submodule* of an R-module; when M is either finitely generated or torsion-free of finite rank, which case we are interested to, then a submodule B of M is basic if and only if:

1) $B = \bigoplus U_i$, where U_i is uniserial;

2) B is pure in M;

3) if V is a uniserial submodule of M, then either $B \cap V \neq 0$, or $B \oplus V$ is *not* pure in M.

Basic submodules are unique up to isomorphism ([3], Th. 3.2, p. 203) hence the number of the uniserial summands of a basic submodule is an invariant of M, which we shall denote by $b(M)$; when M is finitely generated, then $b(M) = g(M) =$ Goldie dimension of M ([3], Cor. 2.2, p. 179). An R-module is said to be *homogeneous* if the uniserial summands of a basic submodule are all isomorphic. If M is finite rank torsion-free, we shall say that M *has co-rank* 1 if $\operatorname{rk}(M) = b(M) + 1$, or, equivalently, if M/B is uniserial, for B basic in M.

In this preliminary section we recall some definitions and results on finitely generated modules given in [10, 11] (see also Ch. IX of [3]), to emphasize the analogies with the discussion on finite rank torsion-free modules of the next section.

Let X be a finitely generated module; the *length* of X, denoted by $l(X)$, is the minimal number of generators of X. We shall deal with the case when X is homogeneous and $l(X) = b(X) + 1 = g(X) + 1 = n + 1$. For the fundamental notion of *annihilator sequence* of a finitely generated module we refer to Ch. IX of [3]; in this case it is enough to note that X uniquely determines two ideals, $A = \operatorname{Ann} X < J$, where J is such that $X/B \cong R/J$ for any basic submodule B of X (X/B is cyclic, because $l(X) = b(X) + 1$). A and J are said to be *the ideals in the annihilator sequence of* X.

We can choose a minimal set of generators $x = \{x_0, x_1, \ldots, x_n\}$ of X, such that:

i) $\langle x_1, \ldots, x_n \rangle = \bigoplus_{i=1}^{n} Rx_i = B$ is basic in M, so that $Rx_i \simeq R/A$, for $i = 1, \ldots, n$;

ii) there exist *units* u_i^r of R, for $i = 1, \ldots, n$ and $r \in J^* = J \setminus \{0\}$. such that

$$rx_0 = r \sum_{i=1}^{n} u_i^r x_i \qquad \text{for all } r \in J^* . \tag{1}$$

Note that $J = \operatorname{Ann}(x_0 + B)$, since $X/B = R(x_0 + B)$.

From ii) it follows that, if $rR \leqslant sR$, with $r, s \in J^*$, then

$$r(u_i^r - u_i^s) \in A \qquad \text{for } i = 1, \ldots, n . \tag{2}$$

Since S is a maximal immediate extension of R, for all $i \leqslant n$ there exists $u_i \in U(S)$ such that

$$u_i - u_i^r \in r^{-1}AS \qquad \text{for all } r \in J^* . \tag{3}$$

In such a way we get an n-tuple $(u_1, \ldots, u_n) \in U(S)^n$; if we set $I = \bigcap_{r \in J^*} r^{-1}A$, by (3) and the definition of breadth ideal, it follows that either $u_i \in R$ or $B(u_i) \leqslant I$, for $i = 1, \ldots, n$. The n-tuple $(u_1, \ldots, u_n)$ is said to be *associated* to X; we also say that the system of generators x *produces* $(u_1, \ldots, u_n)$.

The content of Theorem 6 and Prop. 7 of [10] can be summarized in the following

THEOREM 0 [10]. (1) *Let X be a homogeneous finitely generated module such that $l(X) = b(X) + 1 = n + 1$, let $(u_1, \ldots, u_n) \in U(S)^n$ be an associated n-tuple of X, and let $I = \bigcap_{r \in J^*} r^{-1}A$, where $A < J$ are the ideals in the annihilator sequence of X. Then X is indecomposable if and only if $u_1, \ldots, u_n$ are u-independent over I.* (2) *Let $u_1, \ldots, u_n \in U(S)$ be u-independent over a suitable ideal I of R, and let $B(u_i) \leqslant I$ for $i = 1, \ldots, n$. Then there exists a finitely generated indecomposable homogeneous module X, with $l(X) = b(X) + 1 = n + 1$, such that the n-tuple $(u_1, \ldots, u_n)$ is associated to it.*

REMARK. If X, as above, is indecomposable, X is called *couniform* homogeneous. The ideals A and J and suitable equivalence classes of associated n-tuples provide a complete and independent system of invariants for couniform homogeneous modules (see Th. 3 of [11]; see also [7]).

§ 2. Let us now denote by M a torsion-free module of finite rank $n+1$, *which is homogeneous and has co-rank* 1. We shall look at M as an R-submodule of a vector space $V \simeq Q^{n+1}$. As in the case of finitely generated modules, we want to find suitable systems of generators of M, in order to define n-tuples of units of S associated to M.

For this purpose we proceed by steps.

STEP 1. *A basic submodule B of M is of the form* $B = \bigoplus_{i=1}^{n} Lx_i$, *where $L \geqslant R$ is a suitable R-submodule of Q, and $x_1, \ldots, x_n \in M$.*

Let $B = \bigoplus_{i=1}^{n} U_i$ be a basic submodule of M, with U_i uniserial for all i. Since M is homogeneous, all the U_i's are isomorphic to a suitable torsion-free uniserial module L; in view of Th. 1.1, p. 270 of [3], we can assume that L is an R-submodule of Q, containing R. If $f_i\colon L \to U_i$ is an isomorphism $(i = 1, \ldots, n)$, set $f_i(1) = x_i \in M$ (recall that $1 \in L \geqslant R$). It is then immediate that B has the desired form. ///

STEP 2. *M/B is isomorphic to H, where H is a suitable R-submodule of Q, containing L.* ///

STEP 3. *There exists $x_0 \in V$ such that:*

i) *$\{x_0, x_1, \ldots, x_n\}$ is a basis of V,*

ii) *M can be written, by generators, in the form:*

$$M = \langle B, x_r\colon \text{for all } r^{-1} \in H \setminus L \rangle$$

where $x_r = r^{-1}\left(x_0 + \sum_{i=1}^{n} u_i^r x_i\right)$, *for suitable $u_i^r \in L$.*

In view of Step 2, there exists an isomorphism $f\colon H \to M/B$; for all $r^{-1} \in H \setminus L$, choose $x_r \in M$ such that $f(r^{-1}) = x_r + B$, and choose $x_0 \in M$ such that $f(1) = x_0 + B$. Since M/B is torsion-free, we have $Rx_0 \cap B = 0$, and this ensures that $\{x_0, x_1, \ldots, x_n\}$ is a basis of V.

In particular, for all $r^{-1} \in H \setminus L$, there exists $a_0^r, a_1^r, \ldots, a_n^r \in Q$ such that

$$x_r = a_0^r x_0 + \sum_{i=1}^{n} a_i^r x_i ; \tag{4}$$

from $rf(r^{-1}) = f(1)$ and (4), it follows:

$$rx_r - x_0 \in B \qquad \text{for all } r^{-1} \in H \setminus L ,$$

from which, using the fact that $x_0, x_1, \ldots, x_n$ are linearly independent, we get, for all $r^{-1} \in H \setminus L$ and for all $i \leqslant n$

$$ra_0 = 1 , \qquad \text{i.e. } a_0^r = r^{-1} ;$$

$$ra_i^r \in L , \qquad \text{i.e. } a_i^r = r^{-1} u_i^r \text{ for suitable } u_i^r \in L .$$

It is clear that $M = \langle B, x_r \colon r^{-1} \in H \setminus L \rangle$, and we have proved that x_r is of the form $x_r = r^{-1}\left(x_0 + \sum_{i=1}^{n} u_i^r x_i\right)$. ///

STEP 4. *For a suitable choice of x_0 in Step 3, the u_i^r turn out to be units of R, for all $r^{-1} \in H \setminus L$, and for all $i \leqslant n$.*

In the notation of Step 3, note that, if $H \geqslant s^{-1}R \geqslant r^{-1}R > L$, from $f(r^{-1}) = r^{-1} s f(s^{-1})$ it follows

$$r^{-1} s x_s - x_r \in B \tag{5}$$

from which, by the linear independence of $x_0, x_1, \ldots, x_n$, we get, for all $i \leqslant n$:

$$u_i^r - u_i^s \in rL \qquad (\text{for all } r, s \colon H \geqslant s^{-1}R \geqslant r^{-1}R > L) . \tag{6}$$

Fix now $t^{-1} \in H \setminus L$; if we take an arbitrary $r^{-1} \in H \setminus L$, from (6) it follows that either $u_i^r - u_i^t \in rL$ or $u_i^r - u_i^t \in tL$; in any case $u_i^r - u_i^t \in P$, because rL, $tL \leqslant P$, so that, for all $r^{-1} \in H \setminus L$ and for all $i \leqslant n$,

$$v_i^r = u_i^r - u_i^t - 1 = u_i^r - a_i$$

is a unit of R. Set now

$$y_0 = x_0 + \sum_{i=1}^{n} a_i x_i \,;$$

then the x_r's can be written in the form

$$x_r = r^{-1}\Big(y_0 + \sum_{i=1}^{n} (u_i^r - a_i) x_i\Big) = r^{-1}\Big(y_0 + \sum_{i=1}^{n} v_i^r x_i\Big)\,,$$

where $v_i^r \in U(R)$ for all $r \in H \setminus L$, and for all $i \leqslant n$. The desired conclusion follows. ///

STEP 5. *If $u_i^r \in U(R)$, for all r and i, then there exist $u_i \in U(S)$, $i = 1, \dots, n$, such that*

(7) $$u_i - u_i^r \in rLS \qquad \text{for all } r^{-1} \in H \setminus L\,.$$

Since S is a maximal immediate extension of R, the assertion follows from (6). ///

Note that, differently from the case of finitely generated modules, M determines L and H only up to isomorphism; hence also $I = \bigcap_{r^{-1}\in H\setminus L} rL$ is determined up to isomorphism.

It is clear that $u_1, \dots, u_n$, found in Step 5, depend by the choice of L, H and of the system of generators of M. By definition of breadth ideal, the relations (7) show that either $u_i \in R$ or $B(u_i) \leqslant I$, for all $i \leqslant n$.

The n-tuple $(u_1, \dots, u_n) \in U(S)^n$ is said to be *associated* to M; the ideal $I = \bigcap_{r^{-1}\in H\setminus L} rL$ is said to be *the ideal of the n-tuple* $(u_1, \dots, u_n)$.

By another point of view, we see that if $(u_1, \dots, u_n) \in U(S)^n$ is associated to a homogeneous torsion-free module M of finite rank and co-rank 1, then $M \subseteq Q^{n+1}$ can be written in the form

$$M = \Big\langle \bigoplus_{i=1}^{n} L x_i,\; r^{-1}\Big(x_0 + \sum_{i=1}^{n} u_i^r x_i\Big) : r^{-1} \in H \setminus L \Big\rangle$$

for a suitable choice of $\{x_0, x_1, \dots, x_n\}$ basis of Q^{n+1}, of H and L R-submodules of Q, and of u_i^r units of R which satisfy the relations (7).

The next Proposition 1 is the main ingredient to prove the analog of Theorem 0 for finite rank torsion-free modules. In the proof of it, we shall use the notion of *height* $h_M(x)$ of an element x of M, and its properties; we refer to Ch. VIII of [3] for an extensive treatment on heights.

PROPOSITION 1. *Let M be a homogeneous torsion-free module of finite rank, with co-rank 1; let $B = \bigoplus_{i=1}^{n} Lx_i$ be basic in M, with $L \geqslant R$. If M is decomposable, then there exists $j \leqslant n$ such that Lx_j is a summand of M.*

PROOF. Suppose that $M = M_1 \oplus M_2$ is a non trivial decomposition of M. Since $\operatorname{rk}(M) = b(M) + 1$ and by the uniqueness of basic submodules up to isomorphism, it follows that one of the summands, say M_2, is such that $\operatorname{rk}(M_2) = b(M_2)$. But then M_2 is a direct sum of uniserial modules, all isomorphic to L. So we can assume, without loss of generality, that M_2 is uniserial. Let $\pi_1\colon M \to M_1$, $\pi_2\colon M \to M_2$ be the canonical projections, and, for $i = 1, \ldots, n$, set $x_i' = \pi_1(x_i)$, $x_i'' = \pi_2(x_i)$. It is clear that, if $x_i'' \neq 0$, then the restriction $\pi_2\colon Lx_i \to M_2$ is injective, since each proper quotient of Lx_i is torsion and M_2 is torsion-free. Let us prove that there exists $j \leqslant n$ such that $\pi_2\colon Lx_j \to M_2$ is onto, in which case π_2 restricted to Lx_j will be an isomorphism. By contradiction, assume that for all $i \leqslant n$, $\pi_2\colon Lx_i \to M_2$ is not surjective. In particular, for all $i \leqslant n$, $\pi_2(Lx_i) = Lx_i''$ is either zero, or it is not pure in M_2, because a nonzero pure submodule of a torsion-free uniserial module is the whole module. From this fact we deduce that, for all $i \leqslant n$, $h_M(x_i'') = h_{M_2}(x_i'') > L/R = h_M(x_i)$. But then, from $x_i = x_i' + x_i''$, it follows $h_M(x_i') = h_{M_1}(x_i') = L/R$. Let us prove that the x_i' are linearly independent; in fact, if $\sum_{i=1}^{n} a_i x_i' = 0$, with $a_i \in R$ not all zero, it follows that

$$0 \neq \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i x_i'' ,$$

and this is impossible, because the height of the second member is strictly larger than the height of the first member. Let us now prove that $\bigoplus_{i=1}^{n} Lx_i'$ is pure in M_1; it is enough to check that, for any choice of $a_1, \ldots, a_n \in R$, with some a_i a unit of R, we have $h_{M_1}\left(\sum_{i=1}^{n} a_i x_i'\right) = L/R$.

This is true: in fact,

$$h_{M_1}\Big(\sum_{i=1}^{n} a_i x_i'\Big) = h_M\Big(\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} a_i x_i''\Big) = h_M\Big(\sum_{i=1}^{n} a_i x_i\Big) = L/R\,,$$

since B is pure and $h_M\Big(\sum_{i=1}^{n} a_i x_i''\Big) > L/R$. But if $\bigoplus_{i=1}^{n} Lx_i'$ is pure in M_1, then $\bigoplus_{i=1}^{n} Lx_i' \oplus M_2$ is pure in M, from which $n+1 \leqslant b(M) = n$, which is the desired contradiction.

If then we choose $j \leqslant n$ in such a way that $\pi_2\colon Lx_j \to M_2$ is an isomorphism, we obtain $M = Lx_j \oplus M_1$. This concludes the proof. ///

THEOREM 2. *Let M be a homogeneous torsion-free module of finite rank, with co-rank* 1; *let $(u_1, \ldots, u_n)$ be an n-tuple associated to M, and let I be the ideal of $(u_1, \ldots, u_n)$. If $u_1, \ldots, u_n$ are u-independent over I, then M is indecomposable.*

PROOF. Let us write M in the form

$$M = \Big\langle \bigoplus_{i=1}^{n} Lx_i = B,\ x_r = r^{-1}\Big(x_0 + \sum_{i=1}^{n} u_i^r x_i\Big) \colon r^{-1} \in H \setminus L \Big\rangle\ ;$$

by contradiction, let us suppose that M is decomposable. In view of Prop. 1, we can assume, without loss of generality, that Lx_n is a direct summand of M, i.e. $M = Lx_n \oplus N$. Then we have

$$x_i = b_i x_n + m_i \qquad \text{for } i = 0, 1, \ldots, n-1\,,$$

$$x_r = b_r x_n + m_r \qquad \text{for all } r^{-1} \in H \setminus L\,,$$

where $b_i, b_r \in L$, $m_i, m_r \in N$, for all i and r. We obtain, for all $r^{-1} \in H \setminus L$

$$(8) \qquad rx_r = rb_r x_n + rm_r = x_0 + \sum_{i=1}^{n} u_i^r x_i =$$

$$= \Big(b_0 + \sum_{i=1}^{n-1} b_i u_i^r + u_n^r\Big)x_n + \Big(m_0 + \sum_{i=1}^{n-1} u_i^r m_i\Big)\,.$$

By the uniqueness of the decomposition we get

$$(9)\qquad b_0+\sum_{i=1}^{n-1} b_i u_i^r + u_n^r = rb_r \in rL \qquad \text{for all } r^{-1}\in H\setminus L\,;$$

multiplying, if necessary, (9) for a suitable element of R, we can get a relation

$$(10)\qquad c_0+\sum_{i=1}^{n} c_i u_i^r \in rL\,,$$

where $c_0, c_1, \ldots, c_n \in R$, and *some* c_i *is a unit.* By (10), using (7), we obtain

$$(11)\qquad c_0+\sum_{i=1}^{n} c_i u_i \in \bigcap_{r^{-1}\in H\setminus L} rLS = IS\,;$$

since $u_1, \ldots, u_n$ are u-independent over I, (11) would imply $c_0, c_1, \ldots$ $\ldots, c_n \in P$, contrary to our choice of the c_i's. The desired conclusion follows. ///

Suppose now to have chosen $u_1, \ldots, u_n \in U(S)$, and an ideal I of R such that:

a) $B(u_i) \leqslant I$ for $i = 1, \ldots, n$,

b) $u_1, \ldots, u_n$ are u-independent over I.

As already observed, from *a*) and *b*) it follows $B(u_i) = I$ for all $i \leqslant n$.

In this situation, we ask if there exists an indecomposable finite rank torsion-free module M, which is homogeneous, of co-rank 1, and such that $(u_1, \ldots, u_n)$ is associated to it.

For this purpose, we choose two submodules L, H of Q, with $Q \geqslant H > L \geqslant R$, such that

i) $I = \bigcap_{r^{-1}\in H\setminus L} rL$;

ii) $I < rL$ for all $r^{-1}\in H\setminus L$.

Such a choice is possible in view of the results given in [10, 6, 8]; as a matter of fact, it is enough to take $L = R$, $H = \{r^{-1}\in Q : rL > I\}$; the triple (L, H, I) is said to be *compatible* (see [8, 6]). Since $rL > I$

for all $r^{-1} \in H \setminus L$, and $B(u_i) = I$ for all i, there exists a family $\{u_i^r: i = 1, \dots, n, r^{-1} \in H \setminus L\}$ of units of R, such that, for all i and r,

$$u_i - u_i^r \in rLS . \tag{7}$$

We define by generators an R-submodule of the vector space

$$V = \bigoplus_{i=1}^{n} Qx_i ,$$

in the following way:

$$M = \left\langle \bigoplus_{i=1}^{n} Lx_i = B,\ x_r = r^{-1}\left(x_0 + \sum_{i=1}^{n} u_i^r x_i\right):\ r^{-1} \in H \setminus L \right\rangle .$$

PROPOSITION 3. *In the above notation, M is indecomposable, homogeneous, with co-rank* 1, *and* $(u_1, \dots, u_n)$ *is an associated n-tuple of M.*

PROOF. If we prove that B is basic in M, we are done; in fact, in that case, by the definitions, M has co-rank 1, $(u_1, \dots, u_n)$ is associated to M, and I is the ideal of $(u_1, \dots, u_n)$, so that we can apply Theorem 2, to obtain M indecomposable.

First of all, let us prove that B is pure in M; actually, we will check that M/B is uniserial and torsion-free. Note that $M/B = \langle x_r + B: r^{-1} \in H \setminus L \rangle$. To prove that M/B is uniserial, it is enough to prove that the cyclic submodules $R(x_r + B)$, $r^{-1} \in H \setminus L$, form a chain with respect to inclusion. Let us choose r, s such that $H \geqslant s^{-1}R \geqslant r^{-1}R > L$; then (7) and $rL \geqslant sL$ imply that, for $i = 1, \dots, n$

$$u_i^r - u_i^s \in rL . \tag{12}$$

From (12) it follows

$$sr^{-1}x_s - x_r = r^{-1} \sum_{i=1}^{n} (u_i^s - u_i^r) x_i \in B ; \tag{13}$$

from (13) we reach at once the desired conclusion. Since M/B is uniserial, to prove that it is torsion-free, it is enough to exhibit an element of M/B with zero annihilator. For instance, $x_0 + B \in M/B$, and $Rx_0 \cap B = 0$ implies that $\mathrm{Ann}_{M/B}(x_0 + B) = 0$.

Since $\mathrm{rk}\,(M) = \mathrm{rk}\,(B) + 1$, to conclude that B is basic, it is enough to prove that M is not a direct sum of uniserial modules. Actually, if $M = \bigoplus_{i=1}^{n} U_i$, U_i uniserial, since Lx_n is pure in M, by Th. 5.6, p. 192 of [3], we get that Lx_n is a direct summand of M; using the same argument as in the proof of Theorem 2, we contradict the u-independence of the u_i's. This completes the proof. ///

§ 3. The purpose of this section is to show in which way u-dependence and decomposition of modules are related.

Let $(u_1, \ldots, u_n) \in U(S)^n$ be an n-tuple associated to a homogeneous torsion-free module M, with co-rank 1; let I be the ideal of $(u_1, \ldots, u_n)$. We shall say that u_j *u-depends* by u_i over I, where $i \in F \subseteq \{1, \ldots, n\}$ if

$$u_j \equiv c_0 + \sum_{i \in F} c_i u_i \qquad (\text{mod } IS)$$

with c_0, $c_i \in R$, for all $i \in F$.

If $u_1, \ldots, u_n$ are not u-independent over I, using an easy induction, one proves that there exists a proper subset F of $\{1, \ldots, n\}$, such that the u_i's, $i \in F$, are u-independent over I, and u_j u-depends by u_i over I, for all $j \in \{1, \ldots, n\} \setminus F$. If, possibly, $F = \emptyset$, this simply means that $u_j \in R + IS$, for $1 \leqslant j \leqslant n$. Let us suppose that such an F is nonempty; let $k < n$ be the cardinality of F. Without loss of generality we can assume that $F = \{1, \ldots, k\}$; for $j = k + 1, \ldots, n$, we have

$$u_j \equiv c_{0j} + \sum_{i=1}^{k} c_{ij} u_i \qquad (\text{mod } IS)\,, \tag{14}$$

for suitable c_{0j}, c_{ij} in R.

The following theorem shows that from the relations (14) we can deduce a canonical decomposition into indecomposable summands of the module M, which has $(u_1, \ldots, u_n)$ as associated n-tuple.

THEOREM 4. *Let M be a torsion-free homogeneous module of finite rank $n + 1$, with co-rank 1; let $(u_1, \ldots, u_n)$ be an associated n-tuple of M, and let I be the ideal of $(u_1, \ldots, u_n)$. Let us write $M \subseteq V = \bigoplus_{i=0}^{n} Qx_i$, by generators, in the form*

$$M = \left\langle \bigoplus_{i=1}^{n} Lx_i,\ x_r = r^{-1}\Big(x_0 + \sum_{i=1}^{n} u_i^r x_i\Big) : r^{-1} \in H \setminus L \right\rangle .$$

If the relations (14) *hold for a suitable* $k < n$, *where* $u_1, \dots, u_k$ *are u-independent over* I, *set* $y_i = x_i + \sum_{j=k+1}^{n} c_{ij} x_j$, *for* $i = 0, 1, \dots, k$, *and* $y_j = x_j$ *for* $j > k$. *Then* M *decomposes in the following way:*

$$M = N \oplus Ly_{k+1} \oplus \dots \oplus Ly_n$$

where

$$N = \left\langle \bigoplus_{i=1}^{k} Ly_i,\ y_r = r^{-1}\Big(y_0 + \sum_{i=1}^{k} u_i^r y_i\Big) : r^{-1} \in H \setminus L \right\rangle$$

is indecomposable.

PROOF. Let us note that $\bigoplus_{i=1}^{n} Lx_i = \bigoplus_{i=1}^{n} Ly_i$, as it is immediate to check. From (14) and (7) we get, for all $r^{-1} \in H \setminus L$, and for $j = k + 1, \dots, n$

$$\text{(15)} \qquad u_j^r \equiv c_{0j} + \sum_{i=1}^{k} c_{ij} u_i^r \qquad (\text{mod } rL) .$$

For all $r^{-1} \in H \setminus L$, by the definitions of $y_0, \dots, y_n$ and of y_r, using (15) we obtain:

$$x_r - y_r = r^{-1}\Big(\sum_{i=1}^{n} u_i^r x_i - \sum_{j=k+1}^{n} c_{0j} x_j - \sum_{i=1}^{k} u_i^r\Big(x_i + \sum_{j=k+1}^{n} c_{ij} x_j\Big)\Big) =$$

$$= r^{-1} \sum_{j=k+1}^{n} \Big(u_j^r - c_{0j} - \sum_{i=1}^{k} c_{ij} u_i^r\Big) x_j \in \bigoplus_{j=k+1}^{n} Ly_j .$$

This shows, first of all, that $y_r \in M$ for all r, hence $N \subseteq M$; moreover $x_r - y_r \in \bigoplus_{j=k+1}^{n} Ly_j$, for all r, implies that $M \subseteq N + \bigoplus_{j=k+1}^{n} Ly_j$, so that $M = N + \bigoplus_{j=k+1}^{n} Ly_j$: Since $N \subseteq \bigoplus_{i=0}^{k} Qy_i$, it is also clear that $N \cap \Big(\bigoplus_{j=k+1}^{n} Ly_j\Big) = 0$.

It remains to prove that N is indecomposable. Since $\bigoplus_{i=1}^{k} Ly_i$ is basic in N, and $\text{rk}(N) = k + 1$, N is homogeneous with co-rank 1; by the definitions, $(u_1, \dots, u_k)$ is a k-tuple associated to N, and I is the ideal of $(u_1, \dots, u_k)$. It is then enough to invoke Theorem 2. ///

It is easy to verify that the number of uniserial summands in any indecomposable decomposition of a torsion-free module M of finite rank, is an invariant of M (for example we can use the fact that the endomorphism ring of a uniserial module U is local, so that U has the exchange property).

In view of Theorem 4, we deduce that the positive integer $k = |F|$, where F is as in the discussion before Th. 4, *does not depend neither by the choice of* F, *nor by the n-tuple* $(u_1, \dots, u_n)$.

It is interesting to prove the analog of Theorem 4 for finitely generated modules. The next Theorem 5 will be an improvement of Prop. 7 of [10] (hence of Theorem 0, too).

Let X be a finitely generated homogeneous module such that $l(X) = b(X) + 1 = n + 1$, and let $(u_1, \dots, u_n) \in U(S)^n$ be associated to X. Let $A < J$ be the ideals in the annihilator sequence of X, and let $I = \bigcap_{r \in J^*} r^{-1}A$. As in the above discussion, we can assume that $u_1, \dots, u_k$ are u-independent over I, while $u_{k+1}, \dots, u_n$ u-depend by $u_1, \dots, u_k$, according to the relations (14).

Such relations of u-dependence give a canonical decomposition of X; we have the following

THEOREM 5. *Let X be a finitely generated homogeneous module such that $l(X) = b(X) + 1 = n + 1$; let $A < J$ be the ideals in the annihilator sequence of X, let $I = \bigcap_{r \in J^*} r^{-1} A$, and let $(u_1, \dots, u_n)$ be an associated n-tuple of X. Let $x = \{x_0, x_1, \dots, x_n\}$ be a system of generators of X which produces $(u_1, \dots, u_n)$. If the relations* (14) *hold for a suitable $k < n$, and $u_1, \dots, u_k$ are u-independent over I, set*

$$y_0 = x_0 - \sum_{j=k+1}^{n} c_{0j} x_j \,, \qquad y_i = x_i + \sum_{j=k+1}^{n} c_{ij} x_j$$

for $i = 1, \dots, k$, and $y_j = x_j$, for $j = k + 1, \dots, n$. Then X decomposes in the following way

$$X = Y \oplus R y_{k+1} \oplus \dots \oplus R y_n$$

where $Y = \langle y_0, y_1, \dots, y_k \rangle$ is indecomposable, and $(u_1, \dots, u_k)$ is associated to Y.

PROOF. First of all, note that $y = \{y_0, y_1, \dots, y_n\}$ is a minimal system of generators of X, since the matrix T such that $Tx = y$ is invertible in R. Moreover, $\bigoplus_{i=1}^{n} Rx_i = \langle y_1, \dots, y_n \rangle$, and it is an easy exercise to verify that $y_1, \dots, y_n$ are linearly independent, so that $\bigoplus_{i=1}^{n} Ry_i$ is basic in X. Hence to prove that $X = Y \oplus Ry_{k+1} \oplus \dots \oplus Ry_n$, it is enough to verify that $Y \cap \left(\bigoplus_{j=k+1}^{n} Ry_j \right) = 0$.

By contradiction, let us suppose that

$$ry_0 + \sum_{i=1}^{k} a_i y_i + \sum_{j=k+1}^{n} a_j y_j = 0 \tag{16}$$

with $\sum_{j=k+1}^{n} a_j y_j \neq 0$. From (16) and the definition of y, it follows that $rx_0 \in \langle x_1, \dots, x_n \rangle$, so that $r \in J = \operatorname{Ann}(x_0 + \langle x_1, \dots, x_n \rangle)$. If now $r = 0$, we have an immediate contradiction, since $y_1, \dots, y_n$ are linearly independent. We can thus assume that $r \in J^*$, and, since $\bigoplus_{i=1}^{n} Ry_i$ is pure we can write $a_i = rb_i$, for suitable $b_i \in R$, for $i = 1, \dots, k, k+1, \dots, n$. Then (16) is equivalent to

$$r\left(x_0 - \sum_{j=k+1}^{n} c_{0j} x_j + \sum_{i=1}^{k} b_i x_i + \sum_{i=1}^{k} b_i \left(\sum_{j=k+1}^{n} c_{ij} x_j \right) + \sum_{j=k+1}^{n} b_j x_j \right) = 0 . \tag{17}$$

Now, since $r \in J^*$, we have $rx_0 = r \sum_{i=1}^{n} u_i^r x_i$, where $u_i^r \in U(R)$ are such that $u_i - u_i^r \in r^{-1}AS$, for all i. Thus, substituting, in (17), rx_0 by $r \sum_{i=1}^{n} u_i^r x_i$, we obtain

$$r \sum_{i=1}^{k} (u_i^r + b_i) x_i + r \sum_{j=k+1}^{n} \left(u_j^r - c_{0j} + \sum_{i=1}^{k} b_i c_{ij} + b_j \right) x_j = 0 , \tag{18}$$

from which

$$u_i^r + b_i \equiv 0 \pmod{r^{-1}A} \qquad \text{for } i = 1, \dots, k \tag{19}$$

and

$$u_j^r - c_{0j} + \sum_{i=1}^{k} b_i c_{ij} + b_j \equiv 0 \pmod{r^{-1}A} \qquad j = k+1, \dots, n . \tag{20}$$

From the relations (14), using the fact that $u_i - u_i^r \in r^{-1}AS$, for $i = 1, \ldots, n$, we get

$$u_j^r \equiv c_{0j} + \sum_{i=1}^{k} u_i^r c_{ij} \qquad (\operatorname{mod} r^{-1}A)\,. \tag{21}$$

Substituting (19) and (21) in (20), we get

$$b_j \equiv 0 \quad (\operatorname{mod} r^{-1}A) \qquad \text{for } j = k+1, \ldots, n\,.$$

This implies that

$$\sum_{j=k+1}^{n} a_j y_j = \sum_{j=k+1}^{n} r b_j y_j = 0\,,$$

which is the required contradiction.

It remains to prove that Y is indecomposable, Since $\bigoplus_{i=1}^{k} Ry_i$ is basic in Y, we have, for all $r \in J^*$

$$r y_0 = r \sum_{i=1}^{k} v_i^r y_i \qquad \text{for suitable } v_i^r \in R\,. \tag{22}$$

From (22) we get

$$r\Big(x_0 - \sum_{i=k+1}^{n} c_{0j} x_j\Big) = r \sum_{i=1}^{k} v_i^r \Big(x_i + \sum_{j=k+1}^{n} c_{ij} x_j\Big)\,. \tag{23}$$

From (23), since $r x_0 = r \sum_{i=1}^{n} u_i^r x_i$, we obtain, for all $r \in J^*$

$$u_i^r - v_i^r \in r^{-1}A \qquad \text{for } i = 1, \ldots, k$$

and also

$$u_i - v_i^r \in r^{-1}AS \qquad \text{for } i = 1, \ldots, k \text{ and for all } r \in J^*\,.$$

This implies that $(u_1, \ldots, u_k)$ is associated to Y. Since $u_1, \ldots, u_k$ are u-independent over $I = \bigcap_{r \in J^*} r^{-1}A$, where $A < J$ are the ideals in the annihilator sequence of Y, we can apply Theorem 0 to Y, obtaining that Y is indecomposable, as desired. ///

REMARK. Let us consider a torsion-free module M, with rank $n+1$, containing a submodule $B = \bigoplus_{i=1}^{n} Lx_i$, *which is pure in* M *but not necessarily basic* (in other words: it can happen that $M = B \oplus U$, with U uniserial). Again M can be written by generators in the form

$$M = \left\langle B, x_r = r^{-1}\left(x_0 + \sum_{i=1}^{n} u_i^r x_i\right) : r^{-1} \in H \setminus L \right\rangle$$

(in the discussion of § 2 we only use the fact that B is pure and M/B is uniserial). We can also associate to M an n-tuple $(u_1, \ldots, u_n)$ and consider the ideal I of the n-tuple. It is easy to adapt Theorem 4 to this slightly more general situation, obtaining that *such* M *is a direct sum of uniserial modules if and only if there exist* $c_1, \ldots, c_n \in R$ *such that* $c_i \equiv u_i$ (mod IS) *for* $i = 1, \ldots, n$. Analogous considerations hold for the case of finitely generated modules.

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