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Roberto Peirone

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Homomorphisms between Complete Chains and the Independence of the Axioms of Limitoid.

ROBERTO PEIRONE (*)

G. Greco introduced in [3] the notion of limitoid, which is a generalization of the notions of lim inf and lim sup, and more generally of Γ-limit (see also [4]).

Namely, if X is a set and L is a complete lattice, then an L-limitoid in X is defined to be a map T from L^X into L satisfying the following conditions:

1) *$f \leqslant g$ implies $T(f) \leqslant T(g)$ for all $f, g \in L^X$.*
2) *$T(\psi \circ f) = \psi(T(f))$ whenever $f \in L^X$ and $\psi\colon L \to L$ is a complete homomorphism.*
3) *If L' is a closed sublattice of L and $f \in L^X$ and $\operatorname{Im} f \subseteq L'$, then $T(f) \in L'$.*

Greco also considered the question whether property 3) follows from 1) and 2). In this paper we give a necessary and sufficient condition for the existence of a map from L^X into L satisfying 1) and 2), but not satisfying 3), in case X has at least two elements and L is completely distributive. When L is a complete chain this condition is equivalent to the following one:

a) There exists $a \in L \setminus \{0_L, 1_L\}$ such that every complete homomorphism from $[0_L, a]$ into $[a, 1_L]$ is constant and every complete homomorphism from $[a, 1_L]$ into $[0_L, a]$ is constant.

(*) Indirizzo dell'A.: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56100 Pisa, Italy.

We construct a chain satisfying property *a*), and so in general 3) does not follow from 1) and 2). More generally, for any natural number $n \geqslant 1$, we find n complete chains $L_1, \ldots, L_n$ having the following equivalent properties *b*), *c*):

b) *If* $i \neq j$ *every complete homomorphism from* L_i *into* L_j *is constant.*

c) *If* $i \neq j$ *every* (*non-trivial*) *closed interval of* L_i *is not embeddable in* L_j.

We also prove that 1) follows from 2), provided that L is a complete chain having at least three elements.

1. Notation and definitions.

We now recall some definitions and notation concerning lattices. For other basic notions we refer to [1]. In particular, we require, in contrast to the definitions of [5], that a complete lattice L has a maximum (denoted by 1_L) and a minimum (denoted by 0_L).

A subset M of a complete lattice L is said to be a closed sublattice of L if $\sup A \in M$ and $\inf A \in M$ for every non-empty subset A of M.

If L and L' are complete lattices, then we denote by $\mathrm{Hom}_c(L, L')$ the set of all complete homomorphisms from L into L'. We recall that $\psi\colon L \to L'$ is a complete homomorphism if $\sup_{c\in A} \psi(c) = \psi\big(\sup_{c\in A} c\big)$ and $\inf_{c\in A} \psi(c) = \psi\big(\inf_{c\in A} c\big)$ for every non-empty subset A of L. We remark that, if L is a completely distributive complete lattice, then, for all $a, b \in L$, the map $\varphi_{a,b}$ defined by $\varphi_{a,b}(c) = (c \vee a) \wedge b$ is a complete homomorphism. We denote by $\mathrm{Cons}(L, L')$ the set of all constant maps from L into L'. Clearly, $\mathrm{Cons}(L, L') \subseteq \mathrm{Hom}_c(L, L')$. The expression $\mathrm{Hom}_c(L, L') = \mathrm{Cons}(L, L')$ will be shortened to $\mathrm{Hom}_c(L, L') = \mathrm{Cons}$ and similarly for similar expressions.

Let L and L' be chains. We say that L is embeddable in L' (and write $L \preccurlyeq L'$) if there exists an order-preserving map ψ from L into L' (i.e. $a < b$ implies $\psi(a) < \psi(b)$). We say that L and L' are order-isomorphic (and write $L \simeq L'$) if there exists an order-preserving map from L onto L' (see [5], chap. 1, § 2).

A chain L is said to be dense if, for any $a, b \in L$ with $a < b$, there exists $c \in L$ such that $a < c < b$.

If $L_1, \ldots, L_n$ are chains, we define the sum $\sum_{i \leqslant n} L_i$ according to [5], Definition 1.38. We remark that $\sum_{i \leqslant n} L_i$ is not dense (for $n > 1$) even if L_i is dense for every $i \leqslant n$. If L_i ($i = 1, \ldots, n$) are dense chains then we define $\sum_{d\, i \leqslant n} L_i$, by identifying 1_{L_i} and $0_{L_{i+1}}$ (for $i = 1, \ldots, n-1$) in $\sum_{i \leqslant n} L_i$. Thus $\sum_{d\, i \leqslant n} L_i$ is dense. It is also complete provided that L_i is complete for $i = 1, \ldots, n$.

If L is a chain, then ${}^{\omega}L$ denotes the set of all maps from $\mathbf{N}$ into L ordered lexicographically, i.e., $a < b$ if and only if there exists $n \in \mathbf{N}$ such that $a_i = b_i$ for all $i < n$ and $a_n < b_n$.

$\mathcal{C}(L)$ denotes the completion of the chain L (see [5], Definition 2.31). $\mathbf{2}$ denotes the chain having two elements (denoted by $0, 1$). If L is a chain, then L^* is the chain having the same elements as L such that $a \leqslant b$ in L^* if and only if $a \geqslant b$ in L.

For every $n \in \mathbf{N}$ we denote by ω_n the smallest ordinal of cardinality $\aleph_n$.

Let X be a set and L a complete lattice; then, for every $A \subseteq X$, we denote by χ_A the map from X into L defined by

$$\chi_A(x) = \begin{cases} 0_L & \text{if } x \notin A\,, \\ 1_L & \text{if } x \in A\,. \end{cases}$$

Char (X, L) denotes the set $\{\chi_A \colon A \subseteq X\}$. For further information on chains see [5].

2. General considerations.

Following [3] we denote by $LIM(X, L)$ the set of all L-limitoids in X. We also consider the following weakening of condition 3) of the definition of limitoid:

3′) *If* $f \in \text{Char}(X, L)$, *then* $T(f) \in \{0_L, 1_L\}$.

If L is a completely distributive complete lattice, then T is an L-limitoid in X if and only if it satisfies 1), 2), 3′) (see [3], pp. 157-158).

We shall study the question, previously considered in [3], whether 3) follows from 1) and 2). First, however, we are concerned with the independence of 1) and 2).

2.1. Remark. Conditions 1) and 3) do not imply 2) even for $L = \overline{\mathbf{R}}$ $(= \mathbf{R} \cup \{+\infty, -\infty\})$. It suffices to consider a set X having at least two elements and the map $T\colon \overline{\mathbf{R}}^X \to \overline{\mathbf{R}}$ defined by

$$T(f) = \begin{cases} \inf\limits_{x \in X} f(x) & \text{if } f(x) \leqslant 0 \text{ for every } x \in X\,, \\ \sup\limits_{x \in X} f(x) & \text{otherwise}\,. \end{cases} \quad \blacklozenge$$

2.2. Remark. Conditions 2) and 3) do not imply 1) for $L = \mathbf{2}$. It suffices to consider a set X having at least three elements and the map $T\colon \mathbf{2}^X \to \mathbf{2}$ defined by

$$T(f) = \begin{cases} 1 & \text{if } \operatorname{card} f^{-1}(0) \text{ is } 0 \text{ or } 2\,, \\ 0 & \text{otherwise}. \end{cases} \quad \blacklozenge$$

However the following theorem holds:

2.3. Theorem. *If L is a complete chain having at least three elements, then for every $T\colon L^X \to L$ condition 2) implies 1).*

Proof. We suppose that there exist $T\colon L^X \to L$ which satisfies 2) and $f, g \in L^X$ such that $f \leqslant g$ and $T(f) > T(g)$, and derive a contradiction. We distinguish two cases.

Case A) There exists $a \in L$ with $T(g) < a < T(f)$. We put

$$h(x) = \begin{cases} g(x) & \text{if } g(x) \leqslant a\,, \\ f(x) & \text{if } f(x) > a\,, \\ a & \text{if } f(x) \leqslant a < g(x)\,. \end{cases}$$

Since the maps $\varphi_1, \varphi_2\colon L \to L$ defined by $\varphi_1(c) = c \vee a$ and $\varphi_2(c) = c \wedge a$ are complete homomorphisms, and $\varphi_1 \circ h = \varphi_1 \circ f$ and $\varphi_2 \circ h = \varphi_2 \circ g$, we have:

$$\varphi_1(T(h)) = T(\varphi_1 \circ h) = T(\varphi_1 \circ f) = T(f)$$

and

$$\varphi_2(T(h)) = T(\varphi_2 \circ h) = T(\varphi_2 \circ g) = T(g)\,,$$

and so $T(f) = T(h) = T(g)$, a contradiction.

Case B) $]T(g), T(f)[= \emptyset$. By the hypothesis either $T(g) \neq 0_L$ or $T(f) \neq 1_L$. We suppose $T(g) \neq 0_L$. (If $T(f) \neq 1_L$, the argument is analogous.) We consider $\psi \in \mathrm{Hom}_c(L, L)$ defined by

$$\psi(c) = \begin{cases} 0_L & \text{if } c \leqslant T(g)\,, \\ 1_L & \text{if } c > T(g)\,, \end{cases}$$

and put $f_1 = \psi \circ f$ and $g_1 = \psi \circ g$. Then

$$T(f_1) = T(\psi \circ f) = \psi\big(T(f)\big) = 1_L \text{ and } T(g_1) = T(\psi \circ g) = \psi\big(T(g)\big) = 0_L\,.$$

Thus, since $f_1 \leqslant g_1$ and $T(g_1) < T(g) < T(f_1)$, we may proceed as in Case *A*). ◆

We now return to the independence of 2).

2.4. Definition. *A map* $T\colon L^X \to L$ *is an L-quasi-limitoid in* X *if it satisfies* 1) *and* 2) *of the definition of limitoid. We denote by* $QLIM(X, L)$ *the set of all L-quasi-limitoids in* X.

In the sequel the word « lattice » will mean « completely distributive complete lattice » and the word « chain » will mean « complete chain ».

2.5. Theorem. *If* L *is a lattice,* $a \in L$ *and* X *is a set having at least two elements, then the following conditions* i) *and* ii) *are equivalent*:

i) *There exist* $f \in \mathrm{Char}(X, L)$ *and* $T \in QLIM(X, L)$ *such that* $T(f) = a$.

ii) *For every* $\psi \in \mathrm{Hom}_c(L, L)$ *we have* $\psi(a) = \big(a \vee \psi(0_L)\big) \wedge \psi(1_L)$.

Proof. i) $\Rightarrow$ ii). Let $\psi \in \mathrm{Hom}_c(L, L)$ and put

$$\psi'(c) = \big(c \vee \psi(0_L)\big) \wedge \psi(1_L)\,.$$

Then $\psi' \in \mathrm{Hom}_c(L, L)$ and $\psi'(0_L) = \psi(0_L)$ and $\psi'(1_L) = \psi(1_L)$. Hence 2) yields:

$$\psi(a) = \psi\big(T(f)\big) = T(\psi \circ f) = T(\psi' \circ f) = \psi'\big(T(f)\big) = \psi'(a)\,.$$

Thus $\psi(a) = \big(a \vee \psi(0_L)\big) \wedge \psi(1_L)$, and ii) holds.

ii) $\Rightarrow$ i). We put $T(f) = (a \vee \inf f) \wedge \sup f$ for every $f \in L^X$. Then T satisfies 1). It also satisfies 2). In fact for every $\psi \in \mathrm{Hom}_c(L, L)$ we have:

$$\psi(T(f)) = \psi((a \vee \inf f) \wedge \sup f)$$

$$= [\psi(a) \vee \inf(\psi \circ f)] \wedge \sup(\psi \circ f) \qquad \textit{because } \psi \in \mathrm{Hom}_c(L, L)$$

$$= \Big[[(a \vee \psi(0_L)) \wedge \psi(1_L)] \vee \inf(\psi \circ f)\Big] \wedge \sup(\psi \circ f) \qquad \textit{by ii)}$$

$$= [(a \wedge \psi(1_L)) \vee \psi(0_L) \vee \inf(\psi \circ f)] \wedge \sup(\psi \circ f) \qquad \textit{because } L \textit{ is distributive}$$

$$= [(a \wedge \psi(1_L)) \vee \inf(\psi \circ f)] \wedge \sup(\psi \circ f)$$

$$= (a \vee \inf(\psi \circ f)) \wedge \psi(1_L) \wedge \sup(\psi \circ f) \qquad \textit{because } L \textit{ is distributive}$$

$$= (a \vee \inf(\psi \circ f)) \wedge \sup(\psi \circ f) = T(\psi \circ f).$$

So $T \in QLIM(X, L)$. Since $T(\chi_{\{x\}}) = a$ for every $x \in X$, i) holds. ◆

2.6. Remark. If L is a chain, then condition ii) is equivalent to the following one: *for every $\psi \in \mathrm{Hom}_c(L, L)$ we have $\psi(a) = \psi(0_L)$ if $\psi(a) > a$ and $\psi(a) = \psi(1_L)$ if $\psi(a) < a$.* ◆

2.7. Definition. *Let L be a chain. An element $a \in L$ is a distortion point of L if a satisfies* i) *(or* ii)*) of Theorem* 2.5. *We denote by $D(L)$ the set of all distortion points of L.*

Obviously, $\{0_L, 1_L\} \subseteq D(L)$. Moreover, the following corollary holds.

2.8. Corollary. *Let X have at least two elements and let L be a chain. Then*

$$QLIM(X, L) = LIM(X, L) \qquad \textit{if and only if } D(L) = \{0_L, 1_L\}. \qquad ◆$$

2.9. Remark. $D(L)$ is a closed sublattice of L. In fact, if $a_i \in D(L)$ for every $i \in I$ and $\psi \in \mathrm{Hom}_c(L, L)$, then

$$\psi\Big(\sup_{i \in I} a_i\Big) = \sup_{i \in I} \psi(a_i) = \sup_{i \in I} [(a_i \vee \psi(0_L)) \wedge \psi(1_L)] = \Big(\sup_{i \in I} a_i \vee \psi(0_L)\Big) \wedge \psi(1_L).$$

So $\sup_{i \in I} a_i \in D(L)$, and we can prove similarly that $\inf_{i \in I} a_i \in D(L)$. ◆

In order to prove that condition 3) does not follow from 1) and 2, it is enough to find a chain L and a set X such that $QLIM(X, L) \neq \neq LIM(X, L)$ or, equivalently, $D(L) \neq \{0_L, 1_L\}$. In the sequel of this section we shall give some characterizations of distortion points in terms of homomorphisms between chains, and in the next section we shall use those characterizations to solve a slightly more general problem; namely for every natural number n ($\geqslant 3$) we shall find a (complete) chain having exactly n distortion points.

We remark that it is easy to show that the most usual lattices satisfy $D(L) = \{0_L, 1_L\}$ (cf. [3]). For istance

$$D(\overline{\mathbf{R}}) = \{0_{\mathbf{R}}, 1_{\mathbf{R}}\} = \{-\infty, +\infty\}$$

as, if we put $\psi(a) = a + 1$, then $\psi \in \mathrm{Hom}_c(\overline{\mathbf{R}}, \overline{\mathbf{R}})$ and

$$\psi(a) \neq (a \vee \psi(-\infty)) \wedge \psi(+\infty) \qquad \text{if } -\infty \neq a \neq +\infty .$$

Another example is any chain L which is not dense. In fact, if $a > \sup\{b \in L\colon\ b < a\}$, then we can define $\psi \in \mathrm{Hom}_c(L, L)$ by

$$\psi(b) = \begin{cases} 0_L & \text{if } b < a, \\ 1_L & \text{if } b \geqslant a. \end{cases}$$

So we have that $\psi(a) \neq (a \vee \psi(0_L)) \wedge \psi(1_L)$ if $0_L \neq a \neq 1_L$.

2.10. Remark. In the same way we can prove that, if L is a chain which is not dense and L' has at least two elements, then $\mathrm{Hom}_c(L, L') \neq \mathrm{Cons}$. ◆

2.11. Lemma. *Let L and L' be chains and let L be dense. If $\psi \in \mathrm{Hom}_c(L, L')$ and $a, b \in L$ are such that $a < b$ and $\psi(a) < u < \psi(b)$, then there exists $\xi \in L$ with $a < \xi < b$ and $\psi(\xi) = u$.*

Proof. Let $A = \{c \in L\colon\ \psi(c) \leqslant u\}$ and $\xi = \sup A$. Then $\psi(\xi) = = \psi(\sup A) = \sup_{c \in A} \psi(c) \leqslant u$ (so $\xi < b$). On the other hand, $\psi(\xi) \geqslant u$ for, in the contrary case, $\inf\{\psi(d)\colon\ d > \xi\} = \psi(\xi) < u$ and so there would exist $d > \xi$ such that $\psi(d) < u$, in contrast to the definition of ξ. Thus $\psi(\xi) = u$, and, since ψ is non-decreasing, $a < \xi < b$. ◆

2.12. Remark. If L and L' are chains, $a, b \in L$ and $a < b$, then for every $\psi \in \mathrm{Hom}_c([a, b], L')$ there exists $\psi_1 \in \mathrm{Hom}_c(L, L')$ such that

$\psi(c) = \psi_1(c)$ for all $c \in [a, b]$. In fact, it suffices to put $\psi_1(c) = \psi(a)$ if $c \leqslant a$ and $\psi_1(c) = \psi(b)$ if $c \geqslant b$. ◆

2.13. LEMMA. *Let L be a chain and $a \in L$. Then $a \in D(L)$ if and only if*

$$\mathrm{Hom}_c([0_L, a], [a, 1_L]) = \mathrm{Cons} \quad \textit{and} \quad \mathrm{Hom}_c([a, 1_L], [0_L, a]) = \mathrm{Cons}.$$

PROOF. If L is not dense then the assertion is obvious by Remark 2.10 and because $a \notin D(L)$ for every $a \notin \{0_L, 1_L\}$. So we may suppose that L is dense. If $\psi \in \mathrm{Hom}_c([0_L, a], [a, 1_L]) \setminus \mathrm{Cons}$, then $a \leqslant \psi(0_L) < \psi(a)$. In view of Remarks 2.12 and 2.6 this yields $a \notin D(L)$. If $\psi \in \mathrm{Hom}_c([a, 1_L], [0_L, a]) \setminus \mathrm{Cons}$ we proceed in a similar way and obtain that $a \notin D(L)$. Conversely, if $a \notin D(L)$, then, in view of Remark 2.6, there exists $\psi \in \mathrm{Hom}_c(L, L)$ which satisfies one of the following conditions:

i) $\psi(a) > a \vee \psi(0_L)$,

ii) $\psi(a) < a \wedge \psi(1_L)$.

If i) holds, then, by Lemma 2.11 there exists $b \in [0_L, a[$ such that $\psi(a) > \psi(b) \geqslant a$. So $\psi|_{[b,a]} \in \mathrm{Hom}_c([b, a], [a, 1_L]) \setminus \mathrm{Cons}$ and, by Remark 2.12, there exists $\psi' \in \mathrm{Hom}_c([0_L, a], [a, 1_L]) \setminus \mathrm{Cons}$.

If ii) holds, then we can prove, by proceeding in a similar way, that there exists $\psi' \in \mathrm{Hom}_c([a, 1_L], [0_L, a]) \setminus \mathrm{Cons}$. ◆

2.14. PROPOSITION. *Let L and L' be dense chains. Then*

$$\mathrm{Hom}_c(L, L') = \mathrm{Cons}$$

if and only if

$$\textit{for all } a', b' \in L' \textit{ with } a' < b' \textit{ we have } [a', b'] \not\leqslant L.$$

PROOF. If $\varphi \in \mathrm{Hom}_c(L, L') \setminus \mathrm{Cons}$, then, by Lemma 2.11 we have $\mathrm{Im}\,\varphi = [a', b']$, where $a' = \varphi(0_L)$ and $b' = \varphi(1_L)$. Clearly, $\psi: [a', b'] \to L$ defined by $\psi(c') = \inf \varphi^{-1}(c')$ is an order-preserving map. Conversely, if ψ is an order-preserving map from $[a', b']$ into L, where $a', b' \in L'$ and $a' < b'$, then we put

$$a = \psi(a') \quad \text{and} \quad b = \psi(b'),$$

and define $\varphi: [a, b] \to L'$ by

$$\varphi(c) = \inf\{d' \in [a', b']: \psi(d') \geqslant c\}.$$

Since φ is non-decreasing we have $\varphi\big(\sup_{i\in I} c_i\big) \geqslant \sup_{i\in I} \varphi(c_i)$. Moreover, $]\sup_{i\in I} \varphi(c_i), \varphi(\sup_{i\in I} c_i)[\subseteq [a', b'] \setminus \operatorname{Im} \varphi = \emptyset$, because $x = \varphi(\psi(x))$ for every $x \in [a', b']$. Thus, since L' is dense, we have that $\varphi\big(\sup_{i\in I} c_i\big) = \sup_{i\in I} \varphi(c_i)$ and we can prove similarly that $\varphi\big(\inf_{i\in I} c_i\big) = \inf_{i\in I} \varphi(c_i)$. So, by Remark 2.12, $\mathrm{Hom}_c(L, L') \neq \mathrm{Cons}$. ◆

In the sequel Proposition 2.14 will be often understood.

2.15. Corollary. *Let L be a chain and $a \in L$. Then $a \in D(L)$ if and only if, for all a_1, a_2, a_3 and a_4 in L with $a_1 < a_2 \leqslant a \leqslant a_3 < a_4$, we have:*

$$[a_1, a_2] \not\preccurlyeq [a, 1_L] \quad \textit{and} \quad [a_3, a_4] \not\preccurlyeq [0_L, a]\,.$$

Proof. If L is dense this follows immediately from Lemma 2.13 and Proposition 2.14. If L is not dense, $0_L \neq a \neq 1_L$, then $a \notin D(L)$ and suppose, for example, $a_1 < a_2 \leqslant a$, $]a_1, a_2[= \emptyset$; then $[a_1, a_2]$ is embeddable in $[a, 1_L]$. ◆

2.16. Corollary. *If L_i, $i = 1, \ldots, n$ and L'_j, $j = 1, \ldots, m$ are dense chains, then*

$$\mathrm{Hom}_c\Big(\sum_{i\leqslant n}{}_d\, L_i,\ \sum_{j\leqslant m}{}_d\, L'_j\Big) = \mathrm{Cons}$$

if and only if

$$\mathrm{Hom}_c(L_i, L'_j) = \mathrm{Cons}\ \textit{for all}\ i, j\,.$$

Proof. Let $[a', b'] \subseteq \sum_{j\leqslant m}{}_d\, L'_j$ with $a' < b'$ and φ be an order-preserving map from $[a', b']$ into $\sum_{i\leqslant n}{}_d\, L_i$. Then there exist $j \leqslant m$ and $a'_1, b'_1 \in [a', b'] \cap L'_j$ with $a'_1 < b'_1$; moreover, there exist $i \leqslant n$ and a'_2, b'_2 with $a'_1 \leqslant a'_2 < b'_2 \leqslant b'_1$ such that $\varphi|_{[a'_2, b'_2]}$ is an order-preserving map from $[a'_2, b'_2]$ into L_i. The converse follows from Remark 2.12. ◆

2.17. Corollary. *If L_i, $i = 1, \ldots, n$, are dense chains such that* $D(L_i) = \{0_{L_i}, 1_{L_i}\}$ for $i = 1, \ldots, n$ and $\mathrm{Hom}_c(L_i, L_{i'}) = \mathrm{Cons}$ *for $i \neq i'$, then* $D\Big(\sum_{i\leqslant n}{}_d\, L_i\Big) = \{0_{L_i} : i = 1, \ldots, n\} \cup \{1_{L_n}\}$.

Proof. The inclusion «$\supseteq$» follows from Lemma 2.13 and Corollary 2.16. To prove the other inclusion, observe that, if $c \in L_i \setminus \{0_{L_i}, 1_{L_i}\}$, then either

$$\mathrm{Hom}_c([0_{L_i}, c], [c, 1_{L_i}]) \neq \mathrm{Cons}$$

or

$$\mathrm{Hom}_c([c, 1_{L_i}], [0_{L_i}, c]) \neq \mathrm{Cons}$$

(Lemma 2.13). It follows from Remark 2.12 that either

$$\mathrm{Hom}_c([0_{L_1}, c], [c, 1_{L_n}]) \neq \mathrm{Cons}$$

or

$$\mathrm{Hom}_c([c, 1_{L_n}], [0_{L_1}, c]) \neq \mathrm{Cons}\ .$$

This shows, in view of Lemma 2.13, that $c \notin D\Big(\sum_{i\leqslant n}{}_d L_i\Big)$. ◆

3. Chains with non-trivial distortion points.

We now want to find, for any natural number $n \geqslant 3$, a chain with n distortion points. By Corollary 2.17, it suffices to find $n-1$ chains L_i $(i = 1, \dots, n-1)$ without non-trivial distortion points such that $\mathrm{Hom}_c(L_i, L_j) = \mathrm{Cons}$ for $i \neq j$. We shall solve this problem by studying the sup of the cardinality of well-ordered and well-ordered* subsets of a chain.

3.1. Definition. *For any chain L we put*:

$\mathrm{wo}_+ L = \sup\{\mathrm{card}\, A\colon A \subseteq L$ *and* A *well-ordered*$\}$,

$\mathrm{wo}_- L = \inf\{\mathrm{wo}_+ [a, b]\colon a, b \in L$ *and either* $a < b$ *or* $0_L = a = b = 1_L\}$,

$\mathrm{wo}^*_+ L = \sup\{\mathrm{card}\, A\colon A \subseteq L$ *and* A^* *well-ordered*$\}$,

$\mathrm{wo}^*_- L = \inf\{\mathrm{wo}^*_+ [a, b]\colon a, b \in L$ *and either* $a < b$ *or* $0_L = a = b = 1_L\}$.

We put $\mathrm{wo}(L) = \alpha$ *if* $\mathrm{wo}_+ L = \mathrm{wo}_- L = \alpha$, *and* $\mathrm{wo}^*(L) = \alpha$ *if* $\mathrm{wo}^*_+ L = \mathrm{wo}^*_- L = \alpha$.

3.2. Remark. We have $\mathrm{wo}^*_+ L = \mathrm{wo}_+ L^*$ and $\mathrm{wo}^*_- L = \mathrm{wo}_- L^*$. In the sequel we shall often give results concerning wo and omit the dual ones concerning wo*. ◆

The following proposition is an immediate consequence of Proposition 2.14.

3.3. PROPOSITION. *If* L *and* L' *are dense chains and* $\mathrm{wo}_+ L < < \mathrm{wo}_- L'$, *then* $\mathrm{Hom}_c(L, L') = \mathrm{Cons}$. ◆

3.4. THEOREM. *If* α *and* β *are infinite cardinals, then the chain* $\mathcal{C}_{\alpha,\beta} = \mathcal{C}({}^{\omega}(\beta^* + \alpha))$ *satisfies*:

i) $\mathrm{wo}(\mathcal{C}_{\alpha,\beta}) = \alpha$ *and* $\mathrm{wo}^*(\mathcal{C}_{\alpha,\beta}) = \beta$ (*so* $\mathcal{C}_{\alpha,\beta}$ *is dense*),

ii) $D(\mathcal{C}_{\alpha,\beta}) = \{0_{\mathcal{C}_{\alpha,\beta}}, 1_{\mathcal{C}_{\alpha,\beta}}\}$.

PROOF. i) We fix $\bar{c} \in \beta^* + \alpha$ and put

$$E_n = \{a \in {}^{\omega}(\beta^* + \alpha):\ a_i = \bar{c} \text{ for all } i > n\} \qquad \text{for } n \in \mathbf{N}.$$

We first prove that, if A is well-ordered and contained in E_n, then $\mathrm{card}\, A \leqslant \alpha$. If $n = 0$, this is obvious. Assume that the assertion holds for $n - 1$, and define $p: A \to E_{n-1}$ by

$$(p(a))_i = \begin{cases} a_i & \text{if } i < n, \\ \bar{c} & \text{if } i \geqslant n. \end{cases}$$

Then $p(A)$ is well-ordered and contained in E_{n-1}, whence $\mathrm{card}\, p(A) \leqslant \alpha$. For every $b \in p(A)$ the set $p^{-1}(b)$ is well-ordered and $p^{-1}(b) \preccurlyeq \beta^* + \alpha$. It follows that $\mathrm{card}\, p^{-1}(b) \leqslant \alpha$. Since $A = \bigcup_{b \in p(A)} p^{-1}(b)$, we get that $\mathrm{card}\, A \leqslant \alpha$. We now remark that, if A is well-ordered and contained in $\bigcup_{n \in \mathbf{N}} E_n$, then $\mathrm{card}\, A \leqslant \alpha$. Indeed,

$$A = \bigcup_{n \in \mathbf{N}} (A \cap E_n) \quad \text{and} \quad \mathrm{card}\,(A \cap E_n) \leqslant \alpha \qquad \text{for } n \in \mathbf{N}.$$

Finally, we suppose that A is well-ordered and contained in $\mathcal{C}_{\alpha,\beta}$. Then, for every $a \in A$, we can choose $s(a) \in \bigcup_{n \in \mathbf{N}} E_n$ such that $a < s(a) < a'$ whenever $a' \in A$ and $a' > a$. Hence s is order-preserving. Therefore, $A \preccurlyeq \bigcup_{n \in \mathbf{N}} E_n$, and so $\mathrm{card}\, A \leqslant \alpha$. Thus we have proved that $\mathrm{wo}_+\, C_{\alpha,\beta} \leqslant \alpha$ and one can easily show that $\mathrm{wo}_-\, \mathcal{C}_{\alpha,\beta} \geqslant \alpha$. Hence $\mathrm{wo}(\mathcal{C}_{\alpha,\beta}) = \alpha$ and dually $\mathrm{wo}^*(\mathcal{C}_{\alpha,\beta}) = \beta$.

ii) In view of Corollary 2.15, it is enough to show that $\mathcal{C}_{\alpha,\beta} \leqslant [a, 1_{\mathcal{C}_{\alpha,\beta}}]$ whenever $a \in \mathcal{C}_{\alpha,\beta}$ and $a < 1_{\mathcal{C}_{\alpha,\beta}}$. We may and do assume that $a \in {}^{\omega}(\beta^* + \alpha)$. Take $\bar{a} \in \beta^* + \alpha$ with $\bar{a} > a_1$ and put

$$P = \{c \in {}^{\omega}(\beta^* + \alpha)\colon\ c_1 = \bar{a}\}\,.$$

Clearly, $P \subseteq [a, 1_{\mathcal{C}_{\alpha,\beta}}]$ and $P \simeq {}^{\omega}(\beta^* + \alpha)$. An application of [5], Exercise 2.33(1) completes the proof. ◆

3.5. Corollary. *For every $n \geqslant 3$ there exists a chain L with n distortion points.*

Proof. By Corollary 2.17, Proposition 3.3 (and its dual) and Theorem 3.4, $\operatorname{card} D\Big(\sum_{i \leqslant n-1}^{d} \mathcal{C}_{\omega_i, \omega_{n-i}}\Big) = n$. ◆

3.6. Corollary. *There exists a chain L such that $QLIM(X, L) \neq LIM(X, L)$ provided that X has at least two elements.*

Proof. This follows from Corollaries 2.8 and 3.5. ◆

3.7. Remark. By Corollary 2.15, we may easily show that, under the assumption that $2^{\aleph_0}$ is a regular cardinal, the chain $\mathfrak{A}$ considered by L. Gillman in [2] satisfies $D(\mathfrak{A}) = \mathfrak{A}$. ◆

REFERENCES

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ., Providence, 1967.

[2] L. Gillman, *A continuous exact set*, Proc. Amer. Math. Soc., **9** (1958), pp. 412-418.

[3] G. H. Greco, *Limitoidi e reticoli completi*, Ann. Univ. Ferrara, **29** (1983), pp. 153-164.

[4] G. H. Greco, *Operatori di tipo G su reticoli completi*, Rend. Sem. Mat. Univ. Padova, **72** (1984), pp. 277-288.

[5] J. G. Rosenstein, *Linear Orderings*, Academic Press, New York, 1982.

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