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Uniqueness of the Cauchy problem for a second order operator

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1. Introduction and statement of the result.

Uniqueness and non-uniqueness of the Cauchy problem have been the subject of a large number of studies during the last thirty years (see [8]). In the present communication a uniqueness theorem for a second order operator in two variables with roots with variable multeplicity, is proved.

Let U be an open set of $\boldsymbol{R}^2 = \boldsymbol{R}_x \times \boldsymbol{R}_t$; suppose $0 \in U$. Consider the differential operator:

$$P(x, t, D_x, D_t) = \\ = D_t^2 + a(x, t) D_x^2 + b(x, t) D_x + c(x, t) D_t + d(x, t) \tag{1.1}$$

where $a \in C^\infty(U)$ and $b, c, d \in L^\infty_{\text{loc}}(U)$. Here, as usual, D_x stands for $(1/i)(\partial/\partial x)$.

THEOREM 1. *Let k be an integer positive number. Suppose that*

$$a(x, t) = t^k \alpha(x, t) \tag{1.2}$$

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and, writing $\alpha(x, t) = \alpha_1(x, t) + i\alpha_2(x, t)$, $\alpha_j \in \boldsymbol{R}$ *for* $j = 1, 2$, *suppose:*

$$(1.3) \qquad (\alpha_2(x, t) \equiv 0 \quad \textit{and} \quad \alpha_1(0, 0) \neq 0) \quad \textit{or} \quad \alpha_2(0, 0) \neq 0\,.$$

Suppose also that there exists $C > 0$ *so that*

$$(1.4) \qquad |b(x, t)| \leqslant C t^{k/2-1}\,.$$

Let $u \in \mathcal{C}^\infty(U)$, $\operatorname{supp}(u) \subseteq \{(x, t) \in U : t \geqslant 0\}$ *and* $Pu \equiv 0$ *in* U. *Then* $u \equiv 0$ *in a neighbourhood of* 0, *i.e. there is the uniqueness of the Cauchy problem for* P *with respect to the surface* $\{t = 0\}$ *in* 0.

REMARK 1. If k is even, theorem 1 can be deduced from a more general result of Nakane ([3], theorem 1) concerning operators in several variables and of order m.

REMARK 2. Conditions (1.2) and (1.3) are, in a way, analogous to the conditions on roots in Calderon's uniqueness theorem. Infact if $k = 0$, (1.2) and (1.3) are exactly the requests of Calderon's theorem.

Condition (1.4) on the term of the first order is known as Levi type condition.

REMARK 3. It is of particular interest to make a comparison between theorem 1 and some known result in the case that $\alpha_2 \equiv 0$, i.e. in the case of operators with real principal part.

If $\alpha_1(0, 0) < 0$, the operator (1.1) is weakly hyperbolic: condition (1.4) is then sufficient to imply also the well-posedness of the $\mathcal{C}^\infty$-Cauchy problem ([6]). On the contrary, Nakane ([4]) proves that, when:

$$\tilde{P} = D_t^2 - t^k D_x^2 + t^m D_x$$

if $m < k/2 - 1$ then there exist $d, u \in \mathcal{C}^\infty$ so that

$$0 \in \operatorname{supp}(u) \subseteq \{(x, t) : t \geqslant 0\} \qquad \textit{and} \qquad \tilde{P}u + du \equiv 0$$

i.e. non-uniqueness holds for $\tilde{P}$ with respect to $\{t = 0\}$ (for non-solvability of the Cauchy problem for the operator $\tilde{P}$ see also [2]).

If $\alpha_1(0,0) > 0$ the operator (1.1) is degenerate elliptic with real principal part. Watanabe ([7]) shows that, if k is even, theorem 1 is true without conditions on lower order terms, i.e. without any Levi type condition. If k is odd the only known result is Nirenberg's one ([5]) but in this case theorem 1 is more general, in the way that, Nirenberg's result is a compact-uniqueness result and the condition on $b(x, t)$ is

$$|b(x,t)| \leqslant Ct^{k/2} \tag{1.4)'$$

which is more restrictive than (1.4).

2. Proof of the Theorem.

The technique of the proof is inspired to [9], and it is based on a Carleman estimate in which function t^{-N} is the weight function.

It is possible to apply the Carleman estimate directly to the operator (1.1), but it would give only a particular uniqueness result: in that case we obtain that there exists a fixed neighbourhood, U, of the origin so that there are no solutions, u, of:

$$Pu = 0 \tag{2.1}$$

when $0 \in \operatorname{supp}(u)$ and $\operatorname{supp}(u)$ is a compact in $\{t \geqslant 0\} \cap U$.

To reach our result we need to make this change of variables (singular change of variables: see [1])

$$\begin{aligned} x &= y\,, \\ t &= (r - y^2)\,s\,, \end{aligned} \tag{2.2}$$

where $r > 0$ and it will be fixed later.

With this transformation it is possible to obtain from a solution of (2.1) with support in $t \geqslant 0$ a new solution, w, of:

$$Qw = 0$$

with $\operatorname{supp}(w) \subseteq \{(y, s) : s \geqslant 0,\ |y|^2 \leqslant r\}$; here $Q = (r - y^2)^2 \tilde{P}$ and $\tilde{P}$ is the operator P in the new variables.

Theorem 1 follows, with standard technique, from next lemma.

Lemma 1. *There exist* $S_0>0$ $N_0>0$, $C>0$ *so that if* $0<S<S_0$, $N>N_0$, *and* $u\in C_0^\infty([-\sqrt{r},\sqrt{r}]\times[0,S])$, *then*

$$N^2\|s^{-N-2}u\|^2\leqslant C\|s^{-N}Qu\|^2 \tag{2.3}$$

where $\|\cdot\|=\|\cdot\|_{L^2_{R_y\times R_s}}$.

If we write x and t instead of y and s, Q has the following form:

$$\begin{aligned}Q=(r-x^2)^2\tilde{P}=[1&+4x^2t^{k+2}(r-x^2)^k\alpha(x,t)]D_t^2+\\&+4xt^{k+1}\alpha(x,t)(r-x^2)^{k+1}D_xD_t+\alpha(x,t)(r-x^2)^{k+2}t^kD_x^2+\\&+(r-x^2)^2b(x,t)D_x+c^*(x,t)D_t+d^*(x,t)\end{aligned} \tag{2.4}$$

where $c^*, d^*\in L^\infty_{\text{loc}}(U)$.

Calling q the principal part of Q, we have:

$$q=A\partial_1\partial_2-BD_x \tag{2.5}$$

where

$$\begin{aligned}A(x,t)&=1+4x^2t^{k+2}(r-x^2)^k\alpha(x,t)\,,\\B(x,t)&=A(x,t)\big[D_t(-t^{k/2}(r-x^2)^{k/2+1}\sigma_2)+\\&\qquad+t^{k/2}(r-x^2)^{k/2+1}\sigma_1D_x(-t^{k/2}(r-x^2)^{k/2+1}\sigma_2)\big]\,,\\\partial_j&=D_t-t^{k/2}(r-x^2)^{k/2+1}\sigma_j(x,t)D_x\,,\qquad j=1,2\,,\end{aligned} \tag{2.6}$$

and

$$\sigma_j(x,t)=\frac{-2xt^{k/2+1}\alpha(x,t)(r-x^2)^{k/2}+(-1)^j\sqrt{-\alpha(x,t)}}{1+4x^2t^{k+2}(r-x^2)^k\alpha(x,t)}\,. \tag{2.7}$$

Note that choosing r small enough, if $\text{Im}\,\sigma_j(0,0)\neq0$, there exists $T_0>0$ so that, if $0\leqslant t\leqslant T_0$ and $x\in[-\sqrt{r},\sqrt{r}]$ then

$$|\text{Im}\,(\sigma_j(x,t))|\geqslant\varepsilon>0\,. \tag{28}$$

With the notations stated before, it is possible to prove the following:

Lemma 2. *There exist* $T_0 > 0$, $N_0 > 0$, $C > 0$, $C' > 0$ *so that if* $0 < T < T_0$, $N > N_0$ *and* $u \in C_0^\infty([-\sqrt{r}, \sqrt{r}] \times [0, T])$, *then:*

$$N\|t^{-N-1}u\|^2 \leqslant C\|t^{-N}\partial_j u\|^2 \quad j = 1, 2. \tag{2.9}$$

$$N^2\|t^{-N-2}u\|^2 \leqslant C'\|t^{-N}\partial_1\partial_2 u\|^2 . \tag{2.10}$$

Proof. Let $v = t^{-N}u$, and $u \in C_0^\infty([-\sqrt{r}, \sqrt{r}] \times [0, T])$.
Let:

$$\begin{aligned} (r - x^2)^{k/2+1}\sigma_{j1}(x, t) &= \operatorname{Re}\left((r - x^2)^{k/2+1}\sigma_j(x, t)\right) = A_j(x, t) , \\ (r - x^2)^{k/2+1}\sigma_{j2}(x, t) &= \operatorname{Im}\left((r - x^2)^{k/2+1}\sigma_j(x, t)\right) = B_j(x, t) , \end{aligned} \tag{2.11}$$

note that $(r - x^2)^{k/2+1}\sigma_j$ are C^1 in x and t.
Writing:

$$\begin{aligned} X_j &= D_t - t^{k/2}A_j(x, t)D_x , \\ Y_j &= \frac{N}{t} + t^{k/2}B_j(x, t)D_x , \end{aligned} \tag{2.12}$$

we obtain

$$\|t^{-N}\partial_j u\|^2 = \|X_j v - iY_j v\|^2 = \|X_j v\|^2 + \|Y_j v\|^2 + 2\operatorname{Re}\langle X_j v, -iY_j v\rangle .$$

Let's calculate the last term and we find:

$$2\operatorname{Re}\left\langle D_t v, -i\frac{N}{t}v\right\rangle = N\left\|\frac{v}{t}\right\|^2 , \tag{2.13}$$

$$\left|2\operatorname{Re}\left\langle -t^{k/2}A_j(x, t)D_x v, -i\frac{N}{t}v\right\rangle\right| \leqslant NC_1\|t^{k/4-\frac{1}{2}}v\|^2 , \tag{2.14}$$

$$2\operatorname{Re}\langle -t^{k/2}A_j(x, t)D_x v, -it^{k/2}B_j(x, t)D_x v\rangle = 0 . \tag{2.15}$$

If $B_j \equiv 0$ (i.e. P is weakly hyperbolic) it is easy to reach (2.9) from (2.13), (2.14) and (2.15). Suppose that $B_j(0, 0) \neq 0$; knowing that, in this case, (2.8) is true and using a punctual inequality, we obtain that there exists $C_2 > 0$ so that:

$$\|t^{k/2}A_j(x, t)D_x v\| \leqslant C_2\|t^{k/2}B_j(x, t)D_x v\| \tag{2.16}$$

$$\|t^{k/2}D_t(B_j(x, t))D_x v\| \leqslant C_2\|t^{k/2}B_j(x, t)D_x v\| . \tag{2.17}$$

Consider then:

$$(2.18)\qquad \begin{aligned} &= 2\,\mathrm{Re}\,\langle D_t v, -it^{k/2}B(x,t)D_x v\rangle = \\ &= \langle D_t v, -it^{k/2}B(x,t)D_x v\rangle + \langle -it^{k/2}B(x,t)D^k v, D_t v\rangle = \\ &= i\langle t^{k/2}D_x(B(x,t))D_t v, v\rangle - \frac{k}{2}\langle t^{k/2-1}B(x,t)D_x v, v\rangle + \\ &\qquad - i\langle t^{k/2}D_t(B(x,t))D_x v, v\rangle = a + b + c\,. \end{aligned}$$

Using (2.16) we have:

$$\begin{aligned} |a| &\leqslant C_3T^{k/2}\|D_t v\|\,\|v\| \leqslant C_3T^{k/2}\|D_t v - t^{k/2}A(x,t)D_x v\|\,\|v\| + \\ &\quad + C_3T^{k/2}\|t^{k/2}A(x,t)D_x v\|\,\|v\| \leqslant \\ &\leqslant C_3T^{k/2}\|Xv\|\,\|v\| + C_2C_3T^{k/2}\|t^{k/2}B(x,t)D_x v\|\,\|v\| \leqslant \\ &\leqslant C_3T^{k/2}\|Xv\|\,\|v\| + C_2C_3T^{k/2}\left(\|Yv\|\,\|v\| + N\left\|\frac{v}{t}\right\|\,\|v\|\right) \end{aligned}$$

and:

$$\begin{aligned} b &= -\frac{k}{2}\langle t^{k/2-1}B(x,t)D_x v, v\rangle = \\ &= -\frac{k}{2}\left\langle t^{k/2}B(x,t)D_x v + N\frac{v}{t}, \frac{v}{t}\right\rangle + \frac{k}{2}\left\langle N\frac{v}{t}, \frac{v}{t}\right\rangle = -\frac{k}{2}\left\langle Yv, \frac{v}{t}\right\rangle + \frac{k}{2}N\left\|\frac{v}{t}\right\|^2. \end{aligned}$$

Using (2.17) we have:

$$|c| \leqslant C_2\|t^{k/2}B(x,t)D_x v\|\,\|v\| \leqslant C_2\|Yv\|\,\|v\| + C_2N\left\|\frac{v}{t}\right\|\,\|v\|\,.$$

Finally:

$$\left|a + b + c - \frac{k}{2}N\left\|\frac{v}{t}\right\|^2\right| \leqslant (2.19)$$

where:

$$(2.19)\qquad \begin{aligned} &= 2C_3T^{k/2}(\|Xv\|^2 + \|v\|^2) + 2C_2C_3T^{k/2}(\|Yv\|^2 + \|v\|^2) + \\ &\quad + 2NC_2C_3T^{k/2}\left(\left\|\frac{v}{t}\right\|^2 + \|v\|^2\right) + \frac{k}{2}\left(\varepsilon\|Yv\|^2 + C_\varepsilon\left\|\frac{v}{t}\right\|^2\right) + \\ &\quad + C_2(\varepsilon\|Yv\|^2 + C_\varepsilon\|v\|^2) + C_2N\left(\varepsilon\left\|\frac{v}{t}\right\|^2 + C_\varepsilon\|v\|^2\right) \end{aligned}$$

ε and C_ε are constants.

We obtain:

$$(2.18) \geqslant \frac{k}{2} N \left\| \frac{v}{t} \right\|^2 - (2.19)$$

and from (2.13), (2.14), (2.15) and (2.18) we have:

$$\|t^{-N} \partial_j u\|^2 \geqslant \|Xv\|^2 + \|Yv\|^2 + N\left(1 + \frac{k}{2}\right) \left\| \frac{v}{t} \right\|^2 - \left[(2.19) + NC_1 \|t^{k/4 - \frac{1}{2}} v\|^2\right].$$

It is easy to see that choosing T_0 and N_0^{-1} small enough we have:

$$\|Xv\|^2 + \|Yv\|^2 + \left(\frac{1}{2} + \frac{k}{2}\right) N \left\| \frac{v}{t} \right\|^2 - \left[(2.19) + NC_1 \|t^{k/4 + \frac{1}{2}} v\|^2\right] \geqslant 0$$

by which (2.9) is proved.

Using (2.9) twice we get (2.10). Q.E.D.

Consider $g_1, g_2 \in L^\infty$. Let us define $\mathfrak{R}$ as the set of operators of the kind:

$$R = \frac{g_1(x, t)}{t} D_t + t^{k/2-1}(r - x^2)^{k/2+1} g_2(x, t) D_x .$$

Note that for each $R \in \mathfrak{R}$ there exist $h_1, h_2 \in L^\infty$ so that

$$R = h_1(x, t) \frac{\partial_1}{t} + h_2(x, t) \frac{\partial_2}{t} .$$

If $R \in \mathfrak{R}$, it is easy to see that there exist $C > 0$, $N_0 > 0$ so that if $u \in C_0^\infty([-\sqrt{r}, \sqrt{r}] \times [0, T_0])$ and $N > N_0$ then:

$$(2.20) \qquad N \|t^{-N} Ru\|^2 \leqslant C \|t^{-N} \partial_1 \partial_2 u\|^2$$

with T_0 as in Lemma 2.

Infact from (2.9) we have:

$$(2.21) \qquad N \|t^{-N} Ru\|^2 \leqslant C\left(\|t^{-N} \partial_1 \partial_2 u\|^2 + \|t^{-N} \partial_2 \partial_1 u\|^2\right) ;$$

note now that:

$$\partial_2\partial_1 - \partial_1\partial_2 = S \in \mathfrak{R}\,; \tag{2.22}$$

applying (2.21) to S we deduce:

$$N\|t^{-N}Su\|^2 \leqslant C(\|t^{-N}\partial_1\partial_2 u\|^2 + \|t^{-N}\partial_2\partial_1 u\|^2)$$

so

$$N\|t^{-N}Su\|^2 \leqslant C(\|t^{-N}\partial_1\partial_2 u\|^2 + \|t^{-N}(Su + \partial_1\partial_2 u)\|^2)$$

from which

$$N\|t^{-N}Su\|^2 \leqslant C\|t^{-N}\partial_1\partial_2 u\|^2\,. \tag{2.23}$$

Using (2.22), (2.23) and (2.21) we obtain (2.20).

Let us finally come to the proof of Lemma 1. Using (2,4), (2.5) and (1.4) we can write

$$Qu = A\partial_1\partial_2 u + Ru + t^{-2}\gamma(x,t)\,u$$

where $R \in \mathfrak{R}$ and $\gamma \in L^\infty$.

Note that Q has this form only because the Levi type condition is true.

It follows:

$$2\|t^{-N}Qu\|^2 \geqslant \|t^{-N}A\partial_1\partial_2 u\|^2 - 4(\|t^{-N}Ru\|^2 + \|t^{-N-2}\gamma u\|^2)$$

but applying (2.20), with N sufficently large, we have:

$$\tfrac{1}{4}\|t^{-N}A\partial_1\partial_2 u\|^2 > 4\|t^{-N}Ru\|^2$$

and applying (2.10) we obtain:

$$\tfrac{1}{4}\|t^{-N}A\partial_1\partial_2 u\|^2 > 4(\|\gamma\|_{L^\infty})^2\|t^{-N-2}u\|^2\,.$$

So that again from (2.10):

$$2\|t^{-N}Qu\|^2 \geqslant \frac{1}{2}\|t^{-N}A\partial_1\partial_2 u\|^2 \geqslant \frac{N^2}{2C'}\|t^{-N-2}u\|^2\,. \qquad \text{Q.E.D.}$$

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